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ON h-TOPOLOGICAL GROUPS

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Abstract. In this paper, we introduced new notions namely h-topological groups and h-irresolute topological groups by using h-open sets [given by F. Abbas]. Some of the fundamental characteristics of these newly introduced spaces have been thoroughly studied. It has been observed that the notion of h-irresolute topological group is independent of the notion of topological group. Additionally, h-regular and h-Lindelof spaces are presented and used to further explore h-topological groups and h-irresolute topological groups respectively.

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1. Introduction

A topological group is defined as a group with a topology such that binary operations such as multiplication and inverse mapping are continuous. For more information regarding topological groups, one can refer [6,7,10]. In [3], the notions of S-topological groups and s-topological groups have been explored. Recently, Sharma et.al. [9] introduced a new class of topological vector space namely h-irresolute topological vector spaces. In this study, we generalised h-topological group via h-open sets. Another new notion called as h-irresolute topological groups have also been introduced.

2. Preliminaries

This section deals with some basic definitions that will used in the subsequent sections. X and Y will represent two topological spaces with topologies τ and σ respectively, on which no separation axioms are imposed. $Int(P)$ and $Cl(P)$ are the notations for the interior and closure of a subset P of topological space X. A subset P of a topological space X is said to be h-open if $P \subseteq Int(P \cup V)$, where

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$V \in \tau$ and $V \neq \phi, X$. Complement of an h-open set is called an h-closed set. It is evident from the definition that every open set is h-open but converse need not be true. τ_h will be the notation used for the family of all h-open sets in a topological space (X, τ). $Int_h(P)$ denotes the h-interior of a subset P of X and defined as the union of all h-open sets in X contained in P. Also, $Cl_h(P)$ denotes the h-closure of a subset P of X and defined as the intersection of all h-closed sets in X containing P. For more notions and results on h-open sets, one can see [1]. Further, recent work on h-open sets can be seen in [2,4]. Let us recall some definitions that will be used frequently:

Definition 2.1. [1] *A mapping $g : X \to Y$ is said to be*

(1) *h-continuous if inverse image of every open set in Y is h-open in X.*
(2) *h-open if image of h-open set in X is h-open in Y.*
(3) *h-irresolute if inverse image of every h-open set in Y is h-open in X.*
(4) *h-totally continuous if inverse image of every h-open set in Y is clopen in X.*
(5) *h-homeomorphism if it is bijective, h-continuous and h-open.*

3. h-TOPOLOGICAL GROUPS

Definition 3.1. *Let $(G, *)$ be a group endowed with topology τ on it. Then $(G, *, \tau)$ is said to be an h-**topological group** if group operation $*$ as well as the inverse operation $^{-1}$ are h-continuous. Equivalently,*

(1) *for each open neighborhood W of $p * q$, there exists h-open neighborhoods U and V containing p and q respectively such that $U * V \subseteq W$,*
(2) *for each open neighborhood W of p^{-1}, there exists an h-open neighborhood U containing p such that $U^{-1} \subseteq W$.*

Example 3.1. *Let $G = (Z_3, \oplus)$ and $\tau = \{\phi, \{1, 2\}, Z_3\}$. Now $\tau_h = \{\phi, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, Z_3\}$. Then $(G, \oplus, \tau)$ is an example of h-topological group.*

It is evident from the definition that every topological group is an h-topological group but converse need not be true.

Example 3.2. *Let $G = (Z_2, \oplus)$ and $\tau = \{\phi, \{0\}, Z_2\}$. Here $\tau^h = \{\phi, \{0\}, \{1\}, Z_2\}$. Then $(G, \oplus, \tau)$ is an h-topological group which is not a topological group.*

Note 3.1. *For infinite topological spaces with topologies other than trivial ones, it is difficult to generate all h-open sets and, consequently, an h-topological group. However, we can give one such example of an h-topological group without finding all the h-open subsets of that set.*

Example 3.3. *Consider the group of real numbers $(\mathbb{R}, +)$ with the topology $\tau = \{\phi, \mathbb{Q}^c, \mathbb{R}\}$. Clearly, $\mathbb{Q}$ is h-open as $\mathbb{Q} \subseteq Int\,(Q \cup Q^c) = Int\,(\mathbb{R}) = \mathbb{R}$. Also, $\{\{x\} : x \in \mathbb{Q}^c\}$ is set of h-open sets as $\{x\} \subseteq Int\,(\{x\} \cup \mathbb{Q}^c) =$*

$Int(Q^c) = Q^c$. Next, let $a, b \in \mathbb{R}$. Then for each $a + b \in \mathbb{R}$, consider the following cases:
Case 1: If $a + b \in \mathbb{Q}$, then possible open neighborhood of $a + b$ is $\mathbb{R}$. Proof follows trivially in this case.
Case 2: If $a + b \in \mathbb{Q}^c$, then possible open neighborhood of $a + b$ is $\mathbb{Q}^c, \mathbb{R}$. For open neighborhood $\mathbb{R}$, proof follows trivially. For each $a + b \in \mathbb{Q}^c$ and open neighborhood $\mathbb{Q}^c$ containing $a + b$, we have following sub-cases:
Sub-case(i): $a \in \mathbb{Q},\ b \in \mathbb{Q}^c$, then there exists h-open sets $\mathbb{Q}$ and $\mathbb{Q}^c$ containing a and b respectively such that $\mathbb{Q} + \mathbb{Q}^c \subseteq \mathbb{Q}^c$.
Sub-case(ii): $a \in \mathbb{Q}^c,\ b \in \mathbb{Q}$, then there exists h-open sets $\mathbb{Q}^c$ and $\mathbb{Q}$ containing a and b respectively such that $\mathbb{Q}^c + \mathbb{Q} \subseteq \mathbb{Q}^c$.
Sub-case(iii): $a \in \mathbb{Q}^c,\ b \in \mathbb{Q}^c$, then there exists h-open sets $\{a\}$ and $\{b\}$ containing a and b respectively such that $\{a\} + \{b\} \subseteq \mathbb{Q}^c$.
This proves h-continuity of $+$. Similarly, we can prove h-continuity of $-$. Thus, $(R, +, \tau)$ is an h-topological group. But $(R, +, \tau)$ is not a topological group because for open neighborhood $\mathbb{Q}^c$ of $2 + \sqrt{3}$, neither $\mathbb{R} + \mathbb{R} \subseteq \mathbb{Q}^c$ nor $\mathbb{R} + \mathbb{Q}^c \subseteq \mathbb{Q}^c$.

Next, we shall prove some results and provide a way to generate finite h-topological groups.

Theorem 3.1. *Consider a group $(G, *)$ endowed with topology τ. Suppose τ_h is discrete. Then $(G, *, \tau)$ is an h-topological group.*

Proof. Suppose cardinality of G is 1, then the proof follows trivially. Now we shall prove the result if cardinality of G is greater than one. Let p, q be any two elements of G and W be an open neighborhood of $p * q$. By given hypothesis, $\{p\}$ and $\{q\}$ are h-open neighborhoods of p and q respectively. Also, $\{p\} * \{q\} = \{p * q\} \subseteq W$. Also, let U be an open neighborhood of p^{-1}. Again by the same hypothesis, there exists a h-open neighborhood $\{p\}$ of p such that $\{p\}^{-1} = \{p^{-1}\} \subseteq U$. Hence, the proof. □

Remark 3.1. *Converse of the Theorem 3.1 need not be true as $G = \{1, -1, i, -i\}$, the fourth roots of unity endowed with the topology $\tau = \{\phi, \{1, -1\}, \{i, -i\}, G\}$ and $\tau_h = \{\phi, \{1, -1\}, \{i, -i\}, G\}$ is an h-topological group but τ_h is not discrete.*

Using preceding theorem, we can provide some instances of h-topological groups as follows:

(1) Consider a group $(G, *)$ and any of discrete, indiscrete or seirpinski topology on it. This is an example of h-topological group.
(2) Consider the group $(\mathbb{R}, +)$ with the topology $\tau = \{\phi, \mathbb{R} \setminus \{1\}, \mathbb{R}\}$. We have $\tau^h = \mathcal{P}(\mathbb{R})$. Then $(\mathbb{R}, +, \tau)$ is an h-topological group that is not a topological group.

Theorem 3.2. *Suppose $(G, *)$ be a group of finite order having a subgroup K of index 2. Consider topology $\tau = \{\phi, K, K^c, G\}$ endowed on G. Then $(G, *, \tau)$ is an h-topological group.*

Proof. Proof follows directly from Theorem 3.8 of [8]. □

Theorem 3.3. *Let X be any non-empty set endowed with the topology $\tau = \{\phi, X \setminus \{x\}, X\}$ where x be any arbitrary element of X. Then $\tau_h = \mathcal{P}(X)$.*

Proof. Straightforward. □

Note 3.2. *The only connected finite topological group is a finite group with indiscrete topology. However, we can offer other connected topologies for h-topological group. Using preceding theorem, we can provide one such classical example as follows:*

Example 3.4. *Consider any group $(G, *)$ with cardinality greater than one and $\tau = \{\phi, G \setminus \{e\}, G\}$. Then $\tau_h = \mathcal{P}(G)$. Clearly, $(G, *, \tau)$ is a connected h-topological group with topology other than indiscrete one.*

Note 3.3. *Clearly $\tau = \tau_h$ for discrete and indiscrete topologies. Our endeavour would be to find a topology other that discrete and indiscrete for which $\tau = \tau_h$.*

Proofs of the following two results follows trivially, hence omitted.

Theorem 3.4. *Let (X, τ) be a topological space endowed with the topology $\tau = \{\phi,\ A,\ X \setminus A,\ X\}$, where $A \subseteq X$. Then $\tau = \tau_h$.*

Theorem 3.5. *Let (X, τ) be a disconnected topological space. Then $\tau = \tau_h$.*

Consequently, an h-topological group is a topological group if conditions of Theorem 3.4 or Theorem 3.5 is satisfied by it.

From now onwards, we shall assume that G refers to a topological group with binary operation $*$ and topology τ.

Theorem 3.6. *Consider an h-topological group G. Let U be an open set in G containing identity element e. Then*

(1) *left translation and right translation mappings on G are h-continuous.*
(2) *there exists a h-open set V containing e such that $V * V \subseteq U$ and $V^{-1} \subseteq U$.*

Proof. Straightforward. □

Theorem 3.7. *Consider an open set B in an h-topological group G. Then the following holds:*

(1) *$i * B$ is h-open, $\forall\, i \in G$;*
(2) *$B * i$ is h-open, $\forall\, i \in G$;*
(3) *B^{-1} is h-open.*

Proof. (1) Let $y \in i * B$. Then $y = i * b$ for some $b \in B$. By theorem, we have $i^{-1} * V \subseteq B$, for some h-open set V. Thus, $V \subseteq i * B$. This implies that $y \in\ Int_h(i * B)$. Hence, the proof.

(2) Proof for this part is analogous to the first part.

(3) Consider an element y^{-1} of B^{-1}. Clearly, $y \in B$. Since G is an h-topological group, there exists an h-open neighborhood V of y such that $V^{-1} \subseteq B$. This implies that $V \subseteq B^{-1}$. Thus $y^{-1} \in Int_h(B^{-1})$ and hence the proof. □

Corollary 3.1. *Let G be an h-topological group and $A \in \tau$. Then for any subset B of G, we have $A * B$ and $B * A$ h-open.*

Note 3.4. *The conclusion that the Sorgenfrey line is not an h-topological group can be easily drawn using the preceding theorem as $B = [4, 5)$ is open in this topology but $-B = (-5, -4]$ is not h-open. Thus Sorgenfrey line is an example that is neither a topological group nor an h-topological group.*

Theorem 3.8. *Consider a closed set B in an h-topological group G. Then the following holds:*

(1) *$i * B$ is h-closed, $\forall\, i \in G$;*
(2) *$B * i$ is h-closed, $\forall\, i \in G$;*
(3) *B^{-1} is h-closed.*

Proof. (1): We have to show that $i * B = Cl_h(i * B)$. For this, let $p \in Cl_h(i * B)$ and V be an open neighborhood of $q = i^{-1} * p$. Since G is given to be an h-topological group, there exists h-open sets V_1 and V_2 in G containing i^{-1} and p respectively, such that $V_1 * V_2 \subseteq V$. Also, intersection of h-open neighborhood V_2 and $i * B$ is non-empty as $p \in Cl_h(i * B)$. Suppose r belongs to $V_2 \cap (i * B)$. Then $i^{-1} * r \in B \cap (V_1 * V_2) \subseteq B \cap V$. This implies that $B \cap V \neq \phi \Rightarrow q \in B \Rightarrow p \in i * B$. Hence, the proof.

(2) Proof is similar to the previous part.

(3) Straightforward. □

Theorem 3.9. *Consider an h-topological group G. Let B be any subset of G, then $\forall\, i \in G$, we have:*

(1) *$Cl_h(i * B) \subseteq i * Cl(B)$;*
(2) *$i * Cl_h(B) \subseteq Cl(i * B)$;*
(3) *$i * Int(B) \subseteq Int_h(i * B)$;*
(4) *$Int(i * B) \subseteq i * Int_h(B)$.*

Proof. (1) Consider an element p of $Cl_h(i * B)$. Suppose $q = i^{-1} * p$. By given hypothesis, for every open set U in G containing q, there exists h-open sets U_1 and U_2 in G containing i^{-1} and p respectively such that $U_1 * U_2 \subseteq U$. As $p \in Cl_h(i * B)$, we have $r \in i * B \cap U_2$. Thus, $r \in i * B$ and $r \in U_2$. Clearly, $i^{-1} * r \in B$ and $i^{-1} * r \in U_1 * U_2$. This implies that $i^{-1} * r \in B \cap (U_1 * U_2) \subseteq B \cap U$. Therefore, $q \in Cl(B) \Rightarrow p \in i * Cl(B)$.

(2) Consider an element p of $Cl_h(B)$. Suppose $q = i * p$ and U be an open set containing q. By given hypothesis, $\exists$ h-open sets U_1 and U_2 in G containing i and p respectively such that $U_1 * U_2 \subseteq U$. As

$p \in Cl_h(B)$, we have $B \cap U_2$ non-empty. Let $r \in B \cap U_2 \Rightarrow r \in B$ *and* $r \in U_2$. Now $i * r \in i * B$ and $i * r \in U_1 * U_2$. Therefore, $i * r \in i * B \cap (U_1 * U_2) \subseteq i * B \cap U$. This implies that $i * B \cap U$ is non-empty and we have $p \in Cl(i * B)$.

(3) Follows from Theorem 3.7.

(4) Let $p \in Int(i * B)$ and $p = i * q$, where $q \in B$. Since G is a h-topological group, there exists h-open sets U_1 and U_2 in G containing i and p respectively such that $U_1 * U_2 \subseteq Int(i * B)$. Now $i*U_2 \subseteq U_1*U_2 \subseteq Int(i*B) \subseteq i*B$. Consequently, $i*U_2 \subseteq i*Int_h(B)$. This results in $p \in i*Int_h(B)$. □

Theorem 3.10. *Consider an h-topological group G and any two subsets D and E of G. Then*

(1) $Cl_h(D) * Cl_h(E) \subseteq Cl(D * E)$;

(2) $(Cl_h(D))^{-1} \subseteq Cl(D^{-1})$

Proof. (1)Let $a \in Cl_h(D) * Cl_h(E)$. Then $a = b * c$ for some $b \in Cl_h(D)$ and $c \in Cl_h(E)$. Let U be an open neighborhood in G containing a. Then by h-continuity of $*$, there exists h-open sets U_1 and U_2 containing d and e respectively such that $U_1 * U_2 \subseteq U$. Clearly, $D \cap U_1$ and $E \cap U_2$ are non-empty. Let $u \in D \cap U_1$ and $v \in E \cap U_2$. We can see that $u * v \in (D * E) \cap U$, thus $(D * E) \cap U$ is non-empty. Hence, the proof.

(2) Let $a \in (Cl_h(D))^{-1}$. Then $a = b^{-1}$, where $b \in Cl_h(D)$. Now consider an open neighborhood V of b^{-1}. Then by given hypothesis, there exists an h-open set W of b such that $W^{-1} \subseteq V$. Also, $b \in Cl_h(D)$ implies that W intersects with C and contains d, say. Clearly, $d^{-1} \in V \cap D^{-1}$, which completes the proof. □

Theorem 3.11. *Let G be an h-topological group. Then the following holds:*

(1) *Every left and right translation mapping on G is h-homeomorphism;*

(2) *Every inverse mapping is h-homeomorphism.*

Proof. (1) We shall prove the theorem for the left translation mapping only. Proof for the right translation mapping follows along similar lines. For this, let $i \in G$ and define left translation mapping $c_i : G \to G$ as $c_i(p) = i * p$, where p is an arbitrary element of G. Since every left translation mapping is bijective and h-continuous. We claim that c_i is h-open as well. Let B be an open set in G. Then by Theorem 3.7, $c_i(B) = i * B = \{i\} * B$ is h-open in G. Hence, the claim.

(2) Straightforward. □

Definition 3.2. *Let X be a topological space. Then X is said to be h-**homogeneous** if there exists a h-homeomorphism $g : X \to X$ such that $g(p) = q \; \forall \; p, q \in X$.*

Theorem 3.12. *Every h-topological group is an h-homogeneous space.*

Proof. Let $p, q \in G$ and $r = p^{-1} * q$. Then left translation mapping is a h-homoeomorphism such that $c_r(p) = r * p = q$. □

Theorem 3.13. *Consider an h-topological group G and a subgroup K of G containing a non-empty open set V of G, then K is h-open in G.*

Proof. By given hypothesis and Theorem 3.7, $b * V$ is h-open in G for each b in K. Thus, $K = \bigcup_{b \in K} b * V$ is h-open in G. □

Theorem 3.14. *Consider an h-topological group G and an open subgroup K of G, then K is h-closed. Also, K is an h-topological group.*

Proof. Since K is an open subgroup of G, the family $\mathcal{A} = \{b_i * K : b_i \in G\}$ of all left cosets of K is h-open covering of G. Also, $G = \bigcup_{b_i \in G} b_i * K$ and so for each $b_i \in G$, $b_i * K$ is both h-open as well as h-closed. This implies that $K = e * K$ is h-open as well as h-closed. Next, we shall show that K is an h-topological group. For this, let $x_1, x_2 \in K$ and U be an open neighborhood of $x_1 * x_2$. Now $x_1, x_2 \in K \subseteq G$ and G is an h-topological group, there exists h-open sets V_1 and V_2 in G containing x_1 and x_2 respectively such that $V_1 * V_2 \subseteq U$. Since K is open, there exists h-open neighborhoods $U_1 = K \cap V_1$ and $U_2 = K \cap V_2$ in K containing x and y respectively such that $U_1 * U_2 \subseteq V_1 * V_2 \subseteq U$. Thus, $*$ is h-continuous. Similarly, we can prove continuity of inverse of $*$. Hence, the proof follows. □

Now we shall introduce the concept of h-regular spaces in the same sense p-regular spaces [5] were introduced.

Definition 3.3. *Let (X, τ) be topological space. Then X is said to be h-**regular** if for each closed set K of X not containing p, there exists disjoint h-open sets U_1 and U_2 such that $K \subset U_1$ and U_2 contains p.*

Remark 3.2. *The definition makes it clear that any regular space is an h-regular space. Yet the opposite need not be true.*

Example 3.5. *Let $X = \{1, 2, 3\}$ and $\tau = \{\phi, X, \{2\}\}$, $\tau^h = \{\phi, X, \{2\}, \{1, 3\}\}$. Then (X, τ) is an example of an h-regular space that is not regular.*

Theorem 3.15. *Let (X, τ) be topological space. Then the following are equivalent:*

(1) *X is h-regular.*

(2) *For every element p of X and every $W \in \tau$ containing p, there exists an h-open set U satisfying $p \in U \subset Cl_h(U) \subset W$.*

(3) *For each closed set K of X, $\bigcap\{Cl_h(U) : K \subset U \in HO(X)\} = K$.*

(4) *For each $B \subseteq X$ and each $W \in \tau$ such that B intersects W, there exists h-open set U such that B also intersects U and $Cl_h(U) \subset W$.*

(5) *For each $B \subseteq X$, where $B \neq \phi$ and each closed set K of X such that $B \cap K$ is empty, there exists h-open sets U, V such that B intersects U, $K \subset V$ and $U \cap V$ is empty.*

Proof. (1) $\Rightarrow$ (2): Let $W \in \tau$ contains p. Then $A = X \setminus W$ is closed in X and $p \notin A$. By given hypothesis, $\exists$ disjoint h-open sets U and V such that $p \in U$ and $A \subset V$. By Proposition 2.12 [**?**], $Cl_h \cap V$ is empty, since if $q \in V \cap Cl_h(U)$, then there is an h-open set V containing q whose intersection with U is empty. Thus, $x \in U \subset Cl_h(U) \subset U$.

(2) $\Rightarrow$ (3): Let K be a closed set of X. Clearly, K is h-closed. Thus $K \subset \bigcap\{Cl_h(U) : K \subset U \in HO(X)\}$. To prove reverse inclusion, suppose $p \notin K$. Then $X \setminus K \in \tau$ and $p \in X \setminus K$. By (2), there exists an h-open sets W such that $p \in W \subset Cl_h(W) \subset X \setminus K$. Suppose $U = \{Cl_h(W)\}^c$. Then U is an h-open set containing K such that $Cl_h(U)$ does not contain p. Thus, the proof follows.

(3) $\Rightarrow$ (4): Let $B \subseteq X$ and B intersects W, where $W \in \tau$. Suppose $p \in B \cap W$. Clearly, W^c is a closed set not containing p. Thus, there exists an h-open set V containing W^c and $Cl_h(V)$ does not contain p. Now $U = \{Cl_h(V)\}^c$ is an h-open set and $U \cap B$ contains p. Also, $Cl_h(U) \subset Cl_h(X \setminus V) = X \setminus V \subset W$.

(4) $\Rightarrow$ (5): Let $B \subseteq X$ does not intersect K, where K is closed in X and $B \neq \phi$. Clearly, $K^c \in \tau$ and B is non-empty. Thus there exists an h-open set U such that B intersects U and $Cl_h(U) \subseteq K^c$. Put $V = \{Cl_h(U)\}^c$. Then V is an h-open set containing K and disjoint from U.

(5) $\Rightarrow$ (1): Straightforward. □

Note 3.5. *In the succeeding theorem, γ_e will be the notation used for base at the identity element e.*

Theorem 3.16. *Consider an h-topological group G. Suppose for each element W of γ_e, $\exists$ an open neighborhood U of e such that $U = U^{-1}$ and $U^2 \subset W$. Then G is h-regular at identity element e.*

Proof. By using preceding theorem, it suffices to show that $Cl_h(U)$ is contained in W. For this, consider an element p of $Cl_h(U)$. Clearly, $p * U$ intersects U. Thus, there exists $x, y \in U$ such that $p = y * x^{-1} \in U * U^{-1} = U * U \subset W$. Hence, the required proof. □

4. h-irresolute topological groups

Definition 4.1. *Let $(G, *)$ be a group endowed with topology τ on it. Then $(G, *, \tau)$ is said to be an h-**irresolute topological group** if group operation $*$ as well as the inverse operation $^{-1}$ are h-irresolute. Equivalently,*

(1) *for each h-open neighborhood W of $p * q$, there exists h-open neighborhoods U and V containing p and q respectively such that $U * V \subseteq W$,*

(2) *for each h-open neighborhood W of p^{-1}, there exists an h-open neighborhood U containing p such that $U^{-1} \subseteq W$.*

It is evident from the definition that every h-irresolute topological group is h-topological group but converse need not be true.

- Consider Klein's group $K_4 = \{e,\ a,\ b,\ ab\}$ endowed with the topology $\tau = \{\phi,\ \{a,b\},\ K_4\}$ and $\tau_h = \{\phi,\ \{a,b\},\ \{e,ab\},\ \{e,a,ab\}, \{e,b,ab\},\ K_4\}$. This is an example of h-topological group that is not h-irresolute topological group.
- Example 3.3 is an example of h-topological group that is not an h-irresolute topological group.

Remark 4.1. *It should be noted that the notion of topological group and h-irresolute topological group are independent of each other.*

Example 4.1. (1) *Any group with discrete topology is an example of topological group as well as h-irresolute topological group.*

(2) *Any group with Sierpinski topology is an example of h-irresolute topological group that is not a topological group.*

(3) *Example 3.3 is neither a topological group nor an h-irresolute topological group.*

Theorem 4.1. *Consider an h-irresolute topological group and an h-open set B in G. Then $i * B,\ B * i,\ B^{-1}$ are h-open for all $i \in G$.*

Proof. First, we shall prove $i * B \in \tau_h$. Let $r \in i * B$. Then there exists h-open neighborhoods U_1 and U_2 of i^{-1} and r respectively such that $U_1 * U_2 \subseteq B$. Thus, $r \in Int_h(i * B)$. Similarly, we can prove $B * i \in \tau_h$. Next, let $j = i^{-1} \in B^{-1}$, for some $i \in B$. Clearly, there exists an h-open neighborhood U of j such that $U \subseteq B^{-1}$. Thus, $j \in Int_h(B^{-1})$ and B^{-1} is h-open. □

Theorem 4.2. *Consider an h-irresolute topological group and any subset B of G. Then for all $i \in G$, we have*

(1) $Cl_h(i * B) = i * Cl_h(B)$;

(2) $Cl_h(B^{-1}) = (Cl_h(B))^{-1}$.

Proof. (1)Let $p \in Cl_h(i * B)$ and $q = i^{-1} * p$. Suppose U be an h-open neighborhood of q. Then there exists h-open neighborhoods U_1 and U_2 of i^{-1} and p respectively such that $U_1 * U_2 \subseteq U$. Also, $i^{-1} * U_2 \subseteq U$. As $p \in Cl_h(i * B)$, $i * B$ intersects U_2 and hence, intersects $i * U$. This results in $B \cap U \neq \phi$. Thus, $Cl_h(i * B) \subseteq i * Cl_h(B)$. To prove reverse inclusion, let $q \in Cl_h(B)$ and an open neighborhood V of $i * q$. Then there exists h-open neighborhoods V_1 and V_2 of i and q respectively such that $V_1 * V_2 \subseteq V$. Clearly, B intersects V_2 and hence $i * B$ intersects V. This completes the proof.
(2) Let $p \in Cl_h(B^{-1})$ and U be an h-open neighborhood of p^{-1}. Then there exists an h-open neighborhood V of p such that $V^1 \subseteq U$. Also, we have $B^{-1} \cap V \neq \phi$. Thereby it follows that B intersects V^{-1} and hence $B \cap U \neq \phi$. Thus, $p \in (Cl_h(B))^{-1}$. Next, let $q \in (Cl_h(B))^{-1}$ and U be an h-open neighborhood of q. Then $q = p^{-1}$ for some $p \in Cl_h(B)$. By assumption, there exists an h-open neighborhood V containing p such that $V^{-1} \subseteq U$. Since $p \in Cl_h(B)$, B intersects V. Also, B^{-1} intersects V^{-1} which implies that B^{-1} intersects V. Hence, the proof. □

Theorem 4.3. *Consider an h-irresolute topological group and any subset B of G. Then for all $i \in G$, we have*

(1) $Int_h(i * B) = i * Int_h(B)$;
(2) $Int_h(B^{-1}) = (Int_h(B))^{-1}$.

Proof. (1) Let $p \in Int_h(i * B)$. Then $p = i * b$, for some $b \in B$ and there exists h-open sets U_1 and U_2 containing i and b respectively such that $U_1 * U_2 \subseteq i * B$. Thus $p \in i * Int_h(B)$. Reverse inclusion can be seen using Theorem 4.1.
(2) Let $p \in Int_h(B^{-1})$ and $b \in B$. Then there exists an h-open neighborhood V of b such that $V^{-1} \subseteq B^{-1}$. Now we can easily deduce $Int_h(B^{-1}) \subseteq (Int_h(B))^{-1}$. For reverse inclusion, see Theorem 4.1 . □

Theorem 4.4. *Let G be an h-irresolute topological group and K be an h-open subgroup of G. Then K is h-closed in G.*

Proof. By assumption, K is an h-open subgroup of G. Clearly, $G = \bigcup_{b_i \in G} b_i * K$. Now $b_i * K = \left(\bigcup_{b_j \neq b_i \in G} b_j * K \right)^c$. Thus, $b_i * K$ is h-open as well as h-closed. In particular, $K = e * K$ is h-open as well as h-closed in G. □

Theorem 4.5. *Let G be an h-irresolute topological group and K be an open subgroup of G. Then K is also an h-irresolute topological group.*

Proof. Let p_1, p_2 be any two arbitrary elements of K and V be an h-open neighborhood containing $p_1 * p_2$. Since $K \subseteq G$, there exists h-open neighborhoods V_1 and V_2 in G containing p_1 and p_2 respectively such that $V_1 * V_2 \subseteq V$. Now $W_1 = K \cap V_1$ and $W_2 = K \cap V_2$ are h-open in K containing p_1 and p_2 respectively and satisfying $W_1 * W_2 \subseteq V$. This proves $*$ is h-irresolute. Now we shall prove h-irresoluteness of inverse of $*$. Let $q = p^{-1} \in K$ and U be an h-open neighborhood containing q. Since $K \subseteq G$, there exists an h-open neighborhood V in G containing p such that $V^{-1} \subseteq U$. Also, K is open implies $W = V \cap K$ is h-open in K containing p such that $W^{-1} \subseteq V^{-1} \subseteq U$. □

Note 4.1. *From now onwards, we shall assume that G_1 and G_2 are two groups having binary operations $*_1$ and $*_2$ respectively and endowed with topologies τ_1 and τ_2 respectively. Further, e_1 and e_2 represents identity elements of G_1 and G_2 respectively.*

Theorem 4.6. *Consider two h-irresolute topological groups G_1 and G_2 and a group homomorphism $g : G_1 \to G_2$. If g is h-irresolute at e_1, then g is h-irresolute on G.*

Proof. Let $p \in G$ and W be an h-open set containing $g(p)$. By Theorem , $W *_2 (g(p))^{-1}$ is h-open in G_2 containing $g(e_1) = e_2$. Since g is h-irresolute at e_1. Then there exists an h-open set V containing e_1 such that $g(V) \subseteq W *_2 (g(p))^{-1} \Rightarrow g(V) *_2 g(p) \subseteq W$. Since g is given to be group homomorphism $g(V *_1 p) = g(V) *_2 g(p) \subseteq W$. As $V * p$ is h-open set in G containing p. Thus g is h-irresolute on G. □

Corollary 4.1. *Let G_1 and G_2 be two h-irresolute topological groups and $g : G_1 \to G_2$ be a group homomorphism. If g is h-irresolute at e_1, then g is h-continuous on G.*

Corollary 4.2. *Consider two h-irresolute topological groups G_1 and G_2 and a group homomorphism $g : G_1 \to G_2$. If g is h-totally continuous at e_1, then g is h-continuous on G.*

Theorem 4.7. *Consider an h-irresolute topological group G and a subgroup K of G. Then $Cl_h(K)$ is also a subgroup of G. Further, if K is a normal subgroup of G, then $Cl_h(K)$ is also a normal subgroup of G.*

Proof. Let $a, b \in Cl_h(K)$. Let U be an h-open neighborhood of $a * b$. Then by hypothesis, there exists h-open neighborhoods U_1 and U_2 of a and b respectively such that $U_1 * U_2 \subseteq U$. By definition of $Cl_h(K)$, U_1 and U_2 both intersects K and suppose $x \in U_1 \cap K$ and $y \in U_2 \cap K$. Clearly, $x * y \in U_1 * U_2 \subseteq U$. Since K is a subgroup , we have $x * y \in K$. This implies that $x * y \in U \cap K \Rightarrow a * b \in Cl_h(K)$. Thus, $Cl_h(K) * Cl_h(K) \subseteq Cl_h(K)$. Next, we shall show that if $a \in Cl_h(K)$, then $a^{-1} \in Cl_h(K)$. For this, let V be an h-open neighborhood of a^{-1}. Then there exists an h-open neighborhood U of a such that $U^{-1} \subseteq V$. By definition of $Cl_h(K)$, $U \cap K \neq \phi$ and suppose $x \in U \cap K$. Again, since K is subgroup of G, we have $x^{-1} \in K$. Also, $x^{-1} \in V$. Thus, $V \cap K \neq \phi$. Thus, $a^{-1} \in Cl_h(K)$. Further, let K be a normal subgroup of G. Proof follows from the fact that $Cl_h(gKg^{-1}) = gCl_h(K)g^{-1} \; \forall \; g \in G$. □

Theorem 4.8. *Let $K \neq \phi$ be a subgroup of an h-irresolute topological group G. Then, following are equivalent:*

(1) *K is h-open;*
(2) *$Int_h(K) \neq \phi$.*

Proof. $(1) \Rightarrow (2)$ follows trivially. Conversely, suppose $p \in Int_h(K)$. Then there exists an h-open set U such that $p \in U \subset K$. Clearly, $p * U \subset K$. Now $q * U = q * p^{-1} * p * U \subset K$, for every element q of K. Thus $K = \bigcup_{q \in H} q * U$ is h-open as $q * U$ is h-open. □

Now we shall introduce h-Lindelof spaces and put forth some important results.

Definition 4.2. *Let (X, τ) be a topological space. Then X is said to be h-**Lindelof** if every h-open cover has a countable subcover.*

Proposition 4.1. *Countable union of h-Lindelof spaces is h-Lindelof.*

Proof. Proof is simple, thus omitted. □

Proposition 4.2. *h-irresolute image of h-Lindelof space is h-Lindelof.*

Proof. Let $g : X \to Y$ be an h-irresolute mapping and B be h-Lindelof in X. We have to show that $g(B)$ is h-Lindelof in Y. For this, consider a cover $\{H_\beta : \beta \in \Delta\}$ of h-open sets of $g(B)$ in Y. Since g is h-irresolute, $g^{-1}(H_\beta)$ is h-open in X for each $\beta \in \Delta$. Thus, $\{g^{-1}(H_\beta) : \beta \in \Delta\}$ is a cover of h-open sets of B in X. Now B being h-Lindelof implies that $B \subset \bigcup_{j=1}^{\infty} g^{-1}(H_{\beta_j})$ for some $\beta_1, \beta_2, \beta_3, ... \in \Delta$. Thus, $g(B) \subset \bigcup_{j=1}^{\infty} g(g^{-1}(H_{\beta_j})) \subset \bigcup_{j=1}^{\infty} H_{\beta_j}$. Hence, the proof. □

Theorem 4.9. *Let G be an h-irresolute topological group and A, B be any two subsets of G. Then:*

(1) *If A is h-Lindelof, then A^{-1} is h-Lindelof.*

(2) *If A is h-Lindelof and B is countable, then $A * B$ and $B * A$ are h-Lindelof.*

Proof. (1) Since the inverse mapping is h-irresolute and A is h-Lindelof, proof follows from the above proposition.

(2) Let $b \in B$. Since A is h-Lindelof and left translation mapping is h-irresolute, $b * A$ is h-Lindelof by above Proposition. From Proposition 4.1, it follows that $B * A$ is h-Lindelof as $B * A$ can be written as a countable union of h-Lindelof spaces. In the similar manner, we can prove $A * B$ is h-Lindelof. □

Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] F. Abbas, h-open sets in topological spaces, Bol. Soc. Paran. Mat. 41 (2023), 1–9.

[2] B.S. Abdullah, S.W. Askandar, R.N. Balo, $h\alpha$-open sets in topological spaces, J. Educ. Sci. 31 (2022), 91–98.

[3] M.S. Bosan, M.D. Khan, L.D.R. Kocinac, On s-topological groups, Math. Morav. 18 (2014), 35–44. https://doi.org/10.5937/matmor1402035b.

[4] H. Cakalli, F.I. Dagci, On h-open sets and h-continuous functions, J. Appl. Comp. Math. 10 (2021), 5.

[5] N. El-Deeb, I.A. Hasanein, A.S. Mashhour, T. Noiri, On p-regular spaces, Bull. Math. Soc. Sci. Math. Répub. Soc. Roum., Nouv. Sér. 27 (1983), 311–315. https://www.jstor.org/stable/43680828.

[6] P.J. Higgins, An introduction to topological groups, Cambridge University Press, Cambridge, 1974.

[7] T. Husain, Introduction to topological groups, W. B. Saunders Company, Philadelphia and London, 1966.

[8] A.M. Kumar, P. Gnanachandra, Exploratory results on finite topological groups, JP J. Geom. Topol. 24 (2020), 1–15.

[9] S. Sharma, N. Digra, P. Saproo, S. Billawria, On h-irresolute topological vector spaces, J. Adv. Math. Stud. 16 (2023), 304–319.

[10] D. Spivak, An introduction to topological groups, Lakehead University, Ontario, 2015.