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ON SOME SEPARATION AXIOMS IN BITOPOLOGICAL SPACES

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Abstract. This paper deals with some separation axioms in bitopological spaces. Firstly, we characterize some new low separation axioms in bitopological spaces. Secondly, we introduce the concepts of (τ_1, τ_2)-$\mathscr{D}_0$ spaces, (τ_1, τ_2)-$\mathscr{D}_1$ spaces and (τ_1, τ_2)-$\mathscr{D}_2$ spaces by utilizing $(\tau_1, \tau_2)\mathscr{D}$-sets. Furthermore, some characterizations of (τ_1, τ_2)-$\mathscr{D}_0$ spaces, (τ_1, τ_2)-$\mathscr{D}_1$ spaces and (τ_1, τ_2)-$\mathscr{D}_2$ spaces are established. Finally, we introduce and investigate the notions of (τ_1, τ_2)-$T_{\frac{1}{4}}$ spaces, (τ_1, τ_2)-$T_{\frac{3}{8}}$ spaces and weak (τ_1, τ_2)-R_0 spaces.

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1. Introduction

Separation axioms are one among the most common, important and interesting ideas in topology. Some separation axioms were introduced using generalized open sets. The concept of R_0 topological spaces was first introduced by Shanin [21]. Davis [10] introduced the concept of a separation axiom called R_1. Murdeshwar and Naimpally [17] and Dube [12] studied some of the fundamental properties of the class of R_1 topological spaces. As natural generalizations of the separation axioms R_0 and R_1, the concept of semi-R_0 and semi-R_1 were introduced and studied by Maheshwari and Prasad [16] and Dorsett [11]. Caldas et al. [6] introduced and studied two new weak separation axioms called Λ_θ-R_0 and Λ_θ-R_1 by using the notions of (Λ, θ)-open sets and (Λ, θ)-closure operators. Thongmoon and Boonpok [23] introduced and investigated the concept of (Λ, p)-R_1 topological spaces. In [1], the present authors introduced and studied the notions of $\delta s(\Lambda, s)$-R_0 spaces and $\delta s(\Lambda, s)$-R_1 spaces.

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Furthermore, several characterizations of Λ_p-R_0 spaces and (Λ, s)-R_0 spaces were established in [3] and [2], respectively. Recently, Thongmoon and Boonpok [22] introduced and studied the notion of sober $\delta p(\Lambda, s)$-R_0 spaces. Sarsak [20] introduced and studied weak separation axioms in generalized topological spaces, namely, μ-D_0 spaces, μ-D_1 spaces, μ-D_2 spaces, μ-T_0 spaces, μ-T_1 spaces, μ-T_2 spaces, μ-R_0 spaces, μ-R_1 spaces and weakly μ-D_1 spaces. Moreover, Sarsak [19] introduced and studied new separation axioms in generalized topological spaces, namely, μ-$T_{\frac{1}{4}}$ spaces, μ-$T_{\frac{3}{8}}$ spaces and μ-$T_{\frac{1}{2}}$ spaces. μ-$T_{\frac{1}{4}}$ spaces are strictly placed between μ-T_0 spaces and μ-$T_{\frac{3}{8}}$ spaces, μ-$T_{\frac{3}{8}}$ spaces are strictly placed between μ-$T_{\frac{1}{4}}$ spaces and μ-$T_{\frac{1}{2}}$ spaces and μ-$T_{\frac{1}{2}}$ spaces are strictly placed between μ-$T_{\frac{3}{8}}$ spaces and μ-T_1 spaces. Cammaroto and Noiri [7] defined a weak separation axioms m-R_0 in m-spaces which are equivalent to generalized topological spaces due to Lugojan [15]. In 2006, Noiri [18] introduced the concept of m-R_1 spaces in m-spaces and investigated several characterizations of m-R_0 spaces and m-R_1 spaces. In this paper, we introduce the concepts of (τ_1, τ_2)-$\mathscr{D}_0$ spaces, (τ_1, τ_2)-$\mathscr{D}_1$ spaces, (τ_1, τ_2)-$\mathscr{D}_2$ spaces, (τ_1, τ_2)-$T_{\frac{1}{4}}$ spaces, (τ_1, τ_2)-$T_{\frac{3}{8}}$ spaces and weak (τ_1, τ_2)-R_0 spaces. Furthermore, several characterizations of (τ_1, τ_2)-$\mathscr{D}_0$ spaces, (τ_1, τ_2)-$\mathscr{D}_1$ spaces, (τ_1, τ_2)-$\mathscr{D}_2$ spaces, (τ_1, τ_2)-$T_{\frac{1}{4}}$ spaces, (τ_1, τ_2)-$T_{\frac{3}{8}}$ spaces and weak (τ_1, τ_2)-R_0 spaces are discussed.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called *$\tau_1\tau_2$-closed* [5] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called *$\tau_1\tau_2$-open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the *$\tau_1\tau_2$-closure* [5] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the *$\tau_1\tau_2$-interior* [5] of A and is denoted by $\tau_1\tau_2$-Int(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be *$(\tau_1, \tau_2)r$-open* [24] (resp. *$(\tau_1, \tau_2)s$-open* [4], *$(\tau_1, \tau_2)p$-open* [4], *$(\tau_1, \tau_2)\beta$-open* [4]) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$, $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$).

Lemma 1. [5] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) *$A \subseteq \tau_1\tau_2$-Cl(A) and $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Cl$(A)) = \tau_1\tau_2$-Cl(A).*
(2) *If $A \subseteq B$, then $\tau_1\tau_2$-Cl$(A) \subseteq \tau_1\tau_2$-Cl$(B)$.*
(3) *$\tau_1\tau_2$-Cl(A) is $\tau_1\tau_2$-closed.*
(4) *A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2$-Cl(A).*
(5) *$\tau_1\tau_2$-Cl$(X - A) = X - \tau_1\tau_2$-Int$(A)$.*

3. Characterizations of some new low separation axioms

In this section, we investigate some characterizations of (τ_1, τ_2)-T_0 spaces, (τ_1, τ_2)-T_1 spaces, (τ_1, τ_2)-T_2 spaces, (τ_1, τ_2)-R_0 spaces and (τ_1, τ_2)-R_1 spaces.

Definition 1. [8] *A bitopological space (X, τ_1, τ_2) is said to be:*

(i) (τ_1, τ_2)-T_0 *if for any pair of distinct points in X, there exists a $\tau_1\tau_2$-open set of X containing one of the points but not the other.*

(ii) (τ_1, τ_2)-T_1 *if for any pair of distinct points x, y in X, there exist $\tau_1\tau_2$-open sets U and V of X such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.*

Theorem 1. *A bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-T_0 if and only if $\tau_1\tau_2$-Cl$(\{x\}) \neq \tau_1\tau_2$-Cl$(\{y\})$ for any pair of distinct points x and y of X.*

Proof. Suppose that $x, y \in X$, $x \neq y$ and $\tau_1\tau_2\text{-Cl}(\{x\}) \neq \tau_1\tau_2\text{-Cl}(\{y\})$. Let z be a point of X such that $z \in \tau_1\tau_2\text{-Cl}(\{x\})$ but $z \notin \tau_1\tau_2\text{-Cl}(\{y\})$. We claim that $x \notin \tau_1\tau_2\text{-Cl}(\{y\})$. If $x \in \tau_1\tau_2\text{-Cl}(\{y\})$, then

$$\tau_1\tau_2\text{-Cl}(\{x\}) \subseteq \tau_1\tau_2\text{-Cl}(\{y\}).$$

This contradicts the fact that $z \notin \tau_1\tau_2\text{-Cl}(\{y\})$. Thus, x belongs to the $\tau_1\tau_2$-open set $X - \tau_1\tau_2\text{-Cl}(\{y\})$ to which y does not belong. This shows that (X, τ_1, τ_2) is (τ_1, τ_2)-T_0.

Conversely, let (X, τ_1, τ_2) be (τ_1, τ_2)-T_0 and x, y be any two distinct points of X. There exists a $\tau_1\tau_2$-open set U containing x or y, say x but not y. Then, $X - U$ is a $\tau_1\tau_2$-closed set which does not contain x but contains y. Since $\tau_1\tau_2\text{-Cl}(\{y\})$ is the smallest $\tau_1\tau_2$-closed set containing y, $\tau_1\tau_2\text{-Cl}(\{y\}) \subseteq X - U$ and so $x \notin \tau_1\tau_2\text{-Cl}(\{y\})$. Thus, $\tau_1\tau_2\text{-Cl}(\{x\}) \neq \tau_1\tau_2\text{-Cl}(\{y\})$. □

Theorem 2. *A bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-T_1 if and only if the singletons are $\tau_1\tau_2$-closed sets.*

Proof. Suppose that (X, τ_1, τ_2) is (τ_1, τ_2)-T_1 and x be any point of X. Let $y \in X - \{x\}$. Then, $x \neq y$ and so there exists a $\tau_1\tau_2$-open set U_y such that $y \in U_y$ but $x \notin U_y$. Thus, $y \in U_y \subseteq X - \{x\}$ and hence $X - \{x\} = \cup_{y \in X - \{x\}} U_y$ which is $\tau_1\tau_2$-open.

Conversely, suppose that $\{z\}$ is $\tau_1\tau_2$-closed for each $z \in X$. Let $x, y \in X$ with $x \neq y$. Now $x \neq y$ implies $y \in X - \{x\}$. Thus, $X - \{x\}$ is a $\tau_1\tau_2$-open set containing y but not containing x. Similarly, we have $X - \{y\}$ is a $\tau_1\tau_2$-open set containing x but not containing y. This means that (X, τ_1, τ_2) is a (τ_1, τ_2)-T_1 space. □

Definition 2. *A bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-T_2 if for any pair of distinct points x, y in X, there exist disjoint $\tau_1\tau_2$-open sets U and V of X containing x and y, respectively.*

Remark 1. *Let (X, τ_1, τ_2) be a bitopological space. If (X, τ_1, τ_2) is (τ_1, τ_2)-T_i, then (X, τ_1, τ_2) is (τ_1, τ_2)-T_{i-1}, $i = 1, 2$.*

Definition 3. [13] *A bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-R_0 if for each $\tau_1\tau_2$-open set U and each $x \in U$, $\tau_1\tau_2$-Cl$(\{x\}) \subseteq U$.*

Definition 4. *A bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-symmetric if for each $x, y \in X$, $x \in \tau_1\tau_2$-Cl$(\{y\})$ implies $y \in \tau_1\tau_2$-Cl$(\{x\})$.*

Theorem 3. *A bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-symmetric if and only if for each $x \in X$, $\tau_1\tau_2$-Cl$(\{x\}) \subseteq U$ whenever $x \in U$ and U is $\tau_1\tau_2$-open.*

Proof. Suppose that $x \in \tau_1\tau_2$-Cl$(\{y\})$ but $y \notin \tau_1\tau_2$-Cl$(\{x\})$. This means that the complement of $\tau_1\tau_2$-Cl$(\{x\})$ contains y. Therefore, the set $\{y\}$ is a subset of the complement of $\tau_1\tau_2$-Cl$(\{x\})$. This implies that $\tau_1\tau_2$-Cl$(\{y\})$ is a subset of the complement of $\tau_1\tau_2$-Cl$(\{x\})$. Now the complement of $\tau_1\tau_2$-Cl$(\{x\})$ contains x which is a contradiction.

Conversely, suppose that $\{x\} \subseteq U$ and U is $\tau_1\tau_2$-open in X but $\tau_1\tau_2$-Cl$(\{x\})$ is not a subset of U. This mean that $\tau_1\tau_2$-Cl$(\{x\})$ and the complement of U are not disjoint. Let y belongs to their intersection. Now we have $x \in \tau_1\tau_2$-Cl$(\{y\})$ which is a subset of the complement of U and $x \notin U$. This is a contradiction. □

Theorem 4. *A bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-R_0 if and only if (X, τ_1, τ_2) is (τ_1, τ_2)-symmetric.*

Proof. Suppose that (X, τ_1, τ_2) is (τ_1, τ_2)-R_0. Let $x \in \tau_1\tau_2$-Cl$(\{x\})$ and U be any $\tau_1\tau_2$-open set such that $y \in U$. By the hypothesis, $\tau_1\tau_2$-Cl$(\{y\}) \subseteq U$ and hence $x \in U$. Therefore, every $\tau_1\tau_2$-open set which contains y contain x. Thus, $y \in \tau_1\tau_2$-Cl$(\{x\})$.

Conversely, let U be a $\tau_1\tau_2$-open set and $y \in U$. If $y \notin U$, then $x \notin \tau_1\tau_2$-Cl$(\{y\})$ and by the hypothesis, $y \notin \tau_1\tau_2$-Cl$(\{x\})$. Thus, $\tau_1\tau_2$-Cl$(\{x\}) \subseteq U$ and hence (X, τ_1, τ_2) is (τ_1, τ_2)-R_0. □

Theorem 5. *A bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-T_1 if and only if (X, τ_1, τ_2) is (τ_1, τ_2)-T_0 and (τ_1, τ_2)-R_0.*

Proof. Follows from Remark 1 and Theorem 2.

Conversely, let $x, y \in X$ and $x \neq y$. Since (X, τ_1, τ_2) is (τ_1, τ_2)-T_0, we may assume without loss of generality that $x \in U \subseteq X - \{y\}$ for some $\tau_1\tau_2$-open set U. Thus, $x \notin \tau_1\tau_2$-Cl$(\{y\})$ and by Theorem 4, $y \notin \tau_1\tau_2$-Cl$(\{x\})$. Therefore, $X - \tau_1\tau_2$-Cl$(\{x\})$ is a $\tau_1\tau_2$-open set containing y but not x. This shows that (X, τ_1, τ_2) is (τ_1, τ_2)-T_1. □

Definition 5. [14] *A bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-R_1 if for each points x and y in X with $\tau_1\tau_2$-Cl$(\{x\}) \neq \tau_1\tau_2$-Cl$(\{y\})$, there exist disjoint $\tau_1\tau_2$-open sets U and V such that $\tau_1\tau_2$-Cl$(\{x\}) \subseteq U$ and $\tau_1\tau_2$-Cl$(\{y\}) \subseteq V$.*

Lemma 2. *If a bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-R_1, then it is (τ_1, τ_2)-R_0.*

Theorem 6. *For a bitopological space (X, τ_1, τ_2), the following properties are equivalent:*

(1) (X, τ_1, τ_2) *is* (τ_1, τ_2)-T_2;

(2) (X, τ_1, τ_2) *is* (τ_1, τ_2)-T_1 *and* (τ_1, τ_2)-R_1;

(3) (X, τ_1, τ_2) *is* (τ_1, τ_2)-T_0 *and* (τ_1, τ_2)-R_1.

Proof. (1) $\Rightarrow$ (2): Follows from Remark 1 and Theorem 2.

(2) $\Rightarrow$ (3): Follows from Remark 1.

(3) $\Rightarrow$ (1): It follows from Lemma 2 and Theorem 5 that (X, τ_1, τ_2) is (τ_1, τ_2)-T_1. Since (X, τ_1, τ_2) is (τ_1, τ_2)-R_1, by Theorem 2, (X, τ_1, τ_2) is (τ_1, τ_2)-T_2. □

Corollary 1. *For a* (τ_1, τ_2)-R_0 *bitopological space* (X, τ_1, τ_2), *the following properties are equivalent:*

(1) (X, τ_1, τ_2) *is* (τ_1, τ_2)-T_2;

(2) (X, τ_1, τ_2) *is* (τ_1, τ_2)-T_1;

(3) (X, τ_1, τ_2) *is* (τ_1, τ_2)-T_0.

4. $(\tau_1, \tau_2)\mathscr{D}$-sets and associated separation axioms

In this section, we introduce the concepts of (τ_1, τ_2)-$\mathscr{D}_0$ spaces, (τ_1, τ_2)-$\mathscr{D}_1$ spaces and (τ_1, τ_2)-$\mathscr{D}_2$ spaces. Furthermore, several characterizations of (τ_1, τ_2)-$\mathscr{D}_0$ spaces, (τ_1, τ_2)-$\mathscr{D}_1$ spaces and (τ_1, τ_2)-$\mathscr{D}_2$ spaces are discussed.

Definition 6. *A subset* A *of a bitopological space* (X, τ_1, τ_2) *is called a* $(\tau_1, \tau_2)\mathscr{D}$*-set if there exist* $\tau_1\tau_2$*-open sets* U *and* V *of* X *such that* $U \neq X$ *and* $A = U - V$.

Remark 2. *Let* (X, τ_1, τ_2) *be a bitopological space. Letting* $A = U$ *and* $V = \emptyset$ *in the above definition, it is easy to see that every proper* $\tau_1\tau_2$*-open set* U *is a* $(\tau_1, \tau_2)\mathscr{D}$*-set.*

Definition 7. *A subset* A *of a bitopological space* (X, τ_1, τ_2) *is said to be:*

(i) (τ_1, τ_2)-$\mathscr{D}_0$ *if for any pair of distinct points* x *and* y *of* X, *there exists a* $(\tau_1, \tau_2)\mathscr{D}$*-set* U *of* X *such that* $x \in U$, $y \notin U$ *or* $y \in U$, $x \notin U$;

(ii) (τ_1, τ_2)-$\mathscr{D}_1$ *if for any pair of distinct points* x *and* y *of* X, *there exist* $(\tau_1, \tau_2)\mathscr{D}$*-sets* U *and* V *of* X *such that* $x \in U$, $y \notin U$ *and* $y \in V$, $x \notin V$;

(iii) (τ_1, τ_2)-$\mathscr{D}_2$ *if for any pair of distinct points* x *and* y *of* X, *there exist disjoint* $(\tau_1, \tau_2)\mathscr{D}$*-sets* U *and* V *of* X *containing* x *and* y, *respectively.*

Remark 3. *For a bitopological space* (X, τ_1, τ_2), *the following properties hold:*

(1) *If* (X, τ_1, τ_2) *is* (τ_1, τ_2)-T_i, *then* (X, τ_1, τ_2) *is* (τ_1, τ_2)-$\mathscr{D}_i$, $i = 0, 1, 2$.

(2) *If* (X, τ_1, τ_2) *is* (τ_1, τ_2)-$\mathscr{D}_i$, *then* (X, τ_1, τ_2) *is* (τ_1, τ_2)-$\mathscr{D}_{i-1}$, $i = 1, 2$.

Theorem 7. *For a bitopological space* (X, τ_1, τ_2), *the following properties are equivalent:*

(1) (X, τ_1, τ_2) *is* (τ_1, τ_2)-T_0.

(2) (X, τ_1, τ_2) *is* (τ_1, τ_2)-$\mathscr{D}_0$.

Proof. (1) $\Rightarrow$ (2): Follows from Remark 3 (1).

(2) $\Rightarrow$ (1): Let (X, τ_1, τ_2) is (τ_1, τ_2)-$\mathscr{D}_0$. Then, for each distinct points $x, y \in X$, at least one of x, y, say x, belongs to a $(\tau_1, \tau_2)\mathscr{D}$-set V but $y \notin V$. Suppose that $V = U_1 - U_2$, where $U_1 \neq X$ and U_1, U_2 are $\tau_1\tau_2$-open sets of X. Then, $x \in U_1$ and for $y \notin V$ we have two cases: (i) $y \notin U_1$; (ii) $y \in U_1$ and $y \in U_2$. In case (i), U_1 contains x but does not contain y. In case (ii), U_2 contains y but does not contain x. This shows that (X, τ_1, τ_2) is (τ_1, τ_2)-T_0. □

Corollary 2. *Let* (X, τ_1, τ_2) *be a bitopological space. If* (X, τ_1, τ_2) *is* (τ_1, τ_2)-$\mathscr{D}_1$*, then* (X, τ_1, τ_2) *is* (τ_1, τ_2)-T_0.

Proof. Follows from Remark 3 (2) and Theorem 7. □

Recall that a subset N of a bitopological space (X, τ_1, τ_2) is said to be a $\tau_1\tau_2$*-neighborhood* [5] of a point $x \in X$ if there exists a $\tau_1\tau_2$-open set U such that $x \in U \subseteq N$.

Definition 8. *Let* (X, τ_1, τ_2) *be a bitopological space. A point* $x \in X$ *which has* X *as the* $\tau_1\tau_2$*-neighborhood is said to be* $(\tau_1, \tau_2)\mathscr{D}$*-neat point.*

Theorem 8. *A bitopological space* (X, τ_1, τ_2) *is* (τ_1, τ_2)-$\mathscr{D}_1$ *if and only if* (X, τ_1, τ_2) *is* (τ_1, τ_2)-T_0 *and has no* $(\tau_1, \tau_2)\mathscr{D}$*-neat points.*

Proof. Suppose that (X, τ_1, τ_2) is (τ_1, τ_2)-$\mathscr{D}_1$. Since (X, τ_1, τ_2) is (τ_1, τ_2)-$\mathscr{D}_1$, so each point x of X is contained in a $(\tau_1, \tau_2)\mathscr{D}$-set $G = U - V$ and thus in U. By definition $U \neq X$. Thus, x is not a $(\tau_1, \tau_2)\mathscr{D}$-neat point. (X, τ_1, τ_2) being (τ_1, τ_2)-T_0 follows from Corollary 2.

Conversely, suppose that (X, τ_1, τ_2) is (τ_1, τ_2)-T_0 and has no $(\tau_1, \tau_2)\mathscr{D}$-neat points. Since (X, τ_1, τ_2) is (τ_1, τ_2)-T_0, for each pair of distinct points $x, y \in X$, there exists a $\tau_1\tau_2$-open set U containing x, say but not y. By Remark 2, U is a $(\tau_1, \tau_2)\mathscr{D}$-set. Since X has no $(\tau_1, \tau_2)\mathscr{D}$-neat point, y is not a $(\tau_1, \tau_2)\mathscr{D}$-neat point. Thus, there exists a $\tau_1\tau_2$-open set V containing y such that $V \neq X$, $y \in V - U$, $x \notin V - U$ and $V - U$ is a $(\tau_1, \tau_2)\mathscr{D}$-set. This shows that (X, τ_1, τ_2) is (τ_1, τ_2)-$\mathscr{D}_1$. □

Theorem 9. *For a* (τ_1, τ_2)*-symmetric bitopological space* (X, τ_1, τ_2)*, the following properties are equivalent:*

(1) (X, τ_1, τ_2) *is* (τ_1, τ_2)-T_0;

(2) (X, τ_1, τ_2) *is* (τ_1, τ_2)-$\mathscr{D}_0$;

(3) (X, τ_1, τ_2) *is* (τ_1, τ_2)-$\mathscr{D}_1$.

Proof. (1) $\Leftrightarrow$ (2): Follows from Theorem 7.

(3) $\Rightarrow$ (2): Follows from Remark 3.

(1) $\Rightarrow$ (3): Let $x, y \in X$ and $x \neq y$. By (1), we may assume that $x \in U \subseteq X - \{y\}$ for some $\tau_1\tau_2$-open set U in X. Then, $x \notin \tau_1\tau_2$-Cl$(\{y\})$ and hence $y \notin \tau_1\tau_2$-Cl$(\{x\})$. Thus, there exists a $\tau_1\tau_2$-open set V

such that $y \in V \subseteq X - \{x\}$. Since every $\tau_1\tau_2$-open set is a $(\tau_1, \tau_2)\mathscr{D}$-set, we have that (X, τ_1, τ_2) is a (τ_1, τ_2)-$\mathscr{D}_1$ space. □

Theorem 10. *For a (τ_1, τ_2)-R_0 bitopological space (X, τ_1, τ_2), the following properties are equivalent:*

(1) *(X, τ_1, τ_2) is (τ_1, τ_2)-T_0;*

(2) *(X, τ_1, τ_2) is (τ_1, τ_2)-T_1;*

(3) *(X, τ_1, τ_2) is (τ_1, τ_2)-$\mathscr{D}_1$.*

Proof. (1) ⇒ (2): Follows from Theorem 5.

(2) ⇒ (3): Follows from Remark 3.

(3) ⇒ (1): Follows from Corollary 2. □

Definition 9. *A bitopological space (X, τ_1, τ_2) is said to be weakly $(\tau_1, \tau_2)\mathscr{D}_1$ if $\cap_{x \in X} \tau_1\tau_2$-Cl$(\{x\}) = \emptyset$.*

Theorem 11. *A bitopological space (X, τ_1, τ_2) is weakly (τ_1, τ_2)-$\mathscr{D}_1$ if and only if X has no $(\tau_1, \tau_2)\mathscr{D}$-neat points.*

Proof. Let (X, τ_1, τ_2) be weakly (τ_1, τ_2)-$\mathscr{D}_1$. Suppose that X has a $(\tau_1, \tau_2)\mathscr{D}$-neat point y. Then, we have $y \in \tau_1\tau_2$-Cl$(\{x\})$ for each $x \in X$, which is a contradiction.

Conversely, suppose that (X, τ_1, τ_2) is not weakly (τ_1, τ_2)-$\mathscr{D}_1$. Then, there exists $y \in \cap_{x \in X} \tau_1\tau_2$-Cl$(\{x\})$ and thus, every $\tau_1\tau_2$-open set containing y must be X, that is, y is a $(\tau_1, \tau_2)\mathscr{D}$-neat point of X, which is a contradiction. □

Corollary 3. *A bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-$\mathscr{D}_1$ if and only if (X, τ_1, τ_2) is weakly (τ_1, τ_2)-$\mathscr{D}_1$ and (τ_1, τ_2)-T_0.*

Proof. Follows from Theorem 8 and 11. □

Definition 10. [5] *Let A be a subset of a bitopological space (X, τ_1, τ_2). The set $\cap\{G \mid A \subseteq G$ and G is $\tau_1\tau_2$-open$\}$ is called the $\tau_1\tau_2$-kernel of A and is denoted by $\tau_1\tau_2$-$ker(A)$.*

Lemma 3. [5] *For subsets A, B of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) *$A \subseteq \tau_1\tau_2$-$ker(A)$.*

(2) *If $A \subseteq B$, then $\tau_1\tau_2$-$ker(A) \subseteq \tau_1\tau_2$-$ker(B)$.*

(3) *If A is $\tau_1\tau_2$-open, then $\tau_1\tau_2$-$ker(A) = A$.*

(4) *$x \in \tau_1\tau_2$-$ker(A)$ if and only if $A \cap H \neq \emptyset$ for every $\tau_1\tau_2$-closed set H containing x.*

Lemma 4. *Let (X, τ_1, τ_2) be a bitopological space and $A \subseteq X$. Then,*

$$\tau_1\tau_2\text{-}ker(A) = \{x \in X \mid \tau_1\tau_2\text{-}Cl(\{x\}) \cap A \neq \emptyset\}.$$

Proof. Suppose that $\tau_1\tau_2$-Cl$(\{x\}) \cap A = \emptyset$. Then, $x \notin X - \tau_1\tau_2$-Cl$(\{x\})$ which is a $\tau_1\tau_2$-open set containing A. Thus, $x \notin \tau_1\tau_2$-$ker(A)$ and hence $\tau_1\tau_2$-$ker(A) \subseteq \{x \in X \mid \tau_1\tau_2\text{-Cl}(\{x\}) \cap A \neq \emptyset\}$. Next, we show the opposite implication. Suppose that $x \notin \tau_1\tau_2$-$ker(A)$. Then, there exists a $\tau_1\tau_2$-open set U containing A and $x \notin U$. Therefore, $\tau_1\tau_2$-Cl$(\{x\}) \cap U = \emptyset$. Thus, $\tau_1\tau_2$-Cl$(\{x\}) \cap A = \emptyset$ and hence

$$\tau_1\tau_2\text{-}ker(A) \supseteq \{x \in X \mid \tau_1\tau_2\text{-Cl}(\{x\}) \cap A \neq \emptyset\}.$$

□

Theorem 12. *A bitopological space (X, τ_1, τ_2) is weakly (τ_1, τ_2)-$\mathscr{D}_1$ if and only if $\tau_1\tau_2$-$ker(\{x\}) \neq X$ for each $x \in X$.*

Proof. Let (X, τ_1, τ_2) be weakly (τ_1, τ_2)-$\mathscr{D}_1$. Suppose that there exists a point y in X such that $\tau_1\tau_2$-$ker(\{y\}) = X$. By Lemma 4,

$$y \in \cap_{x\in X}\tau_1\tau_2\text{-Cl}(\{x\}),$$

which is a contradiction.

Conversely, let $\tau_1\tau_2$-$ker(\{x\}) \neq X$ for each $x \in X$. Suppose that (X, τ_1, τ_2) is not weakly (τ_1, τ_2)-$\mathscr{D}_1$. By Theorem 11, X has a $(\tau_1, \tau_2)\mathscr{D}$-neat point y. Thus, $\tau_1\tau_2$-$ker(\{x\}) = X$, which is a contradiction. □

5. (τ_1, τ_2)-$T_{\frac{1}{4}}$ SPACES AND (τ_1, τ_2)-$T_{\frac{3}{8}}$ SPACES

In this section, we introduce the concepts of (τ_1, τ_2)-$T_{\frac{1}{4}}$ spaces and (τ_1, τ_2)-$T_{\frac{3}{8}}$ spaces and weak (τ_1, τ_2)-R_0 spaces. Moreover, some characterizations of (τ_1, τ_2)-$T_{\frac{1}{4}}$ spaces and (τ_1, τ_2)-$T_{\frac{3}{8}}$ spaces and weak (τ_1, τ_2)-R_0 spaces are considered.

Definition 11. [8] *A subset A of a bitopological space (X, τ_1, τ_2) is called a $\Lambda_{(\tau_1,\tau_2)}$-set if $A = \tau_1\tau_2$-$ker(A)$.*

Definition 12. [9] *A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed if $A = U \cap F$, where U is a $\Lambda_{(\tau_1,\tau_2)}$-set and F is a $\tau_1\tau_2$-closed set of X.*

Lemma 5. [8] *For subsets A and $B_\gamma(\gamma \in \Gamma)$ of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) *$\tau_1\tau_2$-$ker(A)$ is a $\Lambda_{(\tau_1,\tau_2)}$-set.*
(2) *If A is a $\tau_1\tau_2$-open set, then A is a $\Lambda_{(\tau_1,\tau_2)}$-set.*
(3) *If B_γ is a $\Lambda_{(\tau_1,\tau_2)}$-set for each $\gamma \in \Gamma$, then $\cup_{\gamma\in\Gamma}B_\gamma$ is a $\Lambda_{(\tau_1,\tau_2)}$-set.*
(4) *If B_γ is a $\Lambda_{(\tau_1,\tau_2)}$-set for each $\gamma \in \Gamma$, then $\cap_{\gamma\in\Gamma}B_\gamma$ is a $\Lambda_{(\tau_1,\tau_2)}$-set.*

Lemma 6. [9] *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties are equivalent:*

(1) *A is $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed;*
(2) *$A = \tau_1\tau_2$-$Cl(A) \cap U$, where U is a $\Lambda_{(\tau_1,\tau_2)}$-set;*
(3) *$A = \tau_1\tau_2$-$Cl(A) \cap \tau_1\tau_2$-$ker(A)$.*

Definition 13. *A bitopological space (X, τ_1, τ_2) is called (τ_1, τ_2)-$T_{\frac{1}{4}}$ if for every finite subset A of X and each $x \in X - A$, there exists a subset B_x of X containing A and disjoint from $\{x\}$ such that B_x is either $\tau_1\tau_2$-open or $\tau_1\tau_2$-closed.*

Theorem 13. *A bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-$T_{\frac{1}{4}}$ if and only if for every finite subset of X is $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed.*

Proof. Let A be a finite subset of X. Since (X, τ_1, τ_2) is (τ_1, τ_2)-$T_{\frac{1}{4}}$, for each $x \in X - A$, there exists a subset B_x of X containing A and disjoint from $\{x\}$ such that B_x is either $\tau_1\tau_2$-open or $\tau_1\tau_2$-closed. Let U be the intersection of all $\tau_1\tau_2$-open sets B_x and F be the intersection of all $\tau_1\tau_2$-closed sets B_x. Then, we have $A = U \cap F$, U is a $\Lambda_{(\tau_1,\tau_2)}$-set and F is a $\tau_1\tau_2$-closed set. Thus, A is $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed.

Conversely, let A be a finite subset of X and $x \in X - A$. By the hypothesis, $A = U \cap F$, where U is a $\Lambda_{(\tau_1,\tau_2)}$-set and F is a $\tau_1\tau_2$-closed set. If $x \notin F$, then we are done. If $x \in F$, then $x \notin U$ and thus $x \notin V$ for some $\tau_1\tau_2$-open set V of X containing A. This shows that (X, τ_1, τ_2) is (τ_1, τ_2)-$T_{\frac{1}{4}}$. □

The proof of the following theorem is similar to that of Theorem 13 and thus omitted.

Theorem 14. *A bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-T_0 if and only if for every singleton of X is $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed.*

Definition 14. *A bitopological space (X, τ_1, τ_2) is called (τ_1, τ_2)-$T_{\frac{3}{8}}$ if for every countable subset A of X and each $x \in X - A$, there exists a subset B_x of X containing A and disjoint from $\{x\}$ such that B_x is either $\tau_1\tau_2$-open or $\tau_1\tau_2$-closed.*

The proof of the following theorem is similar to that of Theorem 13 and thus omitted.

Theorem 15. *A bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-$T_{\frac{3}{8}}$ if and only if for every countable subset of X is $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed.*

Definition 15. *A bitopological space (X, τ_1, τ_2) is said to be weak (τ_1, τ_2)-R_0 if every $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed singleton is a $\Lambda_{(\tau_1,\tau_2)}$-set.*

Theorem 16. *Let (X, τ_1, τ_2) be a bitopological space. If (X, τ_1, τ_2) is (τ_1, τ_2)-R_0, then (X, τ_1, τ_2) is weak (τ_1, τ_2)-R_0.*

Proof. Let $x \in X$ and $\{x\}$ be $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed. By Lemma 6, $\{x\} = \tau_1\tau_2\text{-Cl}(\{x\}) \cap \tau_1\tau_2\text{-}ker(\{x\})$. If $\{x\}$ is not a $\Lambda_{(\tau_1,\tau_2)}$-set, then there exists $y \in \tau_1\tau_2\text{-}ker(\{x\}) - \{x\}$. Thus, $y \notin \tau_1\tau_2\text{-Cl}(\{x\})$. Since (X, τ_1, τ_2) is (τ_1, τ_2)-R_0, $\tau_1\tau_2\text{-Cl}(\{x\}) \cap \tau_1\tau_2\text{-Cl}(\{y\}) = \emptyset$ and hence $x \notin \tau_1\tau_2\text{-Cl}(\{y\})$. Then, there exists a $\tau_1\tau_2$-open set U containing x but not y. This implies that $y \notin \tau_1\tau_2\text{-}ker(\{x\})$, which is a contradiction. Thus, (X, τ_1, τ_2) is weak (τ_1, τ_2)-R_0. □

The following lemma can be verified.

Lemma 7. *For a bitopological space (X, τ_1, τ_2), the following properties are equivalent:*

(1) (X, τ_1, τ_2) *is* (τ_1, τ_2)-T_1.

(2) *Every subset of* X *is a* $\Lambda_{(\tau_1,\tau_2)}$*-set.*

(3) *Every singleton of* X *is a* $\Lambda_{(\tau_1,\tau_2)}$*-set.*

The following theorem is an improvement of Theorem 5.

Theorem 17. *For a bitopological space (X, τ_1, τ_2), the following properties are equivalent:*

(1) (X, τ_1, τ_2) *is* (τ_1, τ_2)-T_1.

(2) (X, τ_1, τ_2) *is* (τ_1, τ_2)-T_0 *and* (τ_1, τ_2)-R_0.

(3) (X, τ_1, τ_2) *is* (τ_1, τ_2)-T_0 *and weak* (τ_1, τ_2)-R_0.

Proof. It suffices to show that (3) $\Rightarrow$ (1). To show that (X, τ_1, τ_2) is (τ_1, τ_2)-T_1, it suffices to show by Lemma 7 that every singleton of X is a $\Lambda_{(\tau_1,\tau_2)}$-set. Let $\{x\}$ be singleton of X. Since (X, τ_1, τ_2) is (τ_1, τ_2)-T_0, it follows from Theorem 14 that $\{x\}$ is $\mathscr{C}$-$\Lambda_{(\tau_1,\tau_2)}$-closed. Since (X, τ_1, τ_2) is weak (τ_1, τ_2)-R_0, $\{x\}$ is a $\Lambda_{(\tau_1,\tau_2)}$-set. □

Definition 16. *A bitopological space (X, τ_1, τ_2) is called (τ_1, τ_2)-$T_{\frac{1}{2}}$ if every g-(τ_1, τ_2)-closed set of X is $\tau_1\tau_2$-closed.*

Corollary 4. *For a weak (τ_1, τ_2)-R_0 bitopological space (X, τ_1, τ_2), the following properties are equivalent:*

(1) (X, τ_1, τ_2) *is* (τ_1, τ_2)-T_0;

(2) (X, τ_1, τ_2) *is* (τ_1, τ_2)-$T_{\frac{1}{4}}$;

(3) (X, τ_1, τ_2) *is* (τ_1, τ_2)-$T_{\frac{3}{8}}$;

(4) (X, τ_1, τ_2) *is* (τ_1, τ_2)-$T_{\frac{1}{2}}$;

(3) (X, τ_1, τ_2) *is* (τ_1, τ_2)-T_1.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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