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COMMON AND FIXED POINT RESULTS FOR ULTRAMETRIC SPACES AND THEIR APPLICATIONS TO WELL-POSEDNESS

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Abstract. This article explores the existence and uniqueness of common fixed points (CFPs) of rational-type mappings in ultrametric space. In addition, we also provide a numerical evaluation to prove that our results are accurate and feasible. Also, as an application, the well-posedness of fixed points and common fixed-point problems (CFPPs) is shown. The findings of our study build upon and broaden the scope of numerous previously established findings in the academic literature.

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1. Introduction

In recent years, the field of fixed-point theory has become an important topic due to its wide application in mathematical analysis, particularly nonlinear functional analysis. The fixed point theory is a powerful and necessary tool for determining the existence and uniqueness of solutions to many mathematical models. It is also an interdisciplinary field that can be applied to various mathematics and other disciplines, such as game theory, mathematical economics, approximation theory, engineering problems, variational inequality, and optimization theory. Recent research has shown that the fixed point theory has been studied and generalized in various spaces. Several researchers have contributed to the development of fixed-point theory. Particularly, the contribution of Banach, who established that

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contraction mapping in a complete metric space possesses a unique fixed point [4]. Since then, the study of contractive mappings has become an essential topic in metric spaces. Recently, as a result, a number of intriguing findings have been made regarding contractive mappings. Among them, Dass and Gupta's [6] discovered rational type contraction in 1975, demonstrating the uniqueness of fixed points in metric spaces. Jaggi [13] also demonstrated the existence and uniqueness of fixed points in metric spaces using rational-type contraction.

The theory of multi-valued mappings is a fascinating combination of analysis, topology, and geometry. It has been receiving significant attention from researchers in various fields of the mathematical sciences. In traditional analysis, all mappings are single-valued, but many problems in applied mathematics require multi-valued mappings. For example, the problems of stability and control theory can be solved with the help of fixed point methods for multi-valued mappings. The inverse of a single-valued map is the first naturally occurring instance of a set-valued map. When a beginner examines the inverse of fundamental trigonometric functions they can understand the importance of multi-valued mappings in solving mathematical problems. Subsequently, in 1969, Nadler [20] proved the existence of a fixed point for a set-valued mapping using Banach's method of iteration. Recently, there has been some refinement to the Nadler fixed point theorem by mathematicians (see [20], [27], [28], [30]).

Over the last 30 years, ultrametric space has developed into a new area of research. In 1978 Van Rooij proposed the concept of ultrametric space. Due to its extensive applications in various mathematical analyses, it has been successfully applied by many researchers (ref. [22], [26], [27], [28]). Gajic [10] used generalized contraction conditions to prove some fixed point results in ultrametric space. The authors of [29] presented some sufficient conditions regarding coincidence points for three and four self-maps by employing generalized contractive conditions. Similar results can be found in the references (see [8] and [14]). Balaanandhan R and Uma J recently utilized the concept of p-adic distance to discover new fixed point theorems on partially ordered ultrametric spaces [3]. Additionally, Almalki et al. [1] found some common fixed point results in modular ultrametric spaces by using different contractions and demonstrating their applicability. These additions have significantly improved non-Archimedean functional analysis, providing valuable new information and results for researchers and mathematicians in this specialized field.

Motivation of this study.

Eshaghi Gordji [9] proved the generalization of Nadler's fixed point theorem in complete metric space stated as follows:

Theorem 1.1. *Let* $(\mathbb{E}, \mathbb{d}_{\mathbb{p}})$ *be a complete metric space and* $\mathcal{F}$ *ba a mapping from* $\mathbb{E}$ *to* $CB(\mathbb{E})$ *such that*

$$\mathbb{d}_p(\mathbb{H}\mathsf{a}, \mathbb{H}\mathsf{b}) \preceq \mu \mathbb{d}_p(\mathsf{a}, \mathsf{b}) + \nu[D(\mathsf{a}, \mathcal{F}\mathsf{a}) + D(\mathsf{b}, \mathcal{F}\mathsf{b})] + \omega[D(\mathsf{a}, \mathcal{F}\mathsf{b}) + D(\mathsf{b}, \mathcal{F}\mathsf{a})]$$

for all $\mathsf{a}, \mathsf{b} \in \mathbb{E}$, *where* $\mu, \nu, \omega \geq 0$ *and* $\mu + 2\nu + 2\omega < 1$. *Then* $\mathcal{F}$ *has a fixed point.*

Later, Rameshkumar and Pitchaimani [23] extended the above results in ultrametric spaces stated as follows:

Theorem 1.2. *Let* $(\mathbb{E}, \mathbb{d}_p)$ *be a complete ultrametric space and* $\mathcal{F} : \mathbb{E} \to CB(\mathbb{E})$ *such that*

$$\mathbb{d}_p(\mathbb{H}\mathsf{a}, \mathbb{H}\mathsf{b}) \preceq \mu \mathbb{d}_p(\mathsf{a}, \mathsf{b}) + \nu[D(\mathsf{a}, \mathcal{F}\mathsf{a}) + D(\mathsf{b}, \mathcal{F}\mathsf{b})] + \omega[D(\mathsf{a}, \mathcal{F}\mathsf{b}) + D(\mathsf{b}, \mathcal{F}\mathsf{a})]$$

for all $\mathsf{a}, \mathsf{b} \in \mathbb{E}$, *where* $\mu, \nu, \omega \geq 0$ *and* $\mu + 2\nu + 2\omega < 1$. *Then* $\mathcal{F}$ *has a fixed point.*

Motivated by the above results in this manuscript, we explore various fixed point results in ultrametric space with the help of rational-type contractions.

Highlights of this study:

The main contribution of our study comprises the following:

1. In the setting of ultrametric space , we prove the existence and uniqueness of a CFP for a pair of single and set-valued mappings using rational-type contractive conditions.
2. The accuracy and the feasibility of results are shown with numerical examples.
3. Finally, The well-posedness of CFP problems is illustrated as an application.

Structure of this study

The paper is structured into five sections for the ease of reading and understanding. Section 1 covers the background and motivation for the study, while Section 2 discusses the main results. In Section 3, the results are illustrated with an example, and in Section 4, applications to well-posedness is examined. The conclusion can be found in Section 5.

The following symbols and their meanings, which we provide in the following table 1, are frequently used throughout this work.

TABLE 1. List of symbols used in this article

Symbol	Representation
$\mathbb{E}, \mathsf{A}^*, \mathsf{B}^*$	Sets
$\mathbb{d}_p$	Ultrametric distance
$\mathcal{F}, \mathcal{G}$	Mappings
$\mathbb{H}$	Hausdorff metric
μ, ν, ω	Scalars
$CB(\mathbb{E})$	Class of closed and bounded subsets of $\mathbb{E}$

In this section, we will start by introduce some definitions that have frequently appeared in our results.

Definition 1.3. [31] An ultrametric space $(\mathbb{E}, \mathbb{d}_p)$ is a metric space that satisfies the following form of stronger inequality, that is for every point a, b and c in $\mathbb{E}$

$$\mathbb{d}_p(\mathsf{a}, \mathsf{c}) \preceq \max\{\mathbb{d}_p(\mathsf{a}, \mathsf{b}), \mathbb{d}_p(\mathsf{b}, \mathsf{c})\},$$

then the pair $(\mathbb{E}, \mathbb{d}_p)$ is said to be an ultrametric space. In ultrametric spaces, a key characteristic is that the distance between any two points is limited by the maximum of the distances from one of those points to the other two which replaces the usual triangle inequality in metric space.

Example 1.4. *Consider* $\mathbb{E} = \{\mu, \nu, \omega, \lambda\}$ *and* $\mathbb{d}_p(\nu, \omega) = \mathbb{d}_p(\mu, \lambda) = 2, \mathbb{d}_p(\mu, \nu) = \mathbb{d}_p(\nu, \lambda) = \mathbb{d}_p(\mu, \omega) = \mathbb{d}_p(\omega, \lambda) = 3$. *Therefore,* $(\mathbb{E}, \mathbb{d}_p)$ *is an ultrametric space.*

Definition 1.5. [23] A space in an ultrametric system is considered complete when every Cauchy sequence converges. Let $(\mathbb{E}, \mathbb{d}_p)$ be an ultrametric space. We define $CB(\mathbb{E})$ as the set of all non-empty, closed, and bounded subsets of $\mathbb{E}$. Let $\mathbb{H}$ be a Hausdorff metric, that is,

$$\mathbb{H}(\mathsf{A}^*, \mathsf{B}^*) = \max\left\{sup_{a\in\mathsf{A}} D(a, \mathsf{B}^*), sup_{b\in\mathsf{B}} D(b, \mathsf{A}^*)\right\},$$

for $\mathsf{A}^*, \mathsf{B}^*$ in $CB(\mathbb{E})$, where $\mathbb{d}_p(x, \mathsf{B}^*) = \inf_{y\in\mathsf{B}^*} \mathbb{d}_p(x, y)$. It is easy to see that this is an ultrametric space.

Definition 1.6. [23] Let $\mathbb{E}$ be a non-empty set and mappings $\mathcal{F} : \mathbb{E} \to CB(\mathbb{E})$ and $\mathsf{g} : \mathbb{E} \to \mathbb{E}$ such that
(a) If $\mathsf{u} = \mathsf{g}(\mathsf{a}) \in \mathcal{F}(\mathsf{a})$, then $\mathsf{u} \in \mathbb{E}$ is called a point of coincidence of g and $\mathcal{F}$.
(b) If $\mathsf{a} = \mathsf{g}(\mathsf{a}) \in \mathcal{F}(\mathsf{a})$, then $\mathsf{a} \in \mathbb{E}$ is said to be a CFP of g and $\mathcal{F}$.
(c) If $\mathsf{a} \in \mathcal{F}(\mathsf{a})$, then $\mathsf{a} \in \mathbb{E}$ is said to be a fixed point of $\mathcal{F}$.

In this work, we establish some fixed point theorems in complete ultrametric spaces by drawing inspiration from the concepts of rational type contraction mappings.

2. Main Results

In this section, we explore the results for CFP theorems under single-valued and multi-valued mappings over non-Archimedean space (Ultrametric space).

2.1. **Common fixed point theorem concerns single-valued mappings.**

Theorem 2.1. *Let* $(\mathbb{E}, \mathbb{d}_p)$ *be a complete ultrametric space and let* $\mathcal{F}, \mathcal{G}$ *be the mappings from* $\mathbb{E}$ *to itself such that*

$$\mathbb{d}_p(\mathcal{F}\mathsf{a}, \mathcal{G}\mathsf{b}) \preceq \mu\frac{\mathbb{d}_p(\mathsf{b}, \mathcal{G}\mathsf{b})(1 + \mathbb{d}_p(\mathsf{a}, \mathcal{F}\mathsf{a}))}{1 + \mathbb{d}_p(\mathsf{a}, \mathsf{b})} + \nu\mathbb{d}_p(\mathsf{a}, \mathcal{G}\mathsf{b}) + \omega\mathbb{d}_p(\mathsf{a}, \mathsf{b}), \tag{2.1}$$

for each $\mathsf{a}, \mathsf{b} \in \mathbb{E}$, *where* $\mu, \nu, \omega \in [0, 1)$ *with* $\mu + \nu + \omega < 1$. *Then* $\mathcal{G}$ *and* $\mathcal{F}$ *have a unique CFP.*

Proof. For a given $\mathsf{a}_0 \in \mathbb{E}$ define a sequence a_n as follows:

$$\mathsf{a}_{n+1} = \mathcal{F}(\mathsf{a}_n) \text{ and } \mathsf{a}_{n+2} = \mathcal{G}\mathsf{a}_{n+1}, \qquad \text{for } n = 0, 1, 2, \cdots,$$

using (2.1), we obtain

$$\begin{aligned}
\mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) &= \mathbb{d}_p(\mathcal{F}(\mathsf{a}_{n-1}), \mathcal{G}(\mathsf{a}_n)) \\
&\preceq \mu \frac{\mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1})(1 + \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n))}{1 + \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n)} + \nu \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_{n+1}) + \omega \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n) \\
&\preceq \mu \mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) + \nu \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_{n+1}) + \omega \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n)
\end{aligned}$$

$$\mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) \preceq \mu \mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) + \nu \ max \{\mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n), \mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1})\} + \omega \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n). \tag{2.2}$$

Case 1: Suppose that $\max \{\mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n), \mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1})\} = \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n)$. Then, we have

$$\mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) \preceq \mu \mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) + \nu \ \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n) + \omega \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n)$$

$$\begin{aligned}
\mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) &\preceq \frac{\nu + \omega}{1 - \mu} \ \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n) \\
\mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) &\preceq \left(\frac{\nu + \omega}{1 - \mu}\right)^n \ \mathbb{d}_p(\mathsf{a}_0, \mathsf{a}_1) \\
\mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) &\preceq \kappa^n \mathbb{d}_p(\mathsf{a}_0, \mathsf{a}_1), \qquad (2.3)
\end{aligned}$$

where $\kappa = \frac{\nu+\omega}{1-\mu} < 1$.

Case 2: On the other hand, suppose that $\max \{\mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n), \mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1})\} = \mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1})$, then we have

$$\begin{aligned}
\mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) &\preceq \mu \mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) + \nu \ \mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) + \omega \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n) \\
\mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) &\preceq \frac{\omega}{(1 - \mu - \nu)} \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n).
\end{aligned}$$

Similarly, continuing this process, we get

$$\begin{aligned}
\mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) &\preceq \left(\frac{\omega}{(1 - \mu - \nu)}\right)^n \mathbb{d}_p(\mathsf{a}_0, \mathsf{a}_1) \qquad (2.4) \\
\mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) &\preceq \mathsf{h}^n \mathbb{d}_p(\mathsf{a}_0, \mathsf{a}_1), \qquad (2.5)
\end{aligned}$$

where, $\mathsf{h} = \frac{\omega}{(1-\mu-\nu)} < 1$.
Hence, from (2.3) and (2.5), we have $\lim_{n\to\infty} \kappa^n = \lim_{n\to\infty} \mathsf{h}^n = 0$ which implies that sequence $\{\mathsf{a}_n\}$ in $\mathbb{E}$ is a Cauchy sequence. Since $\mathbb{E}$ is complete, there exists a point $\mathsf{a} \in \mathbb{E}$ such that

$$\lim_{n\to\infty} \mathsf{a}_n = \mathsf{a}.$$

Therefore,

$$\begin{aligned}\mathbb{d}_p(\mathsf{a},\mathcal{G}\mathsf{a}) &\preceq \max\ \left\{\mathbb{d}_p(\mathsf{a},\mathsf{a}_{n+1}),\mathbb{d}_p(\mathsf{a}_{n+1},\mathcal{G}\mathsf{a})\right\}\\ &\preceq \max\ \left\{\mathbb{d}_p(\mathsf{a},\mathsf{a}_{n+1}),\mu\frac{\mathbb{d}_p(\mathsf{a},\mathcal{G}\mathsf{a})(1+\mathbb{d}_p(\mathsf{a}_n,\mathcal{F}\mathsf{a}_n))}{1+\mathbb{d}_p(\mathsf{a}_n,\mathsf{a})}+\nu\mathbb{d}_p(\mathsf{a}_n,\mathcal{G}\mathsf{a})+\omega\mathbb{d}_p(\mathsf{a}_n,\mathsf{a})\right\}\\ &\preceq \max\ \left\{\mathbb{d}_p(\mathsf{a},\mathsf{a}_{n+1}),\mu\frac{\mathbb{d}_p(\mathsf{a},\mathcal{G}\mathsf{a})(1+\mathbb{d}_p(\mathsf{a}_n,\mathsf{a}_{n+1}))}{1+\mathbb{d}_p(\mathsf{a}_n,\mathsf{a})}+\nu\mathbb{d}_p(\mathsf{a}_n,\mathcal{G}\mathsf{a})+\omega\mathbb{d}_p(\mathsf{a}_n,\mathsf{a})\right\}.\end{aligned}$$

Allowing limit as $n\to\infty$, we obtain

$$\mathbb{d}_p(\mathsf{a},\mathcal{G}\mathsf{a}) \preceq (\mu+\nu)\mathbb{d}_p(\mathsf{a},\mathcal{G}\mathsf{a}).$$

Since $\mu+\nu<1$, which implies that $\mathcal{G}\mathsf{a}=\mathsf{a}$. Thus, it is evident that a is a fixed point of $\mathcal{G}$. Similarly, we can prove that $\mathsf{a}=\mathcal{F}\mathsf{a}$. Hence, a is a CFP of $\mathcal{F}$ and $\mathcal{G}$. Now, we will show that uniqueness of a. Consider another fixed point $'\mathsf{b}'$ of $\mathcal{G}$ other than a,

$$i.e.,\ \mathcal{G}\mathsf{b}=\mathsf{b}. \tag{2.6}$$

Now,

$$\begin{aligned}\mathbb{d}_p(\mathsf{a},\mathsf{b}) &= \mathbb{d}_p(\mathcal{F}\mathsf{a},\mathcal{G}\mathsf{b})\\ &\preceq \mu\frac{\mathbb{d}_p(\mathsf{a},\mathcal{G}\mathsf{a})(1+\mathbb{d}_p(\mathsf{b},\mathcal{F}\mathsf{b}))}{1+\mathbb{d}_p(\mathsf{a},\mathsf{b})}+\nu\mathbb{d}_p(\mathsf{a},\mathcal{G}\mathsf{b})+\omega\mathbb{d}_p(\mathsf{a},\mathsf{b})\end{aligned}$$

which yields

$$\mathbb{d}_p(\mathsf{a},\mathsf{b}) \preceq (\nu+\omega)\mathbb{d}_p(\mathsf{a},\mathsf{b}).$$

Since, $\nu+\omega<1$, we have $\mathsf{a}=\mathsf{b}$. Hence, $\mathcal{F}$ and $\mathcal{G}$ have a unique CFP in $\mathbb{E}$. □

If we set $\omega=0$ in Theorem 2.1, we obtain the following results.

Corollary 2.2. *Let $(\mathbb{E},\mathbb{d}_p)$ be a complete ultrametric space and let $\mathcal{F},\mathcal{G}$ be the mappings from $\mathbb{E}$ to itself such that*

$$\mathbb{d}_p(\mathcal{F}\mathsf{a},\mathcal{G}\mathsf{b}) \preceq \mu\frac{\mathbb{d}_p(\mathsf{b},\mathcal{G}\mathsf{b})(1+\mathbb{d}_p(\mathsf{a},\mathcal{F}\mathsf{a}))}{1+\mathbb{d}_p(\mathsf{a},\mathsf{b})}+\nu\mathbb{d}_p(\mathsf{a},\mathcal{G}\mathsf{b}) \tag{2.7}$$

for each $\mathsf{a},\mathsf{b}\in\mathbb{E}$, where $\mu,\nu\in[0,1)$ with $\mu+\nu<1$. Then $\mathcal{G}$ and $\mathcal{F}$ have a unique CFP.

2.2. Fixed point theorem concerns single-valued and multi valued mappings:

If we set $\mathcal{F}=\mathcal{G}$ in Theorem 2.1, we obtain the following results.

Theorem 2.3. *Let $(\mathbb{E},\mathbb{d}_p)$ be a complete ultrametric space and let $\mathcal{F}$ be the self map from $\mathbb{E}$ to itself such that*

$$\mathbb{d}_p(\mathcal{F}\mathsf{a},\mathcal{F}\mathsf{b}) \preceq \mu\frac{\mathbb{d}_p(\mathsf{b},\mathcal{F}\mathsf{b})(1+\mathbb{d}_p(\mathsf{a},\mathcal{F}\mathsf{a}))}{1+\mathbb{d}_p(\mathsf{a},\mathsf{b})}+\nu\mathbb{d}_p(\mathsf{a},\mathcal{F}\mathsf{b})+\omega\mathbb{d}_p(\mathsf{a},\mathsf{b}), \tag{2.8}$$

for each $\mathsf{a},\mathsf{b}\in\mathbb{E}$, where $\mu,\nu,\omega\in[0,1)$ with $\mu+\nu+\omega<1$. Then $\mathcal{F}$ has a unique fixed point.

Proof. For a given $\mathsf{a}_0 \in \mathbb{E}$ define a sequence $\{\mathsf{a}_n\}$ as follows:

$$\mathsf{a}_{n+1} = \mathcal{F}(\mathsf{a}_n), \text{ for } n = 0, 1, 2, \cdots .$$

Let us consider,

$$\begin{aligned}
\mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) &= \mathbb{d}_p(\mathcal{F}(\mathsf{a}_{n-1}), \mathcal{F}(\mathsf{a}_n)) \\
&\preceq \mu \frac{\mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1})(1 + \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n))}{1 + \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n)} + \nu \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_{n+1}) + \omega \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n) \\
&\preceq \mu \mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) + \nu \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_{n+1}) + \omega \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n)
\end{aligned}$$

$$\mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) \preceq \mu \mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) + \nu \ \max\{\mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n), \mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1})\} + \omega \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n). \tag{2.9}$$

Case 1: Suppose that $\max\{\mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n), \mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1})\} = \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n)$. Then, we have

$$\mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) \preceq \mu \mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) + \nu \ \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n) + \omega \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n)$$

$$\begin{aligned}
\mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) &\preceq \frac{\nu + \omega}{1 - \mu} \ \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n) \\
\mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) &\preceq \left(\frac{\nu + \omega}{1 - \mu}\right)^n \mathbb{d}_p(\mathsf{a}_0, \mathsf{a}_1) \\
\mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) &\preceq \kappa_1^n \mathbb{d}_p(\mathsf{a}_0, \mathsf{a}_1), \qquad (2.10)
\end{aligned}$$

where $\kappa_1 = \frac{\nu+\omega}{1-\mu} < 1$.
Case 2: On the other hand, if $\max\{\mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n), \mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1})\} = \mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1})$. Then, we have

$$\begin{aligned}
\mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) &\preceq \mu \mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) + \nu \ \mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) + \omega \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n) \\
\mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) &\preceq \frac{\omega}{(1 - \mu - \nu)} \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n).
\end{aligned}$$

Similarly, continuing this process, we get

$$\begin{aligned}
\mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) &\preceq \left(\frac{\omega}{(1 - \mu - \nu)}\right)^n \mathbb{d}_p(\mathsf{a}_0, \mathsf{a}_1) \\
\mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_{n+1}) &\preceq \mathsf{h}_1^n \mathbb{d}_p(\mathsf{a}_0, \mathsf{a}_1), \qquad (2.11)
\end{aligned}$$

where $\mathsf{h}_1 = \frac{\omega}{(1-\mu-\nu)} < 1$. Since, $\kappa_1, \mathsf{h}_1 < 1$, in this cases, we have

$$\lim_{n \to \infty} \kappa_1^n = \lim_{n \to \infty} \mathsf{h}_1^n = 0$$

which implies that the sequence $\{\mathsf{a}_n\}$ in $\mathbb{E}$ is a Cauchy sequence. Since $\mathbb{E}$ is complete, there exists a point $\mathsf{a} \in \mathbb{E}$ such that

$$\lim_{n \to \infty} \mathsf{a}_n = \mathsf{a}.$$

Thus,

$$\begin{aligned}
\mathbb{d}_p(\mathsf{a},\mathcal{F}\mathsf{a}) &\preceq \max\ \{\mathbb{d}_p(\mathsf{a},\mathsf{a}_{n+1}),\mathbb{d}_p(\mathsf{a}_{n+1},\mathcal{F}\mathsf{a})\}\\
&\preceq \max\ \left\{\mathbb{d}_p(\mathsf{a},\mathsf{a}_{n+1}),\mu\frac{\mathbb{d}_p(\mathsf{a},\mathcal{F}\mathsf{a})(1+\mathbb{d}_p(\mathsf{a}_n,\mathcal{F}\mathsf{a}_n))}{1+\mathbb{d}_p(\mathsf{a}_n,\mathsf{a})}+\nu\mathbb{d}_p(\mathsf{a}_n,\mathcal{F}\mathsf{a})+\omega\mathbb{d}_p(\mathsf{a}_n,\mathsf{a})\right\}\\
&\preceq \max\ \left\{\mathbb{d}_p(\mathsf{a},\mathsf{a}_{n+1}),\mu\frac{\mathbb{d}_p(\mathsf{a},\mathcal{F}\mathsf{a})(1+\mathbb{d}_p(\mathsf{a}_n,\mathsf{a}_{n+1}))}{1+\mathbb{d}_p(\mathsf{a}_n,\mathsf{a})}+\nu\mathbb{d}_p(\mathsf{a}_n,\mathcal{F}\mathsf{a})+\omega\mathbb{d}_p(\mathsf{a}_n,\mathsf{a})\right\}.
\end{aligned}$$

Allowing limit as $n\to\infty$, we obtain

$$\mathbb{d}_p(\mathsf{a},\mathcal{F}\mathsf{a})\preceq(\mu+\nu)\mathbb{d}_p(\mathsf{a},\mathcal{F}\mathsf{a}).$$

Since $\mu+\nu<1$, we have $\mathcal{F}\mathsf{a}=\mathsf{a}$ Thus, it is evident that a is a fixed point of $\mathcal{F}$.

Uniqueness part: Let us consider $'\mathsf{b}'$ is another fixed point of $\mathcal{F}$.

$$i.e.,\qquad \mathcal{F}\mathsf{b}=\mathsf{b}.$$

Consider

$$\begin{aligned}
\mathbb{d}_p(\mathsf{a},\mathsf{b}) &= \mathbb{d}_p(\mathcal{F}\mathsf{a},\mathcal{F}\mathsf{b})\\
&\preceq \mu\frac{\mathbb{d}_p(\mathsf{a},\mathcal{F}\mathsf{a})(1+\mathbb{d}_p(\mathsf{b},\mathcal{F}\mathsf{b}))}{1+\mathbb{d}_p(\mathsf{a},\mathsf{b})}+\nu\mathbb{d}_p(\mathsf{a},\mathcal{F}\mathsf{b})+\omega\mathbb{d}_p(\mathsf{a},\mathsf{b}),
\end{aligned}$$

which yields

$$\mathbb{d}_p(\mathsf{a},\mathsf{b})\preceq(\nu+\omega)\mathbb{d}_p(\mathsf{a},\mathsf{b})$$

Since, $\nu+\omega<1$, we have $\mathsf{a}=\mathsf{b}$. Hence the proof. □

If we set $\omega=0$ in Theorem 2.3, we obtain the following results.

Corollary 2.4. *Let $\mathcal{F}$ be a mapping of $\mathcal{X}$ into itself such that*

$$\mathbb{d}_p(\mathcal{F}\mathsf{a},\mathcal{F}\mathsf{b})\preceq\mu\frac{\mathbb{d}_p(\mathsf{b},\mathcal{F}\mathsf{b})(1+\mathbb{d}_p(\mathsf{a},\mathcal{F}\mathsf{a}))}{1+\mathbb{d}_p(\mathsf{a},\mathsf{b})}+\nu\mathbb{d}_p(\mathsf{a},\mathcal{F}\mathsf{b}),\tag{2.12}$$

for all $\mathsf{a},\mathsf{b}\in\mathbb{E},\ \ \mu>0,\ \ \nu>0,\ \ \mu+\nu<1$. *Then $\mathcal{F}$ has a unique fixed point.*

Remark 2.5. Let $(\mathbb{E},\mathbb{d}_p)$ be an ultrametric space and $\mathcal{A},\mathcal{B}\in CB(E)$. Then for each $\mathsf{a}\in\mathcal{A}$ and $\epsilon\geq 0$ there exist $\mathsf{b}\in\mathcal{B}$ such that

$$\mathsf{d}_p(\mathsf{a},\mathsf{b})\leq\mathbb{H}(\mathcal{A},\mathcal{B})+\epsilon.$$

Further, we determine the fixed point for a set-valued mapping in the following.

Theorem 2.6. *Let $(\mathbb{E},\mathbb{d}_p)$ be a complete ultrametric space and the mapping $\mathcal{F}$ from $\mathbb{E}$ to $CB(\mathbb{E})$ satisfying the following condition*

$$\mathbb{H}(\mathcal{F}\mathsf{a},\mathcal{F}\mathsf{b})\preceq\mu\frac{D(\mathsf{b},\mathcal{F}\mathsf{b})(1+D(\mathsf{a},\mathcal{F}\mathsf{a}))}{1+\mathbb{d}_p(\mathsf{a},\mathsf{b})}+\nu D(\mathsf{a},\mathcal{F}\mathsf{b})+\omega\mathbb{d}_p(\mathsf{a},\mathsf{b})\tag{2.13}$$

for all $\mathsf{a},\mathsf{b}\in\mathbb{E}$, *where* $\mu,\nu,\omega\in[0,1)$ *and* $\mu+2\nu+\omega<1$. *Then $\mathcal{F}$ has a unique fixed point.*

Proof. Let $\mathsf{a}_0 \in \mathbb{E}$, $\mathsf{a}_1 \in \mathcal{F}\mathsf{a}_0$. Define $\kappa_3 = \frac{\nu+\omega}{1-(\mu+\nu)}$. Suppose that $\kappa_3 = 0$. Then the results are obvious. On the other hand, if $\kappa_3 > 0$, there exists $\mathsf{a}_2 \in \mathcal{F}\mathsf{a}_1$ such that

$$\mathbb{d}_p(\mathsf{a}_1, \mathsf{a}_2) \preceq \mathbb{H}(\mathcal{F}\mathsf{a}_0, \mathcal{F}\mathsf{a}_1) + \kappa_3.$$

Since $\mathcal{F}\mathsf{a}_1, \mathcal{F}\mathsf{a}_2 \in CB(\mathbb{E})$ and $\mathsf{a}_2 \in \mathcal{F}\mathsf{a}_1$ and there exists $\mathsf{a}_3 \in \mathcal{F}\mathsf{a}_2$ such that

$$\mathbb{d}_p(\mathsf{a}_2, \mathsf{a}_3) \preceq \mathbb{H}(\mathcal{F}\mathsf{a}_1, \mathcal{F}\mathsf{a}_2) + \kappa_3^2,$$

proceeding in this manner till a_{n-1}, when $n > 0$ we get $\mathsf{a}_n \in \mathcal{F}\mathsf{a}_{n-1}$ satisfying the following condition

$$\mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n) \preceq \mathbb{H}(\mathcal{F}\mathsf{a}_{n-2}, \mathcal{F}\mathsf{a}_{n-1}) + \kappa_3^{n-1}.$$

For $n \in \mathbb{N}$, we have

$$\begin{aligned}
\mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n) &\preceq \mathbb{H}(\mathcal{F}\mathsf{a}_{n-2}, \mathcal{F}\mathsf{a}_{n-1}) + \kappa_3^{n-1} \\
&\preceq \mu \frac{D(\mathsf{a}_{n-1}, \mathcal{F}\mathsf{a}_{n-1})(1 + D(\mathsf{a}_{n-2}, \mathcal{F}\mathsf{a}_{n-2}))}{1 + \mathbb{d}_p(\mathsf{a}_{n-2}, \mathsf{a}_{n-1})} + \nu D(\mathsf{a}_{n-2}, \mathcal{F}\mathsf{a}_{n-1}) + \omega \mathbb{d}_p(\mathsf{a}_{n-2}, \mathsf{a}_{n-1}) + \kappa_3^{n-1} \\
&\preceq \mu D(\mathsf{a}_{n-1}, \mathsf{a}_n) + \nu D(\mathsf{a}_{n-2}, \mathsf{a}_n) + \omega \mathbb{d}_p(\mathsf{a}_{n-2}, \mathsf{a}_{n-1}) + \kappa_3^{n-1} \\
&\preceq \mu \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n) + \nu\{\mathbb{d}_p(\mathsf{a}_{n-2}, \mathsf{a}_{n-1}) + \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n)\} + \omega \mathbb{d}_p(\mathsf{a}_{n-2}, \mathsf{a}_{n-1}) + \kappa_3^{n-1}
\end{aligned}$$

$$\begin{aligned}
\mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n) &\preceq \frac{\nu + \omega}{1 - \mu - \nu} \mathbb{d}_p(\mathsf{a}_{n-2}, \mathsf{a}_{n-1}) + \frac{\kappa_3^{n-1}}{1 - \mu - \nu} \\
\mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n) &\preceq \kappa_3 \mathbb{d}_p(\mathsf{a}_{n-2}, \mathsf{a}_{n-1}) + \frac{\kappa_3^{n-1}}{1 - \mu - \nu}.
\end{aligned} \tag{2.14}$$

Clearly, we obtain,

$$\mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}_n) \preceq \kappa_3^{n-1} \mathbb{d}_p(\mathsf{a}_0, \mathsf{a}_1) + (n-1) \frac{\kappa_3^{n-1}}{1 - \mu - \nu}.$$

Hence, $\{\mathsf{a}_n\}$ is a Cauchy sequence in $\mathbb{E}$ when $n \to \infty$. Since $\kappa_3 < 1$, due to the completeness of $\mathbb{E}$, the sequence $\{\mathsf{a}_n\} \to \mathsf{a}^* \in \mathbb{E}$, that is,

$$\lim_{n \to \infty} \mathsf{a}_n = \mathsf{a}^*.$$

Now,

$$\begin{aligned}
\mathcal{D}(\mathsf{a}^*, \mathcal{F}\mathsf{a}^*) &\preceq \max\left(\mathcal{D}(\mathsf{a}^*, \mathsf{a}_n), \mathcal{D}(\mathsf{a}_n, \mathcal{F}\mathsf{a}^*)\right) \\
&\preceq \max\left(\mathcal{D}(\mathsf{a}^*, \mathsf{a}_n), \mathbb{H}(\mathcal{F}\mathsf{a}_{n-1}, \mathcal{F}\mathsf{a}^*)\right) \\
&\preceq \max\left\{\mathcal{D}(\mathsf{a}^*, \mathsf{a}_n), \mu \frac{\mathcal{D}(\mathsf{a}^*, \mathcal{F}\mathsf{a}^*)(1 + \mathcal{D}(\mathsf{a}_{n-1}, \mathsf{a}_n))}{1 + \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}^*)} + \nu \mathcal{D}(\mathsf{a}_{n-1}, \mathcal{F}\mathsf{a}^*) + \omega \mathbb{d}_p(\mathsf{a}_{n-1}, \mathsf{a}^*)\right\}
\end{aligned}$$

Allowing limit as $n \to \infty$, we have

$$\mathcal{D}(\mathsf{a}^*, \mathcal{F}\mathsf{a}^*) \preceq \max\left((\mu + \nu)\mathcal{D}(\mathsf{a}^*, \mathcal{F}\mathsf{a}^*)\right).$$

Thus, it is evident that $\mathcal{D}(\mathsf{a}^*, \mathcal{F}\mathsf{a}^*) = 0$ as $\mu + \nu < 1$. Hence $\mathcal{F}$ has a fixed point $\mathsf{a}^* \in \mathbb{E}$.

Next, we want to prove that uniqueness. Let a' be the fixed point such that $\mathsf{a}' \in \mathcal{F}\mathsf{a}'$. Now, using (2.13), we obtain

$$\begin{aligned} \mathbb{d}_p(\mathsf{a}^*, \mathsf{a}') &= \mathbb{H}(\{\mathsf{a}^*\}, \{\mathsf{a}'\}) = \mathbb{H}(\mathcal{F}\mathsf{a}^*, \mathcal{F}\mathsf{a}') \\ &\preceq \mu \frac{\mathcal{D}(\mathsf{a}', \mathcal{F}\mathsf{a}')(1 + \mathcal{D}(\mathsf{a}^*, \mathcal{F}\mathsf{a}^*))}{1 + \mathbb{d}_p(\mathsf{a}^*, \mathsf{a}')} + \nu \mathcal{D}(\mathsf{a}^*, \mathcal{F}\mathsf{a}') + \omega \mathbb{d}_p(\mathsf{a}^*, \mathsf{a}') \\ &= (\nu + \omega)\mathbb{d}_p(\mathsf{a}^*, \mathsf{a}'), \end{aligned}$$

which implies that $\mathsf{a}^* = \mathsf{a}'$ as $\nu + \omega < 1$. Hence the proof. □

If we take $\omega = 0$ in Theorem 2.6, we will obtain the following results.

Corollary 2.7. *Let $(\mathbb{E}, \mathbb{d}_p)$ be a complete ultrametric space and $\mathcal{F}$ from $\mathbb{E}$ to $CB(\mathbb{E})$ satisfying the following condition*

$$\mathbb{H}(\mathcal{F}\mathsf{a}, \mathcal{F}\mathsf{b}) \preceq \mu \frac{D(\mathsf{b}, \mathcal{F}\mathsf{b})(1 + D(\mathsf{a}, \mathcal{F}\mathsf{a}))}{1 + \mathbb{d}_p(\mathsf{a}, \mathsf{b})} + \nu D(\mathsf{a}, \mathcal{F}\mathsf{b}) \tag{2.15}$$

for all $\mathsf{a}, \mathsf{b} \in \mathbb{E}$, where $\mu, \nu \in [0, 1)$ and $\mu + 2\nu < 1$. Then $\mathcal{F}$ has unique fixed point.

3. Examples

The p-adic numbers were introduced by German mathematician Kurt Hensel in 1897. The field of p-adic numbers plays a crucial role in number theory. The p-adic numbers are a subset of the field of rational numbers, which themselves are an extension of the field of real numbers. They can be obtained in the same way as real numbers by defining the concept of distance between rational numbers and proceeding to the completion of the distance. In this section, we will explore the p-adic distance, which is a unique way of measuring distance in mathematics. The p-adic distance is used to define metric spaces called ultrametric spaces. These spaces have interesting properties that make them useful in many areas of mathematics. p-adic distance uses a different way of measuring the distance. This is known as the p-adic metric. By studying fixed points in the context of p-adic distance, mathematicians have gained new insights into the behavior and properties of mappings. This has led to the development of p-adic fixed point theory, which has applications in number theory, algebraic geometry, and mathematical physics.

Definition 3.1. [2] Consider a fixed prime number p. Let $c \in \mathbb{R}$, where $0 < c < 1$ and c will be fixed throughout the discussion. If $\varkappa$ is any rational number other than zero, we can write $\varkappa$ in the form

$$|\varkappa|_p = p^{\alpha} \frac{a}{b}$$

where $\alpha \in \mathbb{Z}$, and $a, b \in \mathbb{Z}$, and $p \nmid a, p \nmid b$, and clearly, α may be positive, negative or zero depending on $\mathbb{E}$. We now define

$$|\varkappa|_p = c^{\alpha} \text{ and } |0|_p = 0$$

It is important to note that, $|\varkappa|_p \geq 0$ and equals 0 if and only if $\varkappa = 0$.

Example 3.2. *A p-adic distance function* $\mathbb{d}_{p^*} : \mathbb{E} \times \mathbb{E} \to [0, \infty)$ *by* $\mathbb{d}_{p^*}(\mathsf{a}, \mathsf{b}) = |\mathsf{a} - \mathsf{b}|_p$*, where* $\mathbb{E} = [0, 1]$*. Define the two self mappings* $\mathcal{F}, \mathcal{G}$ *on* $\mathbb{E}$ *by*

$$\mathcal{F}\mathsf{a} = \frac{\mathsf{a}}{2} + \frac{1}{8} \qquad \textit{and} \quad \mathcal{G}\mathsf{a} = 2\mathsf{a} - \frac{1}{4},$$

satisfying the condition (2.1)*. Then* $\mathcal{F}, \mathcal{G}$ *have a CFP in* $\mathbb{E}$*.*

Proof. Now for any $\mathsf{a}, \mathsf{b} \in \mathbb{E}$ and when $p < 7$

$$\begin{aligned}
\mathbb{d}_{p^*}(\mathcal{F}\mathsf{a}, \mathcal{G}\mathsf{b}) &= |(\frac{\mathsf{a}}{2} + \frac{1}{8}) - (2\mathsf{b} - \frac{1}{4})|_p = \frac{1}{2}|\mathsf{a} - 4\mathsf{b} + \frac{3}{4}|_p \\
&\preceq \frac{\mu}{32}\frac{|1 - 4\mathsf{b}|_p|4\mathsf{a} + 7|_p}{|1 + \mathsf{a} - \mathsf{b}|_p} + \frac{\nu}{4}|4\mathsf{a} - 8\mathsf{b} - 1|_p + \omega|\mathsf{a} - \mathsf{b}|_p \\
&\preceq \mu\frac{|\mathsf{b} - (2\mathsf{b} - \frac{1}{4})|_p(1 + |\frac{\mathsf{a}}{2} - \frac{1}{8}|_p)}{1 + |\mathsf{a} - \mathsf{b}|_p} + \nu|\mathsf{a} - (2\mathsf{b} + \frac{1}{4})|_p + \omega|\mathsf{a} - \mathsf{b}|_p \\
&\preceq \mu\frac{\mathbb{d}_{p^*}(\mathsf{b}, \mathcal{G}\mathsf{b})(1 + \mathbb{d}_{p^*}(\mathsf{a}, \mathcal{F}\mathsf{a}))}{1 + \mathbb{d}_{p^*}(\mathsf{a}, \mathsf{b})} + \nu\mathbb{d}_{p^*}(\mathsf{a}, \mathcal{G}\mathsf{b}) + \omega\mathbb{d}_{p^*}(\mathsf{a}, \mathsf{b})
\end{aligned}$$

Hence, the contraction condition is satisfied. Now, by selecting proper values of $\mu, \nu, \omega \in [0, 1)$ such that $\mu + \nu + \omega < 1$, we get $\mathcal{G}$ and $\mathcal{F}$ have a CFP and is equal to $\frac{1}{4} \in \mathbb{E}$.

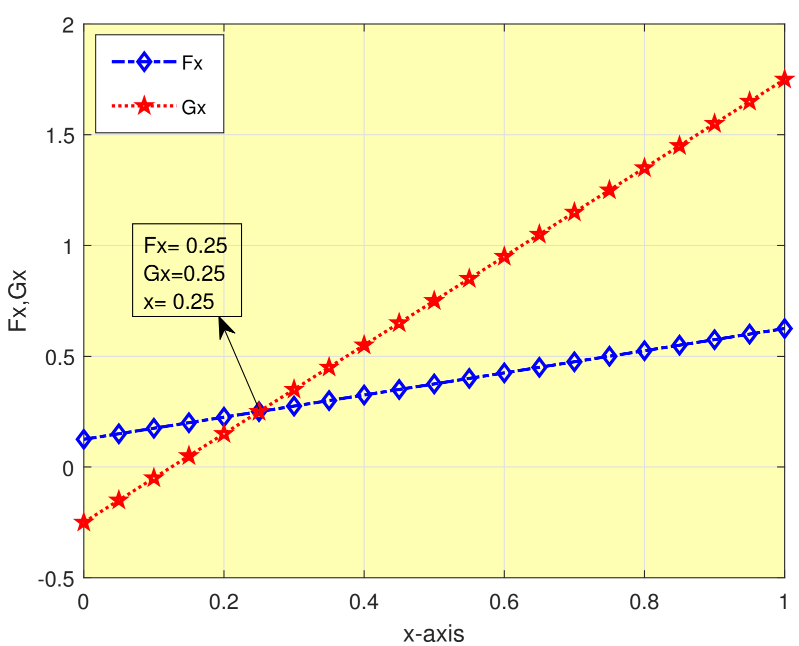

Figure 1. From the above figure, it is clear that $\mathcal{F}(\frac{1}{4}) = \mathcal{G}(\frac{1}{4}) = \frac{1}{4}$.

To check the numerical value for a and b we get the following results:

Consider $\mathsf{a} = \frac{1}{3}$ and $\mathsf{b} = \frac{1}{5}$.

TABLE 2. Performing calculations using the p-adic rational type contraction represented by the expression (2.1).

S.No	Distances	2-adic	3-adic	5-adic
1	$\mathbb{d}_{p^*}(\mathcal{F}\mathsf{a}, \mathcal{G}\mathsf{b}) = \frac{1}{7}$	1	1	1
2	$\mathbb{d}_{p^*}(\mathsf{b}, \mathcal{G}\mathsf{b}) = \frac{1}{20}$	4	1	5
3	$\mathbb{d}_{p^*}(\mathsf{a}, \mathcal{F}\mathsf{a}) = \frac{1}{24}$	8	3	1
4	$\mathbb{d}_{p^*}(\mathsf{a}, \mathcal{G}\mathsf{b}) = \frac{19}{60}$	4	3	5
5	$\mathbb{d}_{p^*}(\mathsf{a}, \mathsf{b}) = \frac{2}{15}$	$\frac{1}{2}$	3	5

Case(i): If $p = 2$, we get the results from (2.1) of Theorem 2.1, we get

$$1 \preceq 24\mu + 4\nu + \frac{1}{2}\omega.$$

Case (ii): If $p = 3$, we get the results from (2.1) of Theorem 2.1, we get

$$1 \preceq \mu + 3\nu + 3\omega.$$

Case(iii): If $p = 5$, we get the results from (2.1) of Theorem 2.1, we get

$$1 \preceq \frac{5}{3}\mu + 5\nu + 5\omega.$$

The above three cases hold the contractions (2.1) in Theorem 2.1 for some $\mu, \omega \in [0, 1)$ and $\nu = \frac{2}{5}$ such that $\mu + \nu + \omega < 1$. Therefore, $\mathcal{G}$ and $\mathcal{F}$ have a unique CFP and is equal to $\frac{1}{4} \in \mathbb{E}$. □

Example 3.3. *Define an ultrametric* $\mathbb{d}_{p^*} : \mathbb{E} \times \mathbb{E} \to \mathbb{R}^+$ *by* $\mathbb{d}_{p^*}(\mathsf{a}, \mathsf{b}) = |\mathsf{a} - \mathsf{b}|_p$, *where* $\mathbb{E} = [0, 1]$. *Let us define the self map* $\mathcal{F}$ *on* $\mathbb{E}$ *by* $\mathcal{F}\mathsf{a} = \frac{\mathsf{a}}{2} + \frac{1}{8}$, *then* $\mathcal{F}$ *has a unique fixed point* $\frac{1}{4} \in \mathbb{E}$.

4. Applications to well-posedness

The concept of well-posedness is essential when studying fixed-point problems. It enables us to analyze the existence, uniqueness, and stability of solutions in a rigorous framework. When a fixed-point problem is well-posed, it has a single solution. Jacques Hadamard, a French mathematician, introduced this concept in the early 20th century to differentiate between mathematically significant problems and poorly defined ones. For a problem to be considered well-posed, it must have three fundamental properties: existence, uniqueness, and stability. Well-posedness is particularly important in numerical analysis and scientific computing, as it ensures accurate and efficient solutions through numerical methods. The solutions obtained from well-posed problems can be relied upon to make predictions and draw meaningful conclusions. To show that a fixed-point problem is well-posed, you need to look at the properties of the mapping or function that it is based on using mathematical tools and methods like contraction mapping theorems and topological methods. The study of the well-posedness of fixed point problems has been presented by Blasi and Myjak [7]. This study was motivated by the work of several other mathematicians, and Blasi and Myjak explored additional research in this field ([5,11,12,15,17,18]). In this section, we investigate the well-posedness of CFP problems.

Definition 4.1. [1,27] Let $\mathcal{F}$ be a self-mapping of $\mathbb{E}$ and the pair $(\mathbb{E}, \mathbb{d}_p)$ be an ultrametric space. The fixed point problems of $\mathcal{F}$ is called well-posed if

- $\mathcal{F}$ has precisely one fixed point $\mathsf{a}_0 \in \mathbb{E}$,
- for every sequence $\{\mathsf{a}_n\} \subset \mathbb{E}$ and $\lim\limits_{n\to\infty} \mathbb{d}_p(\mathsf{a}_n, \mathcal{F}\mathsf{a}_n) = 0$ we have $\lim\limits_{n\to\infty} \mathbb{d}_p(\mathsf{a}_n, \mathsf{a}_0) = 0$.

Let CFP$(\mathcal{F}, \mathcal{G})$ denote the set of all common fixed points of $\mathcal{G}$ and $\mathcal{F}$.

Definition 4.2. [1,27] Let $(\mathbb{E}, \mathbb{d}_p)$ be an ultrametric space and $\mathcal{F}, \mathcal{G} : \mathbb{E} \to \mathbb{E}$. Then the CFP$(\mathcal{F}, \mathcal{G})$ is called well-posed if

- for every sequence $\mathsf{a}_n \in \mathbb{E}$ with
$$\mathsf{a}^* \in CFP(\mathcal{F}, \mathcal{G}) \text{ and } \lim_{n\to\infty} \mathbb{d}_p(\mathsf{a}_n, \mathcal{F}\mathsf{a}_n) = 0 = \lim_{n\to\infty} \mathbb{d}_p(\mathsf{a}_n, \mathcal{G}\mathsf{a}_n),$$
implies $\mathsf{a}^* = \lim\limits_{n\to\infty} \mathsf{a}_n$.
- CFP$(\mathcal{F}, \mathcal{G})$ is a singleton.

Theorem 4.3. *Let $(\mathbb{E}, \mathbb{d}_p)$ be a complete ultrametric space and $\mathcal{G}$ and $\mathcal{F}$ be self mappings on $\mathbb{E}$ such that*
$$\mathbb{d}_p(\mathcal{F}\mathsf{a}, \mathcal{G}\mathsf{b}) \preceq \mu \frac{\mathbb{d}_p(\mathsf{b}, \mathcal{G}\mathsf{b})(1 + \mathbb{d}_p(\mathsf{a}, \mathcal{F}\mathsf{a}))}{1 + \mathbb{d}_p(\mathsf{a}, \mathsf{b})} + \nu \mathbb{d}_p(\mathsf{a}, \mathcal{G}\mathsf{b}) + \omega \mathbb{d}_p(\mathsf{a}, \mathsf{b}), \tag{4.1}$$
for all $\mathsf{a}, \mathsf{b} \in \mathbb{E}$. Then the CFP of $\mathcal{G}$ and $\mathcal{F}$ is well-posed.

Proof. By Theorem 2.1, we have provided that the CFP of $\mathcal{G}$ and $\mathcal{F}$ exists and unique. Let a^* be a unique CFP of $\mathcal{G}$ and $\mathcal{F}$. Let $\{\mathsf{a}_n\}$ be a sequence in $\mathbb{E}$ and
$$\lim_{n\to\infty} \mathbb{d}_p(\mathsf{a}_n, \mathcal{F}\mathsf{a}_n) = 0 = \lim_{n\to\infty} \mathbb{d}_p(\mathsf{a}_n, \mathcal{G}\mathsf{a}_n).$$
Without loss of generality, assume that $\mathsf{a}^* \neq \mathsf{a}_n$ for any non-negative integer n. Using (4.1), we get
$$\begin{aligned}
\mathbb{d}_p(\mathsf{a}^*, \mathsf{a}_n) &\preceq \max\Big\{\mathbb{d}_p(\mathsf{a}^*, \mathcal{F}\mathsf{a}_n), \mathbb{d}_p(\mathcal{F}\mathsf{a}_n, \mathsf{a}_n)\Big\} \\
&\preceq \max\Big\{\mathbb{d}_p(\mathcal{F}\mathsf{a}^*, \mathcal{F}\mathsf{a}_n), \mathbb{d}_p(\mathcal{F}\mathsf{a}_n, \mathsf{a}_n)\Big\} \\
&\preceq \max\Big\{\mu \frac{\mathbb{d}_p(\mathsf{a}_n, \mathcal{G}\mathsf{a}_n)(1 + \mathbb{d}_p(\mathsf{a}^*, \mathcal{F}\mathsf{a}^*))}{1 + \mathbb{d}_p(\mathsf{a}^*, \mathsf{a}_n)} + \nu \mathbb{d}_p(\mathsf{a}^*, \mathcal{G}\mathsf{a}_n) + \omega \mathbb{d}_p(\mathsf{a}^*, \mathsf{a}_n), \mathbb{d}_p(\mathcal{F}\mathsf{a}_n, \mathsf{a}_n)\Big\}.
\end{aligned}$$

Taking the limit as $n \to \infty$, we have

$$\mathbb{d}_p(\mathsf{a}^*, \mathsf{a}_n) \preceq (\nu + \omega)\mathbb{d}_p(\mathsf{a}^*, \mathsf{a}_n),$$

which yields that $\lim_{n\to\infty} \mathbb{d}_p(\mathsf{a}^*, \mathsf{a}_n) = 0$, since $\nu + \omega < 1$. □

Theorem 4.4. *Let $\mathbb{E}$ be a non-empty set and Let $\mathcal{F}$ be a mapping of $\mathbb{E}$ into itself such that*

$$\mathbb{d}_p(\mathcal{F}\mathsf{a}, \mathcal{F}\mathsf{b}) \preceq \mu \frac{\mathbb{d}_p(\mathsf{b}, \mathcal{F}\mathsf{b})(1 + \mathbb{d}_p(\mathsf{a}, \mathcal{F}\mathsf{a}))}{1 + \mathbb{d}_p(\mathsf{a}, \mathsf{b})} + \nu\mathbb{d}_p(\mathsf{a}, \mathcal{F}\mathsf{b}) + \omega\mathbb{d}_p(\mathsf{a}, \mathsf{b}) \tag{4.2}$$

for all $\mathsf{a}, \mathsf{b} \in \mathbb{E}$. This indicates that the well-posedness of the fixed point problems in F holds.

Theorem 4.5. *Let $(\mathbb{E}, \mathbb{d}_p)$ be a complete ultrametric space and the mapping $\mathcal{F}$ from $\mathbb{E}$ to $CB(\mathbb{E})$ satisfying the following condition*

$$\mathbb{H}(\mathcal{F}\mathsf{a}, \mathcal{F}\mathsf{b}) \preceq \mu \frac{D(\mathsf{b}, \mathcal{F}\mathsf{b})(1 + D(\mathsf{a}, \mathcal{F}\mathsf{a}))}{1 + \mathbb{d}_p(\mathsf{a}, \mathsf{b})} + \nu D(\mathsf{a}, \mathcal{F}\mathsf{b}) + \omega\mathbb{d}_p(\mathsf{a}, \mathsf{b}) \tag{4.3}$$

for every $\mathsf{a}, \mathsf{b} \in \mathbb{E}$. Then the fixed point problems $\mathcal{F}$ is well-posed.

Remark 4.6. It is obvious to see that corollaries 2.4 and 2.7 are well posed on $\mathbb{E}$.

5. Conclusion

In the mid-19th century, mathematicians researched multi-valued mappings and found that they needed more than just single-valued ones. There have been interesting results in the fixed point theorem over metric space using rational contraction. In this study, we focus on fixed-point results in complete ultrametric space using both single-valued and multi-valued mappings with rational-type contraction. We provide an example to illustrate our main results and strengthen our proofs. Additionally, we provide an application of one of our results in the context of well-posedness.

In addition, our work indicates encouraging avenues for future research efforts. By exploring various contractive conditions that are specifically designed for the unique characteristics of ultrametric spaces, numerous new fixed-point results can be discovered. This could yield a deeper comprehension of the fundamental mathematical structures and their real-world implications, with the possibility of applications in diverse fields.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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