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ON A NEW TYPE OF TOPOLOGICAL TRANSFORMATION GROUP

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Abstract. In this paper, the notion of $\mathcal{S}$-topological transformation group is defined and studied. $\mathcal{S}$-topological transformation group results when a topological transformation group is weakened by semi totally continuity in lieu of continuity. For a map $\Psi_h : X \to X$ given by $\Psi_h(x) = \Psi(h, x)$, it is ascertained that Ψ_h and $\Psi_{h^{-1}}$ are semi totally continuous and the collection of all semi totally continuous functions of X onto itself, denoted by STC$_G(X)$ constitutes a paratopological group under composition. The extremally disconnectedness property of STC$_G(X)$ creates a Moscow topological group structure on STC$_G(X)$ and it contains an open boolean subgroup. Some basic properties of the $\mathcal{S}$-topological transformation group are explored and we show that a $\mathcal{S}$-topological transformation group implies topological transformation group but the converse is not necessarily true.

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1. Introduction and Preliminaries

The exploration of topological transformation group was sparked by Montogomery and Zippin, which includes Hilbert's fifth problem [16]. Topological transformation group is a structure moulded by interconnecting topological group and topological space with a continuous action. In 1966, William J. Gray [8] discussed a topological transformation group having an end (fixed) point. J. De Vries [6] described a universal topological transformation group in terms of the actions of any infinite locally compact group. Dimension of a topological transformation group was given by Hsu-Tung Ku and Mei-Chin Ku [11]. In 1990, David B. Ellis [7] analyzed the suspensions of topological transformation group. R. A. Alsabeh and E. Y. Abdullah [2] defined distal in topological transformation group and

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used the concept of automorphism to obtain strongly distal. In 2017, E. Y. Abdullah [1] established the properties of extensive set by semi group of topological transformation group. G. Oguz [13] discussed the soft topological transformation group in 2020. The class of Moscow spaces are introduced and their characterisations based on pesudocompactness and extremally disconnectedness were established in chapter 6 of [3]. Also the author proved that the class of locally pesudocompact topological group is a subclass of Moscow topological group. p-topological group [9], β-ideal topological group [10] are the structures which motivates us to undergone with an innovative topological transformation group structure in the view of semi totally continuous function. A new generalization of M. Stone's strong continuous function is semi totally continuous, which is closed under composition and the relation between semi totally continuous and totally continuous were established by S. S.Benchalli and Umadevi I Neeli [4].

In this paper, a novel structure called $\mathcal{S}$-topological transformation group has been induced by semi totally continuous function. $\mathcal{S}$-topological transformation group is a structure formed by concatenating topological group and topological space with a semi totally continuous action. For a $\mathcal{S}$-topological transformation group, it is proved that the set of all semi totally continuous functions (Ψ_h) on X forms a group structure and is denoted by STC$_G(X)$. Later, the map Φ forms a homomorphism between the topological group G and STC$_G(X)$. The kernel of homomorphism Φ from a topological group to STC$_G(X)$ will be known as the kernel of the action $\Psi : G \times X \to X$ and this kernel is a normal subgroup of G. Subsequently, it is proved that the quotient map $G/\text{Ker}\Psi$ is isomorphic to $\Phi(G)$. The homomorphism Φ provokes a G-action $\Psi' : G \times X \to X$ given by $\Psi'(h, x) = \Phi(h)(x)$. The existence of semi totally continuous map Ψ' leads to have a continuous map from a topological group to a subset of a mapping space and vice-versa. Furthermore, it is proved that for an abelian group G, the group structure STC$_G(X)$ forms a paratopological group. Also it is established that STC$_G(X)$ is an extremally disconnected space and proved that for a finite abelian group G, the structure STC$_G(X)$ is a Moscow topological group. Finally, it is proved that the extremally disconnected topological group STC$_G(X)$ contains an open boolean subgroup.

Definition 1.1. [14] A nonempty set G is said to be a topological group if G satisfies the following conditions,
(1) G forms a group.
(2) G is a topological space.
(3) The maps $\varphi : G \times G \to G$ and $\alpha : G \to G$ defined by $\varphi(g, h) = gh$ and $\alpha(g) = g^{-1}$ are continuous.

Definition 1.2. [5] A triplet (G, X, ζ) is called a topological transformation group in which G is a topological group, X is a topological space, and $\zeta : G \times X \to X$ is a continuous map satisfying the following conditions,

(1) $\zeta\ (e, x) = x$, for all $x \in X$, where e is the identity element of G.

(2) $\zeta(h_2, \zeta(h_1, x)) = \zeta(h_2h_1, x)$, for every $h_1, h_2 \in G$ and $x \in X$. The space X, along with a given action ζ of G, is called a G-space.

Definition 1.3. [12] Let X be a topological space and A the subset of X. If $A \subseteq cl(intA)$, then A is said to be semiopen.

Definition 1.4. [4] A semi-totally continuous function f is a map from a topological space X into a topological space Y such that the inverse image of all semi-open subset of Y is clopen in X.

Definition 1.5. [3] A group G is said to be a paratopological group if G is a topological space and the map $\varphi : G \times G \to G$ defined by $\varphi(g, h) = gh$ is continuous.

Theorem 1.6. [4] Semi-totally continuous function is closed under a composition.

Definition 1.7. [3] A space X is extremally disconnected if the closure of any open subset of X is open.

Proposition 1.8. [3] Every discrete space is extremally disconnected.

Proposition 1.9. [3] A space X which is extremally disconnected is Moscow.

Theorem 1.10. [15] Let X be an extremally disconnected Hausdorff space and let $\gamma : X \to X$ be a homeomorphism. Then the set $C = \{x \in X|\gamma(x) = x\}$ of all fixed points of γ is clopen.

Lemma 1.11. [15] Let H be a topological group. If $B(H) = \{h \in H|h^2 = 1\}$ is a neighborhood of 1, then H contains an open boolean subgroup.

2. $\mathcal{S}$-Topological Transformation Group

$\mathcal{S}$-Topological Transformation Group is defined and some of its basic properties are studied. This section aims to obtain a group structure on the set of all semi totally continuous functions on X.

Definition 2.1. A triplet (G,X,Ψ) called $\mathcal{S}$-topological transformation group when G is a topological group, X be a topological space and $\Psi : G \times X \to X$ a semi totally continuous map and it satisfies the following conditions,

(1) $\Psi\ (e, x) = x$, for every $x \in X$, where e represents the identity element of G.

(2) $\Psi(h_2, \Psi(h_1, x)) = \Psi(h_2h_1, x)$, for every $h_1, h_2 \in G$ and $x \in X$.

Remark 2.2. Any group G with discrete topology acts on itself under group operation forms a $\mathcal{S}$-topological transformation group.

Example 2.3. Let $G = \mathbb{Z}_2 = \{[0],[1]\}$ be a topological group equipped with a discrete topology. Let U be the disconnected graph equipped with a discrete topology with two components A_1 and A_2 each of them is a 2-simplex, where $A_i = (V_i, E_i)$ such that $V_i = \{v_{i1}, v_{i2}, v_{i3}\}$, i = 1,2 and $E_i = \{e_{i1}, e_{i2}, e_{i3}\}$, i = 1,2. Then the map $\Psi : \mathbb{Z}_2 \times U \to U$ defined by $\Psi([0], v_{i1}) = v_{i1}$, $\Psi([0], v_{i2}) = v_{i2}$, $\Psi([0], v_{i3}) = v_{i3}$, where $i = 1, 2.$ and $\Psi([1], v_{11}) = v_{21}$, $\Psi([1], v_{12}) = v_{22}$, $\Psi([1], v_{13}) = v_{23}$, $\Psi([1], v_{21}) = v_{11}$, $\Psi([1], v_{22}) = v_{12}$, $\Psi([1], v_{23}) = v_{13}$ forms a $\mathcal{S}$-topological transformation group.

Proposition 2.4. Every $\mathcal{S}$-topological transformation group is a topological transformation group.

Proof. Since every semitotally continuous function is continuous, Ψ is continuous. Hence (G,X,Ψ) is a topological transformation group. □

The converse of the above proposition need not be true, which is provided in the following example.

Example 2.5. Let $G = \mathbf{R}$ be the set of real numbers and it forms a topological group under addition. Let $X = \mathbf{R}$ is a topological space equipped with standard topology. Then the map $\Psi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be given by $\Psi(x, y) = x + y$ is continuous. Thus $(\mathbf{R},\mathbf{R},\Psi)$ forms a topological transformation group. Since $\Psi^{-1}(\mathbf{R})$ is not clopen, $(\mathbf{R},\mathbf{R},\Psi)$is not a $\mathcal{S}$-topological transformation group.

Theorem 2.6. For $\mathcal{S}$-topological transformation group (G, X, Ψ), $h \in G$, let a map $\Psi_h : X \to X$ be defined by $\Psi_h(x) = \Psi(h, x)$. Then Ψ_h and its inverse are semi totally continuous.

Proof. Given $h \in G$, let $i_h : X \to G \times X$ be the map defined by $i_h(x) = (h, x)$. According to the definition of product topology, i_h is continuous. Let $\Psi_h = \Psi \circ i_h$ and V be a semiopen set in X, then $(\Psi \circ i_h)^{-1}(\mathrm{V}) = (i_h)^{-1}(\Psi^{-1}(\mathrm{V}))$ and $\Psi^{-1}(\mathrm{V})$ is clopen, as Ψ is semi totally continuous. Since i_h is continuous, $(i_h)^{-1}(\Psi^{-1}(\mathrm{V}))$ is clopen and thus Ψ_h is semi totally continuous. Let $\Psi_h : X \to X$ be given by $\Psi_h(x) = \Psi(h, x) = hx$. Now we show that Ψ_h is bijective.

(i) The Injectivity condition : Let $x, y \in X$,

$$\begin{aligned} \Psi_h(x) &= \Psi_h(y) \\ h^{-1}(hx) &= h^{-1}(hy) \\ (h^{-1}h)x &= (h^{-1}h)y \\ x &= y. \end{aligned}$$

Therefore Ψ_h is injective.

(ii) The Surjectivity condition : Let $x \in X$,

$$\begin{aligned} x &= ex \\ &= (hh^{-1})x \\ &= h(h^{-1}x) \\ &= hy \\ &= \Psi_h(y) \end{aligned}$$

Therefore, Ψ_h is surjective.

$\implies \Psi_h$ is bijective and inverse exists, $(\Psi_h)^{-1} = \Psi_{h^{-1}}$.

Let V be a semiopen set in X, then $(\Psi \circ (i_{h^{-1}})^{-1}(V)) = (i_{h^{-1}})^{-1}(\Psi^{-1}(V))$ and $\Psi^{-1}(V)$ is clopen, as Ψ is semi totally continuous. Since i_h is continuous, $(i_{h^{-1}})^{-1}(\Psi^{-1}(\mathrm{V}))$ is clopen and $(\Psi_h)^{-1}$ is semi totally continuous. Hence Ψ_h and $(\Psi_h)^{-1}$ are semi totally continuous. □

Since, the semi totally continuous functions are closed under composition, the closure holds. Composition is always associative. Since $\Psi_h \circ \Psi_{h^{-1}} = id$, inverse and identity exists. This implies, $\mathrm{STC}_G(X)$ forms a group under composition.

Proposition 2.7. Let $\Phi : G \to \mathrm{STC}_G(X)$ defined by $h \mapsto \Psi_h$. Then G is homomorphic to $\mathrm{STC}_G(X)$.

Proof. For any $h_1, h_2 \in G$, we have

$$\begin{aligned} \Phi(h_1h_2) &= \Phi(h_1h_2)(x) \\ &= \Psi_{h_1h_2}(x) \\ &= \Psi \circ i_{h_1h_2}(x) \\ &= \Psi(h_1, \Psi \circ i_{h_2})(x) \\ &= (\Psi_{h_1} \circ \Psi_{h_2})(x) \\ &= \Phi(h_1) \circ \Phi(h_2). \end{aligned}$$

Therefore G is homomorphic to $\mathrm{STC}_G(X)$. □

Remark 2.8.

(i) The kernel of homomorphism Φ is the kernel of action Ψ.

(ii) Ker Ψ is a normal subgroup of G.

(iii) If $\Phi : G \to \mathrm{STC}_G(X)$ is a group homomorphism with $P = Ker\Phi$, then G/P is isomorphic to $\Phi(G)$.

2.1. **STC$_G(X)$ - a Paratopological group.** This section is focused to obtain a paratopological and Moscow topological group structure on STC$_G(X)$.

Proposition 2.9. Given a homomorphism $\Phi : G \to$ STC$_G(X)$ defined by $\Phi(h)(x) = \Psi^{'}(h, x)$ where $\Psi^{'} : G \times X \to X$. Then $\Psi^{'}$ is a G-action.

Proof. Straightforward. □

Corollary 2.10. $\Psi^{'}$ is a G-action if and only if Φ is a homomorphism of groups.

Proof. The proof follows from the Remark 2.8 (i) and Proposition 2.9. □

For a topological spaces X and Y, the set of semi totally continuous map $f : X \to Y$ is denoted by the Map (X, Y). The Map(X, Y) is called a mapping space. Now, for a subset $F \subset$ Map(X, X), let $W(C, S) = \{\Psi \in F/\Psi(C) \subset S\}$ where C and S are the given clopen subset and a semiopen subset of X. The clopen-semiopen topology on F has a sub-basis, the sets $W(C, S)$ for $C \subset X$ clopen and $S \subset X$ semiopen. It is possible to have a mapping space with the clopen-semiopen topology.

Theorem 2.11. Let F be a subset of Map (X, X) where X is a compact topological space. For a map Φ from a topological group G into F, we define a map $\Psi' : G \times X \to X$ by $\Psi'(h, x) = \Phi(h)(x)$. Then Ψ' is semi totally continuous if and only if Φ is continuous.

Proof. Given a semi totally continuous function Ψ', for every $(h, x) \in G \times X$ and all semi open set $S \in X$ with $\Phi(h, x) \in S$, we have a clopen set C in $G \times X$ such that $(h, x) \in C$ and $\Phi(C) \subset S$. For each $h \in G$ and for all open neighborhood S of $\Phi(h)$, we get an open neighborhood C of h such that $\Phi(C) \subset S$. For any clopen set C of X and any semiopen set S of X, we have to prove that $\Phi^{-1}(W(C, S))$ is open in G. For this, it is enough to show that for any point h of $\Phi^{-1}(W(C, S))$ there is a open neighborhood U of G with $\Phi(U) \subset W(C, S)$. Fix an arbitrary point h_0 of $\Phi^{-1}W(C, S)$. For $x \in C$, we have $\Psi'(h_0, x) = \Phi(h_0)(c_0) \in S$. Since Ψ' is semi totally continuous, we have a clopen set U_x containing h_0 and a clopen set V_x containing x such that $\Psi'(U_x \times V_x) \subset S$. Since h_0 is fixed and x may vary, we write U_x. If we consider this for each $x \in C$, we have trivially $\bigcup\limits_{x \in C} V_x \supset C$. There exists a finite set $\{x_1, ..., x_n\} \subset C$ with $\bigcup\limits_{r=1}^{n} V_{x_i} \supset C$ as X is compact. Now we put, $U = \bigcap\limits_{r=1}^{n} U_{x_r}$, $V = \bigcup\limits_{r=1}^{n} V_{x_r}$. Then U is an open neighborhood of h_0 and V is an open set of x containing C. In the following, we show that $\Psi'(U \times V) \subset S$. For any point $(h', x') \in U \times V$, choose r with $x' \in V_{x_r}$. Since $h' \in U \subset U_{x_r}$, we have $\Psi'(h', x') \in \Psi'(U_{x_r} \times V_{x_r}) \subset S$. Hence $\Psi'(U \times C) \subset \Psi'(U \times V) \subset S$ and $\Phi(U) \subset W(C, S)$. Therefore, Φ is continuous.

Conversely, assume that Φ is continuous. On the contrary, assume that Ψ' is not semi totally continuous. For a clopen set C, there exists no semiopen set containing $\Psi'(C)$. That is, there is no semiopen set containing $\Phi(g)(x)$, for every $(g,x) \in C$, which is a contradiction, as Φ is continuous, $\Phi(g)(x)$ contained in a semiopen set. Therefore Ψ' is semi totally continuous. □

Example 2.12. Let $G = \mathbb{Z}_3 = \{[0],[1],[2]\}$ be a topological group under addition modulo 3 with a topology $\sigma = \{\emptyset, \mathbb{Z}_3, \{[0]\}, \{[1]\}, \{[0],[1]\}, \{[0],[2]\}\}$. Let $X = \{0,1\}$ equipped with sierpinski topology and $\gamma = \{\emptyset, X, \{0\}, \{1\}\}$ denotes the semiopen collection. Now, let $F = \{C_0, C_1\}$ such that $C_0(x) = 0, C_1(x) = 1, \forall\, x \in X$ and $\Phi([0]) = C_0 = W(x,0), \Phi([1]) = C_1 = W(x,1), \Phi([2]) = C_0 = W(x,0)$. Since $\Psi(g,x) = \Phi(g)(x)$, we have $\Psi(0,0) = 0, \Psi(0,1) = 0, \Psi(1,0) = 1, \Psi(1,1) = 1, \Psi(2,0) = 0, \Psi(2,1) = 0$. Therefore Ψ is semi totally continuous if and only if Φ is continuous.

Theorem 2.13. $\mathrm{STC}_G(X)$ is a paratopological group.

Proof. Let the map $\alpha : \mathrm{STC}_G(X) \times \mathrm{STC}_G(X) \to \mathrm{STC}_G(X)$ be defined by $\alpha(\Psi_g, \Psi_h) = \Psi_g \circ \Psi_h$. Taking any $W(C,S)$ containing $\Psi_g \circ \Psi_h$, we have $\Psi_g \circ \Psi_h(C) \subset S$, where C is a clopen set and S is a semiopen set. Now there exists $W(C_1,S_1)$ and $W(C_2,S_2)$ containing Ψ_g and Ψ_h such that $\alpha\big(W(C_1,S_1) \times W(C_2,S_2)\big) \subset W(C,S)$. Since $\Psi_g \circ \Psi_h(C) \subset S$, $\Psi_h(C) \subset \Psi_g^{-1}(S)$. Since G is abelian, $\Psi_h \circ \Psi_g(C) \subset S$. Therefore $\Psi_g(C) \subset \Psi_h^{-1}(S)$. Thus there exists $W\big(C, \Psi_h^{-1}(S)\big)$ and $W\big(C, \Psi_g^{-1}(S)\big)$ containing Ψ_g and Ψ_h such that $\alpha\big(W\big(C,\Psi_h^{-1}(S)\big) \times W\big(C,\Psi_g^{-1}(S)\big)\big) \subset W(C,S)$. Hence $\mathrm{STC}_G(X)$ is a paratopological group. □

Proposition 2.14. For every subgroup H of G, there exists a subgroup $\mathrm{STC}_H(X)$ in $\mathrm{STC}_G(X)$ forms a paratopological group under a subspace topology.

Proof. Let H be a subgroup of G. For $h \in H$, we have $\Psi_h : X \to X$ be defined by $\Psi_h(x) = \Psi(h,x)$, a semi totally continuous function and the collection of all semi totally continuous function is denoted by $\mathrm{STC}_H(X)$. Now, let $\Psi_h, (\Psi_g)^{-1} \in \mathrm{STC}_H(X)$, then $\Psi_h \circ \Psi_{g^{-1}} \in \mathrm{STC}_H(X)$ and this implies $\mathrm{STC}_H(X)$ is a subgroup of $\mathrm{STC}_G(X)$. By subspace topology, $\mathrm{STC}_H(X)$ forms a paratopological group. □

Proposition 2.15. For a finite topological group G of order n and $h_1 \in G$, $\Psi_{h_1} \in \mathrm{STC}_G(X)$. Then there exist atleast one clopen set C and a semiopen set S such that $W(C,S) = \{\Psi_{h_1}\}$.

Proof. Since Ψ_{h_1} is semi totally continuous, for $h_1 \in G$. Then there exists a clopen set C_1 and a semiopen set S_1 such that $\Psi_{h_1} \in W(C_1,S_1)$. Assume that $W(C_1,S_1) = \{\Psi_{h_1}, \Psi_{h_2}, ..., \Psi_{h_i}\}$. Suppose $\Psi_{h_1} \neq \Psi_{h_2}$, then there exists a clopen set C_2 and a semiopen set S_2 such that $\Psi_{h_1} \in W(C_2,S_2)$, $\Psi_{h_2} \notin W(C_2,S_2)$ and this implies $\Psi_{h_1} \in W(C_1,S_1) \cap W(C_2,S_2)$. Next, if $\Psi_{h_3} \in W(C_1,S_1) \cap W(C_2,S_2)$ and $\Psi_{h_1} \neq \Psi_{h_3}$, then there exists a clopen set C_3 and a semiopen set S_3 such that $\Psi_{h_1} \in W(C_3,S_3)$, $\Psi_{h_3} \notin W(C_3,S_3)$ and so, $\Psi_{h_1} \in W(C_1,S_1) \cap W(C_2,S_2) \cap W(C_3,S_3)$. Proceeding in the same way, we have $W(C_1,S_1) \cap W(C_2,S_2)$ $\cap\, W(C_3,S_3) \cap ... \cap W(C_i,S_i) = \{\Psi_{h_1}\}$. □

Theorem 2.16. $\mathrm{STC}_G(X)$ is a discrete space.

Proof. The proof is obvious by the Proposition 2.15. □

Corollary 2.17. $\text{STC}_G(X)$ is extremally disconnected.

Proof. Since every discrete space is extremally disconnected, $\text{STC}_G(X)$ is extremally disconnected. □

Corollary 2.18. $\text{STC}_G(X)$ is a Moscow space.

Proof. Proof follows from the Corollary 2.17. □

Corollary 2.19. $\text{STC}_G(X)$ is a topological group.

Proof. Since all discrete space are locally compact and every locally compact paratopological group is a topological group. Hence $\text{STC}_G(X)$ is a topological group. □

Corollary 2.20. $\text{STC}_G(X)$ is a Moscow topological group.

Proof. By the Corollary 2.18 and Corollary 2.19, the proof follows. □

Proposition 2.21. Let Γ be a bijective function from a group G to a group $G^{'}$ defined by $\Gamma(h) = h^{'}$ be a continuous homomorphism and inverse is also continuous. Then $\Phi^{'} : \text{STC}_G(X) \to \text{STC}_{G'}(X)$ defined by $\Phi^{'}(\Psi_h) = \Psi_{\Gamma(h)}$ is also a bijective continuous homomorphism and inverse continuous.

Proof.
(i) The Injectivity condition : Let Ψ_h and $\Psi_k \in \text{STC}_G(X)$,

$$\begin{aligned}
\Phi^{'}(\Psi_h) &= \Phi^{'}(\Psi_k) \\
\Psi_{\Gamma(h)} &= \Psi_{\Gamma(k)} \\
\Psi(\Gamma(h), x) &= \Psi(\Gamma(k), x) \\
\Gamma(h)x &= \Gamma(k)x \\
\Psi_h &= \Psi_k.
\end{aligned}$$

Therefore $\Phi^{'}$ is injective.

(ii) The Surjectivity condition : For every $\Psi_{h_i'} \in \text{STC}_{G'}(X)$, there exists $\Psi_{h_i} \in \text{STC}_G(X)$ such that $\Phi^{'}(\Psi_{h_i}) = \Psi_{\Gamma(h_i)} = \Psi_{h_i'}$. Therefore, $\Phi^{'}$ is surjective.

(iii) Homomorphism: Let Ψ_h and $\Psi_k \in \mathrm{STC}_G(X)$,

$$\begin{aligned}
\Phi^{'}(\Psi_h \circ \Psi_k) &= \Phi^{'}(\Psi_h \circ \Psi_k)(x) \\
&= \Psi_{\Gamma(hk)}(x) \\
&= \Psi_{\Gamma(h)\Gamma(k)}(x) \\
&= \Psi_{h^{'}k^{'}}(x) \\
&= \Phi^{'}(\Psi_h) \circ \Phi^{'}(\Psi_k).
\end{aligned}$$

Therefore $\Phi^{'}$ is a homomorphism.

Now, the continuity of $\Phi^{'}$ follows from the commutative diagram.

$$\begin{tikzcd}
G \arrow[r, "\Phi"] \arrow[d, "\Gamma"'] \arrow[dr, "\Gamma \circ \gamma"] & \mathrm{STC}_G(X) \arrow[d, "\Phi^{'}"] \\
G^{'} \arrow[r, "\gamma"'] & \mathrm{STC}_{G^{'}}(X)
\end{tikzcd}$$

Since Γ and γ are continuous homomorphism, so is $\Gamma \circ \gamma$ and hence $\Phi^{'}$ is continuous. The inverse continuous of $\Phi^{'}$ can be verified by the following commutative diagram.

$$\begin{tikzcd}
G^{'} \arrow[r, "\Omega"] \arrow[d, "\zeta"'] \arrow[dr, "\zeta \circ \delta"] & \mathrm{STC}_{G^{'}}(X) \arrow[d, "\eta"] \\
G \arrow[r, "\delta"'] & \mathrm{STC}_G(X)
\end{tikzcd}$$

Since ζ and δ are continuous homomorphism, so is $\zeta \circ \delta$ and hence η is continuous. □

Proposition 2.22. Let Γ be a function from a topological group G to a topological group $G^{'}$ defined by $\Gamma(h) = h^{'}$ be a bijective continuous homomorphism and inverse continuous. Then $\Phi^{'} : \mathrm{STC}_G(X) \to \mathrm{STC}_{G^{'}}(X)$ defined by $\Phi^{'}(\Psi_h) = \Psi_{\Gamma(h)}$ gives an isomorphism of topological groups.

Proof. By the Proposition 2.21 and Corollary 2.19, the proof follows. □

Proposition 2.23. Let $\varsigma : \mathrm{STC}_G(X) \to \mathrm{STC}_G(X)$ defined by $\varsigma(\Psi_h) = \Psi_{h^{-1}}$ is a homeomorphism.

Proof. Since Ψ_h is homeomorphism, so is ς. □

Theorem 2.24. $\mathrm{STC}_G(X)$ contains an open boolean subgroup.

Proof. By Theorem 2.16, Corollary 2.17 and Corollary 2.19, $\mathrm{STC}_G(X)$ is an extremally disconnected topological group and discrete space. Since every discrete space is hausdorff, $\mathrm{STC}_G(X)$ is hausdorff and by Proposition 2.23, any map from $\mathrm{STC}_G(X)$ to $\mathrm{STC}_G(X)$ given by $\Psi_h \mapsto \Psi_{h^{-1}}$ is a homeomorphism. Then $H = \{\Psi_h | \varsigma(\Psi_h) = \Psi_h\}$ is a clopen subset of $\mathrm{STC}_G(X)$ and hence H is an open boolean subgroup of $\mathrm{STC}_G(X)$. □

Conclusion

We defined a new structure of topological transformation group called $\mathcal{S}$-topological transformation group and the relation between topological transformation group and $\mathcal{S}$-topological transformation group have been analyzed. Basic algebraic and topological properties of $\mathcal{S}$-topological transformation group have been discussed. Also, it is established that the set of all semi totally continuous functions involved in $\mathcal{S}$-topological transformation group acts as a group under composition as well as a paratopological group and a Moscow topological group. By Proposition 2.22, we can conclude that the isomorphism of groups defines an equivalence relation on the set of all $\mathrm{STC}_G(X)$. Finally, it is shown that $\mathrm{STC}_G(X)$ has an open boolean subgroup. In a future work, we explore some other properties of $\mathcal{S}$-topological transformation groups.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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