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NONLINEAR ELLIPTIC EQUATION WITH VARIABLE EXPONENTS AND INTEGRABLE DATA

MOHAMMED EL FATRY*, MOUNIR MEKKOUR, YOUSSEF AKDIM

Department of Mathematics, Sidi Mohamed Ben Abdellah University, Morocco

*Corresponding author: mohammed.elfatry36@gmail.com

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Abstract. In this article, we prove the existence of a renormalised solution to the problem of the nonlinear elliptic equation:

$$\begin{cases} -\operatorname{div}(a(x,u,Du)) = f \ \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Where $a(x,u,Du)$ is below-up for a finite value m of the unknown u and the data $f \in L^1(\Omega)$.

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1. Introduction

Recently, interest in variable exponent Sobolev spaces has grown rapidly because they have many physical applications such image processing (underline the borders, eliminate the noise) and electro-rheological fluids, for more detail, we invite the reader to see([17,18]).

Our objective in this paper is to establish that a renormalized solution exists for a class of elliptic-type problems with the following form.

$$\begin{cases} -\operatorname{div}(a(x,u,Du)) = f \ \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N (N \geq 2)$, and the data f in $L^1(\Omega)$. The operator $-\operatorname{div}(a(x,u,Du))$ is a Leray-Lions operator defined on the variable exponent Sobolev space and below-up when $u \to m^-$, with m is strictly positive real number.

When we look at the problem (1.1), we come up against two kinds of difficulties. First the assumptions $f \in L^1(\Omega)$. To overcome this difficulty, we are using the framework of renormalised solutions. This concept was presented by DiPerna and Lions [8], to study of Boltzmann equation see also Lions [10] for a few applications to fluid mechanics models. We invite the reader to see [6,11,12] for elliptic problems

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and to [3,4] for parabolic equations. The second dificulty due to the function $a(x,s,\xi)$ below up when $s \to m^-$, this makes the task of giving meaning to this function in the set $\{x \in \Omega;\ u(x) = m\}$ difficult.

The problem (1.1), has been investigated in a few cases by several authors. In the case where the operator a is replaced by a symmetric matrix, the existence of renormalized solutions has been proved for elliptic problem in (see [4]). Next, If the flux defined on the weighted sobolev spaces , the existence of renormalized solutions have been proved in [14].

This paper is broken down as follows: section 2, we present the variable exponent Sobolev space and some of its properties. Section 3, we make assumptions and provide the main result. Section 3, we provide the main result and we etablish the proof of main result.

2. Preliminaries

Let $p : \bar{\Omega} \to [1, \infty]$ be a measurable function, where Ω is a domain of $\mathbb{R}^n$, satisfy the log-Hölder continuity condition

$$|p(x) - p(y)| \leq \frac{A}{\log \frac{1}{|x-y|}}, \quad \text{for all } x, y \in \Omega \text{ with } |x-y| < r, \tag{2.1}$$

where $A > 0$ and $0 < r < 1$, and $1 < p_- < p_+ < N$, where

$$p_- = \min_{x\in\bar{\Omega}} p(x) \ \text{ and } \ p_+ = \max_{x\in\bar{\Omega}} p(x)$$

We define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \{u : \Omega \to R :\ u \text{ is measurable in } \Omega \text{ and } \int_\Omega |u(x)|^{p(x)} dx < \infty\ \}.$$

The space $L^{p(x)}(\Omega)$ equipped with the Luxemburg-type norm

$$\|u\|_{p(x)} = \inf\left\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1\right\}$$

becomes a Banach space [15]. The relation between the modular $\int_\Omega |f|^{p(x)}\, dx$ and the norm follows from

$$\min(\|f\|^{p^-}_{p(x)}, \|f\|^{p^+}_{p(x)}) \leq \int_\Omega |f|^{p(x)}\, dx \leq \max(\|f\|^{p^-}_{p(x)}, \|f\|^{p^+}_{p(x)}).$$

For all $f \in L^{p(x)}(\Omega)$, $g \in L^{p'(x)}(\Omega)$ with

$$p(x) \in (1, \infty), \quad p'(x) = \frac{p(x)}{p(x)-1},$$

the generalized Hölder inequality holds,

$$\int_\Omega |f\, g|\, dx \leq \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)\|f\|_{p(x)}\|g\|_{p'(x)} \leq 2\|f\|_{p(x)}\|g\|_{p'(x)}\,.$$

The variable exponent Sobolev space is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : \nabla u \text{ exists and } |\nabla u| \in L^{p(x)}(\Omega)\},$$

with respect to the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$

The space $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$. In addition the spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces. For $u \in W_0^{1,p(x)}(\Omega)$, we can define an equivalent norm

$$\|u\|_{1,p(x)} = \|\nabla u\|_{p(x)}.$$

Lemma 2.1. *Let $g \in L^{p(x)}(\Omega)$ and $g_n \in L^{p(x)}(\Omega)$ with $\|g_n\|_{p(x)} < C$. If*

$$g_n \to g \text{ a.e. in } \Omega,$$

then

$$g_n \rightharpoonup g \text{ in } L^{p(x)}(\Omega).$$

For more detail, we invite the reader to see [13,15,16].

3. Assumptions on the Data and Notations

Let Ω be a bounded open set of $\mathbb{R}^N (N \geq 2)$, and

$$a : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N \tag{3.1}$$

is a Carathéodory function, such that there exists a positive function $b \in C^0((-\infty, m))$ which satisfies

$$b^{p(x)-1}(s) \geq \alpha > 0 \quad \forall s \in (-\infty, m)\,;\ \lim_{s \longrightarrow m^-} b(s) = +\infty, \tag{3.2}$$

$$\int_0^m b(s)ds < +\infty, \tag{3.3}$$

and

$$a(x,s,\xi).\xi \geq b(s)^{p(x)-1}|\xi|^{p(x)},\ a(x,s,0) = 0. \tag{3.4}$$

$$|a(x,s,\xi)| \leq \beta\left[L(x) + b(s)^{p(x)-1}|\xi|^{p(x)-1}\right], \tag{3.5}$$

$$\left[a(x,s,\xi) - a(x,s,\xi')\right]\left[\xi - \xi'\right] \geq 0,$$

where L is a non negative function in $L^{p'(x)}(\Omega)$, and $\beta > 0$, for almost every $x \in \Omega$, for every $s \in \Omega$ and $\xi \in \mathbb{R}^N$. Finally the data

$$f \text{ is an element of } L^1(\Omega). \tag{3.6}$$

Definition 3.1. *A mesuarble function u defined on Ω is a renormalized solution of problem (1.1) if*

$$T_k(u) \in W_0^{1,p(x)}(\Omega), \tag{3.7}$$

$$u \le m \text{ a.e. in } \Omega, \tag{3.8}$$

$$a(x, T_m^k(u), DT_m^k(u))\chi_{\{u<m\}} \in (L^{p'(x)}(\Omega))^N, \tag{3.9}$$

$$\lim_{s \to +\infty} \int_{\{-s-1 \le u(x) \le -s\}} a(x, u, Du)Du dx = 0, \tag{3.10}$$

$$\lim_{\delta \to 0} \frac{1}{\delta} \int_{\{m-2\delta \le u(x) \le m-\delta\}} a(x, u, Du)Du dx = \int_{\{u(x)=m\}} f dx, \tag{3.11}$$

and, for every function S in $W^{1,\infty}(\mathbb{R})$ such that $supp(S)$ is compact and $S(m) = 0$, u satisfies

$$\int_\Omega a(x, u, Du)D(S(u)\varphi)dx = \int_\Omega f S(u)\varphi dx, \tag{3.12}$$

for every $\varphi \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$.

Remark 3.2. Conditions (3.7) and (3.9) to show that all term in (3.12) are well defined. The assumption (3.11), has been establissed in [5].

The following notations will be used throughout the paper. For any $k > 0$ and $\varepsilon > 0$, we define

$$T_\varepsilon^k(s) = \begin{cases} -k, & \text{if } s \le -k, \\ r, & \text{if } -k \le r \le \varepsilon, \\ \varepsilon, & \text{if } \ r \ge \varepsilon. \end{cases}$$

For $l \ge 1$ fixed, we define

$$\theta_l(s) = T_1(r - T_l(r))$$

and $h_l(s) = 1 - |\theta_l(s)|$,for all $s \in \mathbb{R}$.

4. The existence theorem

Theorem 4.1. *Assume that (3.1)-(3.6) hold true there existe a renormalized solution u of problem (1.1).*

Proof of theorem

4.1. **Step 1: Approximate problem and a priori estimates.**

Let us introduce the following regularisation of the data: for a fixed $n \ge 1$. let

$$b_n(r) = b(T_{m-1/n}^n(r)),$$

$$a^n(x,s,\xi) = a(x, T^n_{m-1/n}(s), \xi),$$

$$f_n \in L^\infty(\Omega), \text{ such that } f_n \to f \text{ strongly in } L^1(\Omega), \text{ as } n \text{ tends to } +\infty, \tag{4.1}$$

Let us now consider the following regularized problem

$$\begin{cases} -\operatorname{div}\left(a^n(x,u^n,Du^n)\right) = f_n \text{ in } \Omega, \\ u^n = 0 \text{ on } \partial\Omega, \end{cases} \tag{4.2}$$

As a result, demonstrating the existence of a weak solution $u^n \in W_0^{1,p(.)}(\Omega)$ of (4.2) is an easy task (see [9]).

We choose $T_k(u^n)$ as a test function in (4.2), we get

$$\int_\Omega a^n(x,u^n,Du^n)DT_k(u^n)dx = \int_\Omega f_n T_k(u^n)dx.$$

Since (3.2) and (3.4), we have

$$\int_\Omega |DT_k(u^n)|^{p(x)}\,dx < C, \tag{4.3}$$

where C does not depend on n and k.

By a classical argument (see e.g [7]), for a subsequence still indexed by n, from (4.3), we have

$$u^n \to u \; a.e. \text{ in } \Omega, \tag{4.4}$$

and

$$T_k\left(u^n\right) \rightharpoonup T_k\left(u\right) \text{ weakly in } W_0^{1,p(x)}(\Omega). \tag{4.5}$$

Taking now $Z^n = \int_0^{T^k_m(u^n)} b_n(s)ds$ as a test function in (4.2), we give

$$\int_\Omega a^n(x,u^n,Du^n)DZ^n(u^n)dx = \int_\Omega f_n Z^n(u^n)dx. \tag{4.6}$$

By the assumptions (3.4) of a^n and (3.2), we deduce that

$$\int_\Omega |DZ^n(u^n)|^{p(x)}\,dx < C. \tag{4.7}$$

For every $k > 0$, we write

$$\left|a^n(x, T^k_m(u^n), DT^k_m(u^n))\right| \le \beta\left[L(x) + |DZ^n(u^n)|^{p(x)-1}\right]. \tag{4.8}$$

Putting together (4.7) and (4.8), we deduce that

$$a^n(x, T^k_m(u^n), DT^k_m(u^n) \text{ is bounded in } L^{p'(x)}(\Omega),$$

then there exists a function $\varphi_k \in L^{p'(x)}(\Omega)$ such that

$$a^n(x, T^k_m(u^n), DT^k_m(u^n) \rightharpoonup \varphi_k \text{ weakly in } L^{p'(x)}(\Omega). \tag{4.9}$$

Using $T^+_{2m}(u^n) - T^+_m(u^n)$ as a test function in (4.2) leads to

$$\int_\Omega a^n(x, u^n, Du^n).D\left(T^+_{2m}(u^n) - T^+_m(u^n)\right) dx = \int_\Omega f_n \left(T^+_{2m}(u^n) - T^+_m(u^n)\right) dx,$$

thanks to (3.4), we have

$$b(m - \frac{1}{n})^{p(x)-1} \int_\Omega \left|D\left(T^+_{2m}(u^n) - T^+_m(u^n)\right)\right|^{p(x)} dx \leq C, \tag{4.10}$$

we can pass to the limit in (4.10) as n tends to $+\infty$, to deduce that

$$\begin{aligned} T^+_{2m}(u) - T^+_m(u) &= 0 \qquad a.e. \text{ in } Q, \\ u &\leq m \qquad a.e. \text{ in } Q. \end{aligned} \tag{4.11}$$

Taking now $T_k(v^n)$ as a test function in (4.2), with $v^n = \int_0^{u^n)} b_n(s)ds$. We obtain

$$\int_\Omega a^n(x, u^n, Du^n) DT_k(v^n) dx \leq C.$$

By (3.2) and (3.4), we have

$$\int_\Omega |DT_k(v^n)|^{p(x)} dx \leq C. \tag{4.12}$$

By a classical argument (see e.g [7]), for a subsequence still indexed by n, from (4.12), we have

$$v^n \to v \text{ a.e. in } \Omega, \tag{4.13}$$

and

$$T_k(v^n) \rightharpoonup T_k(v) \text{ weakly in } W_0^{1,p(x)}(\Omega).$$

We choose $\theta_k(v^n)$ as a test function in (4.2), it gives

$$\lim_{n\to 0} \int_\Omega a^n(x, u^n, Du^n) D\left(\theta_k(v^n)\right) dx = \int_\Omega f_n \theta_k(v) dx.$$

Since $f \in L^1(\Omega)$, Lebesgue's convergence theorem, we have

$$\lim_{k\to 0} \lim_{n\to 0} \int_{\{n \leq |v^n| \leq n+1\}} a^n(x, u^n, Du^n).Dv^n dx = 0. \tag{4.14}$$

4.1.1. *Step 2: the monotonicity estimate and the weak limit.*

Lemma 4.2. *For any* $k \geq 0$, *we have*

$$\begin{aligned} \lim_{n\to 0} \int_\Omega &\left[\frac{a^n(x, T^k_m(u^n), DT^k_m(u^n))}{b_n(u^n)^{p(x)-1}} - \frac{a^n(x, T^k_m(u^n), DT^k_m(u))}{b_n(u^n)^{p(x)-1}}\right] \\ &\times \left(DT^k_m(u^n) - DT^k_m(u)\right) dx = 0. \end{aligned} \tag{4.15}$$

Proof. Let $k \geq 0$ be fixed. Equality (4.15) is split into

$$\int_\Omega \left[\frac{a^n(x, T_m^k(u^n), DT_m^k(u^n))}{b_n(u^n)^{p(x)-1}} - \frac{a^n(x, T_m^k(u^n), DT_m^k(u))}{b_n(u^n)^{p(x)-1}}\right] \times \left(DT_m^k(u^n) - DT_m^k(u)\right) dx = I_1^n + I_2^n + I_3^n, \tag{4.16}$$

where

$$I_1^n = \int_\Omega \frac{a^n(x, T_m^k(u^n), DT_m^k(u^n))}{b_n(u^n)^{p(x)-1}} DT_m^k(u^n) dx,$$

$$I_2^n = -\int_\Omega \frac{a^n(x, T_m^k(u^n), DT_m^k(u^n))}{b_n(u^n)^{p(x)-1}} DT_m^k(u) dx,$$

$$I_3^n = -\int_\Omega \frac{a^n(x, T_m^k(u^n), DT_m^k(u))}{b_n(u^n)^{p(x)-1}} \left(DT_m^k(u^n) - DT_m^k(u)\right) dx.$$

In what follows we pass to the limit as n tends to $+\infty$ in (4.16).

Limit of I_1^n

We choose $h_l(v^n)\int_0^{T_m^k(u)} \frac{1}{b(s)^{p(x)-1}} ds$ as a test function in (4.2) to obtain

$$\int_\Omega h_l(v^n) a^n(x, u^n, Du^n) \frac{DT_m^k(u)}{b(u)^{p(x)-1}} dx + \int_\Omega a^n(x, u^n, Du^n) Dh_l(v^n). \left(\int_0^{T_m^k(u)} \frac{1}{b(s)^{p(x)-1}} ds\right) dx = \int_\Omega f_n h_l(v^n) \int_0^{T_m^k(u)} \frac{1}{b(s)^{p(x)-1}} ds dx. \tag{4.17}$$

Since h_k have a compact support, we have for a large n

$$|a^n(x, u^n, Du^n) h_l(v^n)| \leq \beta \left[L(x) + |DT_{l+1}(v^n)|^{p(x)-1}\right]. \tag{4.18}$$

From (4.18) and (4.12), we deduce that

$$a^n(x, T_{(l+1)/\alpha}(u^n), DT_{(l+1)/\alpha}(u^n)) h_l(v^n) \text{ is bounded in } L^{p'(x)}(\Omega), \tag{4.19}$$

for every large n.

We first use the estimate (4.19) to extract another subsequence, still indexed by l, such that

$$a^n(x, T_{(l+1)/\alpha}(u^n), DT_{(l+1)/\alpha}(u^n)) h_l(v^n) \rightharpoonup \psi_l \text{ weakly in } L^{p'(x)}(\Omega), \tag{4.20}$$

as n tends to $+\infty$.

Now for $\max(k, m) \leq l/\alpha$, we have

$$a^n(x, T_{(l+1)/\alpha}(u^n), DT_{(l+1)/\alpha}(u^n)) h_l(v^n)\, \chi_{\{-k<u^n<m\}} = h_l(v^n) a^n(x, T_m^k(u^n), DT_m^k(u^n))\, \chi_{\{-k<u^n<m\}}$$

$a.e.$ in Ω. Using the covergences (4.20), (4.13), and (4.20) and letting n tends to $+\infty$, we have for

$$\psi_l DT_m^k(u) = h_l(v)\varphi_k DT_m^k(u) \ a.e. \text{in } \Omega. \tag{4.21}$$

Letting now n tends to $+\infty$ and l tends to $+\infty$. The first term in (4.17) yeilds

$$\lim_{l\to+\infty}\lim_{n\to+\infty}\int_\Omega h_l(v^n)a^n(x,u^n,Du^n)\frac{DT_m^k(u)}{b(u)^{p(x)-1}}dx=\int_\Omega \varphi_k\frac{DT_m^k(u)}{b(u)^{p(x)-1}}dx. \tag{4.22}$$

The second term of (4.17)

$$\left|a^n(x,u^n,Du^n)Dh_l(v^n).\left(\int_0^{T_m^k(u)}\frac{1}{b(s)^{p(x)-1}}ds\right)\right|\le\frac{\max(m,k)}{\alpha}\left|a^n(x,u^n,Du^n)Dv^n\right|.$$

Since (4.14), we deduce that

$$\lim_{l\to+\infty}\lim_{n\to+\infty}\int_\Omega a^n(x,u^n,Du^n)Dh_l(v^n).\left(\int_0^{T_m^k(u)}\frac{1}{b(s)^{p(x)-1}}ds\right)dx=0. \tag{4.23}$$

Due to (4.22) and (4.23) , we have

$$\int_\Omega \varphi_k\frac{DT_m^k(u)}{b(u)^{p(x)-1}}dx=\int_\Omega f\left(\int_0^{T_m^k(u)}\frac{1}{b(s)^{p(x)-1}}ds\right)dx. \tag{4.24}$$

Take $\int_0^{T_m^k(u^n)}\frac{1}{b_n(s)^{p(x)-1}}ds$ as a test fuction in (4.2), we get

$$\int_\Omega a^n(x,u^n,Du^n)\frac{DT_m^k(u^n)}{b_n(u^n)^{p(x)-1}}dx=\int_\Omega f_n\int_0^{T_m^k(u^n)}\frac{1}{b_n(s)^{p(x)-1}}dsdx. \tag{4.25}$$

Passing to the limit as n tends to $+\infty$ in (4.25), in view (4.24), we have

$$\lim_{n\to+\infty}I_1^n=\int_\Omega \varphi_k\frac{DT_m^k(u)}{b(u)^{p(x)-1}}dx. \tag{4.26}$$

Limit of I_2^n

By the assumptiom of b_n, we remark

$$\frac{1}{b_n(u^n)^{p(x)-1}}\to\frac{1}{b_n(u^n)^{p(x)-1}}\ a.e.\text{in }\Omega, \tag{4.27}$$

as n tends to $+\infty$. Since (4.5), (4.9), and (4.27), we have

$$\lim_{n\to+\infty}I_2^n=-\int_\Omega \varphi_k\frac{DT_m^k(u)}{b(u)^{p(x)-1}}dx. \tag{4.28}$$

Limit of I_3^n

We notice (3.1), (3.2), and (4.5), we show

$$\frac{a^n(x,T_m^k(u^n),DT_m^k(u))}{b_n(u^n)^{p(x)-1}}\to\frac{a(x,T_m^k(u),DT_m^k(u))}{b(u)^{p(x)-1}}\ a.e.\text{in }\Omega, \tag{4.29}$$

as n tends to $+\infty$, and

$$\left|\frac{a^n(x,T_m^k(u^n),DT_m^k(u))}{b_n(u^n)^{p(x)-1}}\right|\le\frac{1}{\alpha}\left[\beta\left[L(x)+\left|DT_m^k(u^n)\right|^{p(x)-1}\right]\right]\ a.e.\text{in }\Omega, \tag{4.30}$$

uniformly with respect to n. by (4.29), and (4.30), we deduce

$$\frac{a^n(x,T_m^k(u^n),DT_m^k(u))}{b_n(u^n)^{p(x)-1}}\to\frac{a(x,T_m^k(u),DT_m^k(u))}{b(u)^{p(x)-1}}\ \text{weakly in } L^{p'(x)}(\Omega), \tag{4.31}$$

as n tends to $+\infty$.

From (4.5), we conclude

$$DT_m^k(u^n) - DT_m^k(u) \to 0 \text{ weakly in } L^{p(x)}(\Omega). \tag{4.32}$$

Due to (4.31), and (4.32) imply that

$$\lim_{n\to+\infty} I_3^n = 0. \tag{4.33}$$

Combining (4.16) with (4.26)–(4.33), we etablish 4.15

□

Lemma 4.3. *For fixed $k \geq 0$, one has*

$$\varphi_k = a(x, T_m^k(u), DT_m^k(u)) \text{ a.e.in } \{x \in \Omega\,;\, u(x) < m\}. \tag{4.34}$$

And

$$\frac{a^n(x, T_m^k(u^n), DT_m^k(u^n))}{b_n(u^n)^{p(x)-1}} DT_m^k(u^n) \to \frac{a(x, T_m^k(u), DT_m^k(u))}{b(u)^{p(x)-1}} DT_m^k(u) \text{ weakly in } L^1(\Omega), \tag{4.35}$$

when $n \to +\infty$.

Proof. Let $k \geq 0$ be fixed, by (4.4) and (4.31), we have

$$\lim_{n\to+\infty} \int_\Omega \frac{a^n(x, T_m^k(u^n), DT_m^k(u^n))}{b_n(u^n)^{p(x)-1}} DT_m^k(u^n)\, dx = \int_\Omega \frac{\varphi_k}{b(u)^{p(x)-1}} DT_m^k(u)\, dx.$$

Since (4.27) and (3.2), we have for every ψ

$$\begin{aligned}
0 &\leq \lim_{n\to+\infty} \int_\Omega \left[\frac{a^n(x, T_m^k(u^n), DT_m^k(u^n))}{b_n(u^n)^{p(x)-1}} - \frac{a^n(x, T_m^k(u^n), \psi)}{b_n(u^n)^{p(x)-1}}\right] \left[DT_m^k(u^n) - \psi\right] dx \\
&= \lim_{n\to+\infty} \int_\Omega \frac{a^n(x, T_m^k(u^n), DT_m^k(u^n))}{b_n(u^n)^{p(x)-1}} \left[DT_m^k(u^n) - \psi\right] dx \\
&\quad - \lim_{n\to+\infty} \int_\Omega \frac{a^n(x, T_m^k(u^n), \psi)}{b_n(u^n)^{p(x)-1}} \left[DT_m^k(u^n) - \psi\right] dx \\
&= \int_\Omega \left(\frac{\varphi_k}{b(u)^{p(x)-1}} - \frac{a(x, T_m^k(u), \psi)}{b(u)^{p(x)-1}}\right) \left[DT_m^k(u^n) - \psi\right] dx.
\end{aligned} \tag{4.36}$$

By Minty trick lemma, we conclude that for any

$$\frac{\varphi_k}{b(u)^{p(x)-1}} = \frac{a(x, T_m^k(u), DT_m^k(u))}{b(u)^{p(x)-1}} \text{ a.e.in } \Omega. \tag{4.37}$$

Since (4.37) and (4.11), we deduce (4.34).

To prove (4.35), we observe that the monotone character of a and (4.15) give

$$\left[\frac{a^n(x, T_m^k(u^n), DT_m^k(u^n))}{b_n(u^n)^{p(x)-1}} - \frac{a^n(x, T_m^k(u^n), DT_m^k(u^n))}{b_n(u^n)^{p(x)-1}}\right] \times \left[DT_m^k(u^n) - DT_m^k(u^n)\right] \to 0$$

strongly in $L^1(\Omega)$ as n tends to $+\infty$. From (4.5), (4.9), (4.31), and (4.34), we conclude when $n \to +\infty$

$$\frac{a^n(x, T_m^k(u^n), DT_m^k(u^n))}{b_n(u^n)^{p(x)-1}} DT_m^k(u) \to \frac{a(x, T_m^k(u), DT_m^k(u))}{b(u)^{p(x)-1}} DT_m^k(u) \text{ weakly in } L^1(\Omega), \tag{4.38}$$

and

$$\frac{a^n(x, T_m^k(u^n), DT_m^k(u))}{b_n(u^n)^{p(x)-1}} DT_m^k(u^n) \to \frac{a(x, T_m^k(u), DT_m^k(u))}{b(u)^{p(x)-1}} DT_m^k(u) \text{ weakly in } L^1(\Omega), \tag{4.39}$$

and

$$\frac{a^n(x, T_m^k(u^n), DT_m^k(u))}{b_n(u^n)^{p(x)-1}} DT_m^k(u) \to \frac{a(x, T_m^k(u), DT_m^k(u))}{b(u)^{p(x)-1}} DT_m^k(u) \text{ weakly in } L^1(\Omega). \tag{4.40}$$

By the convergences (4.38), (4.39), and (4.40), we obtain that for any $k \geq 0$

$$\frac{a^n(x, T_m^k(u^n), DT_m^k(u^n))}{b_n(u^n)^{p(x)-1}} DT_m^k(u^n) \to \frac{a(x, T_m^k(u), DT_m^k(u))}{b(u)^{p(x)-1}} DT_m^k(u) \text{ weakly in } L^1(\Omega),$$

as n tends to $+\infty$.

4.2. **Step 5: End of the proof.** Taking $T_m^{s+1}(u^n) - T_m^s(u^n)$ as a test function in (4.2) gives

$$\int_\Omega a^n(x, u^n, Du^n).D\left(T_m^{s+1}(u^n) - T_m^s(u^n)\right) dx = \int_\Omega f_n \left(T_m^{s+1}(u^n) - T_m^s(u^n)\right) dx. \tag{4.41}$$

Since $supp\left(T_m^{s+1}(.) - T_m^s(.)\right) \subset [-(s+1), s]$, we obtain

$$\int_{\{-1-s\leq|u^n|\leq -s\}} a^n(x, u^n, Du^n).Du^n dx \tag{4.42}$$

$$\begin{aligned}
&= \int_\Omega a^n(x, u^n, Du^n) D\left(T_m^{s+1}(u^n) - T_m^s(u^n)\right) dx \\
&= \int_\Omega \frac{a^n(x, u^n, Du^n)}{b_n(u^n)^{p(x)-1}} D\left(T_m^{s+1}(u^n) - T_m^s(u^n)\right) b_n(T_{m-1}^{s+1}(u^n))^{p(x)-1} dx \\
&= \int_\Omega \frac{a^n(x, T_m^{s+1}(u^n), DT_m^{s+1}(u^n))}{b_n(T_m^{s+1}(u^n))^{p(x)-1}} D\left(T_m^{s+1}(u^n)\right) b_n(T_{m-1}^{s+1}(u^n))^{p(x)-1} \\
&\quad - \int_\Omega \frac{a^n(x, T_m^s(u^n), DT_m^s(u^n))}{b_n(u^n)^{p(x)-1}} D\left(T_m^s(u^n)\right) b_n(T_{m-1}^{s+1}(u^n))^{p(x)-1}
\end{aligned} \tag{4.43}$$

We deduce from (4.4) and (4.35) that

$$\begin{aligned}
&\lim_{n\to+\infty} \int_{\{-1-s\leq|u^n|\leq -s\}} a^n(x, u^n, Du^n).Du^n dx \\
&= \int_\Omega a^n(x, u^n, Du^n) D\left(T_m^{s+1}(u^n) - T_m^s(u^n)\right) dx \\
&= \int_\Omega \frac{a(x, T_m^{s+1}(u), DT_m^{s+1}(u))}{b_n(T_m^{s+1}(u^n))^{p(x)-1}} D\left(T_m^{s+1}(u)\right) b_n(T_{m-1}^{s+1}(u))^{p(x)-1} \\
&\quad - \int_\Omega \frac{a(x, T_m^s(u), DT_m^s(u))}{b(u)^{p(x)-1}} D\left(T_m^s(u)\right) b_n(T_{m-1}^{s+1}(u))^{p(x)-1} \\
&= \int_{\{-1-s\leq|u|\leq -s\}} a(x, u, Du) Du dx.
\end{aligned} \tag{4.44}$$

Taking the limit as s tends to $+\infty$ in (4.41) and using the estimates (4.43) and (4.44) show that u satisfies (3.8).

Choosing $h_l(v^n)\frac{1}{\delta}(T^+_{m-\delta}(u) - T^+_{m-2\delta}(u))$ as a test function in (4.2), we have

$$\frac{1}{\delta}\int_\Omega a^n(x,u^n,Du^n)D(h_l(v^n)\left(T^+_{m-\delta}(u) - T^+_{m-2\delta}(u)\right))dx$$
$$= \frac{1}{\delta}\int_\Omega f_n h_l(v^n)(T^+_{m-\delta}(u) - T^+_{m-2\delta}(u))dx. \tag{4.45}$$

Since $supp\,(h_l) \subset [-(l+1), l+1]$, we obtain

$$\frac{1}{\delta}\int_\Omega h_l(v^n)a^n(x,u^n,Du^n)D(T^+_{m-\delta}(u) - T^+_{m-2\delta}(u))dx =$$
$$\frac{1}{\delta}\int_\Omega h_l(v^n)a^n(x,T_{(l+1)/\alpha}(u^n), DT_{(l+1)/\alpha}(u^n))D(T^+_{m-\delta}(u) - T^+_{m-2\delta}(u))dx. \tag{4.46}$$

In addition, using the same procedures as above, we deduce

$$\lim_{l\to+\infty}\lim_{n\to+\infty}\frac{1}{\delta}\int_\Omega h_l(v^n)a^n(x,u^n,Du^n)D(T^+_{m-\delta}(u) - T^+_{m-2\delta}(u))dx$$
$$= \frac{1}{\delta}\int_{\{m-2\delta\le|u|\le m-\delta\}} a(x,u,Du)Du dx. \tag{4.47}$$

Taking the limit as s tends to $+\infty$ in (4.45) and using the estimates (4.46) and (4.47) show that u satisfies (3.8).

Let S be a function in $W^{1,\infty}(\mathbb{R})$ such that S has a compact support and $S(m) = 0$ and let $\varphi \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$. Take $S(u)\, h_l(v^n)\varphi$ as a test fuction in (4.2), we get

$$\int_\Omega h_l(v^n)a^n(x,u^n,Du^n)D\left(S(u)\varphi\right)dx + \int_\Omega S(u)\varphi a^n(x,u^n,Du^n)Dh_l(v^n)dx \tag{4.48}$$
$$= \int_\Omega f_n S(u)h_l(v^n)\varphi dx.$$

Taking the limit as n tends to $+\infty$ and l tends to $+\infty$ in (4.48).

Limit of first term in (4.48)

Since $supp\,(h_l) \subset [-(l+1), l+1]$, we obtain

$$a^n(x,T_{(l+1)/\alpha}(u^n), DT_{(l+1)/\alpha}(u^n))h_l(v^n)) = h_l(v^n)a^n(x,u^n,Du^n)\ a.e.\text{in }\Omega.$$

From (4.11), (4.20), (4.21) and (4.34), we get

$$\lim_{l\to+\infty}\lim_{n\to+\infty}\int_\Omega h_l(v^n)a^n(x,u^n,Du^n)D\left(S(u)\varphi\right)dx$$
$$= \lim_{l\to+\infty}\int_\Omega h_l(v)a(x,T^k_m(u^n), DT^k_m(u^n))D\left(S(u)\varphi\right)dx$$
$$= \int_\Omega a(x,u,Du)D\left(S(u)\varphi\right)dx.$$

Limit of second term in (4.48)

As a consequence of (4.14), we conclude

$$\lim_{l\to+\infty}\lim_{n\to+\infty}\int_{\Omega} S(u)\varphi a^n(x,u^n,Du^n)Dh_l(v^n)dx = 0.$$

Limit of the Right-Hand Side of (4.48)

From (4.1) and (4.13)

$$\lim_{l\to+\infty}\lim_{n\to+\infty}\int_{\Omega} f_n S(u)h_l(v^n)\varphi dx = \int_{\Omega} fS(u)\varphi dx.$$

Then, u satisfies (3.12). □

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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