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SOME NEW ESTIMATES FOR THE JACOBI TRANSFORM IN THE SPACE $L_2(\mathbb{R}^+, J^{\alpha,\beta}(x)dx)$

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ABSTRACT. For the Jacobi transform in the space $L_2(\mathbb{R}^+, J^{\alpha,\beta}(x)dx)$, new estimates are proved in certain class of functions characterized by a generalized continuity modulus, using a generalized operator.
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(1) Introduction

In this paper, we prove new estimates in certain classes of functions characterized by a generalized continuity modulus and connected with the Jacobi transform in the space $L_2(\mathbb{R}^+, J^{\alpha,\beta}(x)dx)$. For this purpose, we use a generalized translation operator which was defined by Flensted-Jensen an Koornwinder (See [4]).

In Section 2, we give some definitions and preliminaries concerning the Jacobi transform. The new estimates are proved in section 3.

(2) Preliminaries on the Jacobi Transform

Let $\alpha > \frac{-1}{2}$, $\alpha \geq \beta \geq \frac{-1}{2}$, $\rho = \alpha + \beta + 1$ and let

$$D_{\alpha,\beta} := \frac{d^2}{dx^2} + ((2\alpha+1)\coth x + (2\beta+1)\tanh x)\frac{d}{dx}$$

be the Jacobi differential operator and denote by $\varphi_\lambda^{(\alpha,\beta)}(x)$ $(\lambda, x \in \mathbb{R}^+)$ the Jacobi function of order (α, β), the function $\varphi_\lambda^{(\alpha,\beta)}(x)$ satisfier the differential equation

$$D_{\alpha,\beta}\varphi + (\lambda^2 + \rho^2)\varphi = 0$$

with the initial conditions $\varphi(0) = 1$ and $\varphi'(0) = 0$

Lemma 1. The following inequalities are valid for Jacobi functions $\varphi_\lambda^{(\alpha,\beta)}(x)$

1. $| \varphi_\lambda^{(\alpha,\beta)}(x) | \leq 1$
2. $|1 - \varphi_\lambda^{(\alpha,\beta)}(x)| \leq x^2(\lambda^2 + \rho^2)$

Proof: See [7, lemmas 3.1-3.2]

Lemma 2. For $|\eta| \leq \rho$, there exists a positive constant $c_1 = c(\alpha, \beta)$ such that

$$|1 - \varphi_{\mu+i\eta}^{(\alpha,\beta)}(x)| \geq c_1|1 - j_\alpha(\mu x)|$$

where $j_\alpha(x)$ is a normalized Bessel function of the first Kind.

Proof. See [2, Lemma 9]

Consider the Hilbert space $L_2(\mathbb{R}^+) = L_2(\mathbb{R}^+, J^{\alpha,\beta}(x)dx)$ with the norm

$$\|f\| = \|f\|_{2,(\alpha,\beta)} = \left(\int_0^{+\infty} \mid f(x) \mid^2 \ J^{\alpha,\beta}(x)dx\right)^{\frac{1}{2}}$$

where

$$J^{\alpha,\beta}(x) = (2\sinh x)^{2\alpha+1}(2\cosh x)^{2\beta+1}$$

The Jacobi transform of a function $f \in L_2(\mathbb{R}^+)$ is defined by

$$g(\lambda) = \int_0^{+\infty} f(x)\varphi_\lambda^{(\alpha,\beta)}(x)J^{\alpha,\beta}(x)dx$$

The inversion formula (See[5]) is

$$f(x) = \int_0^{+\infty} g(\lambda)\varphi_\lambda^{(\alpha,\beta)}(x)d\mu(\lambda)$$

where $d\mu(\lambda) := \frac{1}{2\pi} \mid C(\lambda) \mid^{-2} d\lambda$ and the C-function $C(\lambda)$ is defined by

$$C(\lambda) = \frac{2^\rho\Gamma(i\lambda)\Gamma(\frac{1}{2}(1+i\lambda))}{\Gamma(\frac{1}{2}(\rho+i\lambda))\Gamma(\frac{1}{2}(\rho+i\lambda)-\beta)}$$

The Plancherel formula for Jacobi transform (See [5]) written as

$$\|f\| = \| f \|_{L_2(\mathbb{R}^+, J^{\alpha,\beta}(x)dx)} = \|g\|_{L_2(\mathbb{R}^+, d\mu(\lambda))}$$

Recall from [4] the generalized translation T_h of a suitable function f on $\mathbb{R}^+$, defined by

$$T_h f(x) = \int_0^{+\infty} f(z)K(x,h,z)J^{\alpha,\beta}(z)dz$$

where the Kernel K is explicitly Known (see [5]).

The finite differences of the first and higher orders are defined as follows:

$$\Delta_h f(x) = T_h f(x) - f(x) = (T_h - I)f(x)$$

where I is the identity operator in $L_2(\mathbb{R}^+)$, and

$$\Delta_h^k f(x) = \Delta_h(\Delta_h^{k-1} f(x)) = (T_h - 1)^k f(x) = \sum_{i=0}^k (-1)^{k-1}\binom{k}{i} T_h^i f(x)$$

where $T_h^0 f(x) = f(x)$, $T_h^i f(x) = T_h(T_h^{i-1} f(x))$, ($i = 1, 2, ..., k$ and $k = 1, 2, ...$).

The k^{th} order generalized modulus of continuity of a function $f \in L_2(\mathbb{R}^+)$ is defined by

$$\Omega_k(f,\delta) = \sup_{0<h\leq\delta} \|\Delta_h^k f\|, \delta > 0$$

Denote by $L_2^r(D_{\alpha,\beta})$, r=0,1,...; the class of functions $f \in L_2(\mathbb{R}^+)$ that have on $\mathbb{R}^+$ generalized derivatives $f'(x), f''(x), ..., f^{(2r)}(x)$ in the sense of Levi (See [8]) and belong to $f \in L_2(\mathbb{R}^+)$ with $D_{\alpha,\beta}^r f \in L_2(\mathbb{R}^+)$.

Since in [2]

$$T_h f(x) = \int_0^\infty \varphi_\lambda^{(\alpha,\beta)}(h) g(\lambda) \varphi_\lambda^{(\alpha,\beta)}(x) d\mu(\lambda) \quad \text{and} \quad f(x) = \int_0^{+\infty} g(\lambda) \varphi_\lambda^{(\alpha,\beta)}(x) d\mu(\lambda)$$

it follows

$$T_h f(x) - f(x) = \int_0^\infty (1 - \varphi_\lambda^{(\alpha,\beta)}(h)) g(\lambda) \varphi_\lambda^{(\alpha,\beta)}(x) d\mu(\lambda) \tag{1}$$

The Plancherel equality and formula (1) give

$$\|T_h f(x) - f(x)\|^2 = \int_0^\infty |1 - \varphi_\lambda^{(\alpha,\beta)}(h)|^2 |g(\lambda)|^2 d\mu(\lambda)$$

Hence, for $f \in L_2^r(D_{\alpha,\beta})$, we have

$$\|\Delta_h^k D_{\alpha,\beta}^r f\|^2 = \int_0^\infty (\lambda^2 + \rho^2)^{2r} |1 - \varphi_\lambda^{(\alpha,\beta)}(h)|^{2k} |g(\lambda)|^2 d\mu(\lambda) \tag{2}$$

(3) New Estimates for the Jacobi transform

The goal of this work is to prove several new estimates for the integral

$$\delta_N^2(f) = \int_N^\infty |g(\lambda)|^2 d\mu(\lambda), \quad N > 0$$

in certain classes of functions in $L_2(\mathbb{R}^+)$.

Theorem 1. Let $f \in L_2^r(D_{\alpha,\beta})$. Then

$$\delta_N(f) = O(N^{-2r} \Omega_k(D_{\alpha,\beta}^r f, cN^{-1})), \quad N \to \infty$$

where r=0,1,...; k=0,1,...; and $c > 0$ is a fixed constant.

Proof. In the terms of $j_\alpha(x)$, the normalized Bessel function of the first kind, we have (see[3])

$$1 - j_\alpha(x) = O(1), \quad x \geq 1 \tag{3}$$

$$1 - j_\alpha(x) = O(x^2), \quad 0 \leq x \leq 1 \tag{4}$$

$$j_\alpha(x) = O(x^{-\alpha - \frac{1}{2}}), \quad x \geq 0 \tag{5}$$

we use the same scheme as in the proof of theorem 1 in [1]. It can be sketched as follows.

By virtue of the Hölder inequality, we have

$$\int_N^\infty |g(\lambda)|^2 d\mu(\lambda) - \int_N^\infty j_\alpha(\lambda h) |g(\lambda)|^2 d\mu(\lambda) \leq \frac{(N^2 + \rho^2)^{\frac{-r}{k}}}{c_1} \left(\int_N^\infty |g(\lambda)|^2 d\mu(\lambda) \right)^{\frac{2k-1}{2k}}$$
$$\times \left(\int_N^\infty (\lambda^2 + \rho^2)^{2r} |1 - \varphi_\lambda^{(\alpha,\beta)}(h)|^{2k} |g(\lambda)|^2 d\mu(\lambda) \right)^{\frac{1}{2k}}$$

from (2), we have

$$\int_N^{\infty} (\lambda^2+\rho^2)^{2r}|1-\varphi_\lambda^{(\alpha,\beta)}(h)|^{2k}|g(\lambda)|^2 d\mu(\lambda) \leq \|\Delta_h^k D_{\alpha,\beta}^r f\|^2$$

therefore

$$\begin{aligned}\int_N^{\infty} |g(\lambda)|^2 d\mu(\lambda) &\leq \int_N^{\infty} j_\alpha(\lambda h)|g(\lambda)|^2 d\mu(\lambda)\\ &+ \frac{(N^2+\rho^2)^{\frac{-r}{k}}}{c_1}\left(\int_N^{\infty}|g(\lambda)|^2 d\mu(\lambda)\right)^{\frac{2k-1}{2k}} \|\Delta_h^k D_{\alpha,\beta}^r f\|^{\frac{1}{k}}\end{aligned}$$

From (5), we obtain

$$\begin{aligned}&\int_N^{\infty} |g(\lambda)|^2 d\mu(\lambda)\\ &= O\left(\int_N^{\infty} (\lambda h)^{-\alpha-\frac{1}{2}}|g(\lambda)|^2 d\mu(\lambda) + N^{-\frac{2r}{k}}\left(\int_N^{\infty}|g(\lambda)|^2 d\mu(\lambda)\right)^{\frac{2k-1}{2k}} \|\Delta_h^k D_{\alpha,\beta}^r f\|^{\frac{1}{k}}\right)\\ &= O\left((Nh)^{-\alpha-\frac{1}{2}}\int_N^{\infty} |g(\lambda)|^2 d\mu(\lambda) + N^{-\frac{2r}{k}}\left(\int_N^{\infty}|g(\lambda)|^2 d\mu(\lambda)\right)^{\frac{2k-1}{2k}} \|\Delta_h^k D_{\alpha,\beta}^r f\|^{\frac{1}{k}}\right)\end{aligned}$$

or

$$(1-(Nh)^{-\alpha-\frac{1}{2}})\int_N^{\infty} |g(\lambda)|^2 d\mu(\lambda) = O(N^{-\frac{2r}{k}})\left(\int_N^{\infty}|g(\lambda)|^2 d\mu(\lambda)\right)^{\frac{2k-1}{2k}} \|\Delta_h^k D_{\alpha,\beta}^r f\|^{\frac{1}{k}}$$

Setting $h=\frac{c}{N}$, in the last inequality and choosing $c>0$ such that $1-O(c^{-\alpha-\frac{1}{2}})\geq \frac{1}{2}$, we obtain

$$\int_N^{\infty} |g(\lambda)|^2 d\mu(\lambda) = O(N^{-\frac{2r}{k}})\left(\int_N^{\infty}|g(\lambda)|^2 d\mu(\lambda)\right)^{\frac{2k-1}{2k}} \|\Delta_{cN^{-1}}^k D_{\alpha,\beta}^r f\|^{\frac{1}{k}}$$

We have

$$\int_N^{\infty} |g(\lambda)|^2 d\mu(\lambda) = O(N^{-4r})\Omega_k^2(D_{\alpha,\beta}^r f, cN^{-1})$$

Or

$$\delta_N(f) = O(N^{-2r}\Omega_k(D_{\alpha,\beta}^r f, cN^{-1}))$$

Theorem 2. Let $f\in L_2(\mathbb{R}^+)$. Then

$$\Omega_k(f,\delta) = O\left(N^{-2k}\left(\sum_{l=1}^{N} l^{4k-1}\delta_l^2(f)\right)^{\frac{1}{2}}\right), \quad N\to\infty$$

where $k=0,1,...$

Proof. From (2), we have

$$\|\Delta_h^k f\|^2 = \int_0^{\infty} |1-\varphi_\lambda^{(\alpha,\beta)}(h)|^{2k}|g(\lambda)|^2 d\mu(\lambda)$$

Let $N=[\frac{1}{h}]$, where $0<h<1$.By lemma 1, we have

$$\begin{aligned}
\int_0^\infty &= \int_0^N + \int_N^\infty \\
&= O\left(h^{4k}\int_0^N (\lambda^2+\rho^2)^{2k}|g(\lambda)|^2 d\mu(\lambda) + \int_N^\infty |g(\lambda)|^2 d\mu(\lambda)\right) \\
&= O(N^{-4k})\left[\int_0^N (\lambda^2+\rho^2)^{2k}|g(\lambda)|^2 d\mu(\lambda) + N^{4k}\delta_N^2(f)\right] \\
&= O(N^{-4k})\left[\sum_{l=1}^{N}\int_{l-1}^{l} (\lambda^2+\rho^2)^{2k}|g(\lambda)|^2 d\mu(\lambda) + N^{4k}\delta_N^2(f)\right] \\
&= O(N^{-4k})\left[\sum_{l=1}^{N} l^{4k}\int_{l-1}^{l} |g(\lambda)|^2 d\mu(\lambda) + N^{4k}\delta_N^2(f)\right] \\
&= O(N^{-4k})\left[\sum_{l=1}^{N} l^{4k}\left(\delta_{l-1}^2(f)-\delta_l^2(f)\right) + N^{4k}\delta_N^2(f)\right] \\
&= O(N^{-4k})\left[\sum_{l=0}^{N-1}(l+1)^{4k}\delta_l^2(f) - \sum_{l=1}^{N} l^{4k}\delta_l^2(f) + N^{4k}\delta_{N-1}^2(f)\right] \\
&= O(N^{-4k})\left[\sum_{l=0}^{N}(l+1)^{4k}\delta_l^2(f) - \sum_{l=0}^{N} l^{4k}\delta_l^2(f)\right] \\
&= O(N^{-4k})\left[\sum_{l=0}^{N}((l+1)^{4k}-l^{4k})\delta_l^2(f)\right] \\
&= O(N^{-4k})\left[\sum_{l=1}^{N} l^{4k-1}\delta_l^2(f)\right]
\end{aligned}$$

Thus

$$\|\Delta_h^k f\|^2 = O(N^{-4k})\sum_{l=1}^{N} l^{4k-1}\delta_l^2(f)$$

which implies

$$\Omega_k(f,\delta) = O\left(N^{-2k}\left(\sum_{l=1}^{N} l^{4k-1}\delta_l^2(f)\right)^{\frac{1}{2}}\right)$$

Theorem 3. Let $f\in L_2(\mathbb{R}^+)$. If the series

$$\sum_{l=1}^{\infty} l^{2r-1}\delta_l(f), \quad r=1,2,...$$

converges, then $f \in L_2^r(D_{\alpha,\beta})$ and

$$\Omega_k(D^r_{\alpha,\beta}f,\delta) = O\left(N^{-4k}\sum_{l=1}^{N} l^{4(k+r)-1}\delta_l^2(f)\right)^{\frac{1}{2}} + O\left(\sum_{l=[\frac{N}{2}]}^{\infty} l^{2r-1}\delta_l(f)\right)$$

Where $k = 1, 2, ..,$ and $N \to \infty$

Proof. Since the series

$$\sum_{l=1}^{\infty} l^{2r-1}\delta_l(f), \quad r = 1, 2, ...$$

converges, there exists a function $f \in L_2^r(D_{\alpha,\beta})$ such that

$$D^r_{\alpha,\beta}f(x) = (-1)^r \int_0^{+\infty} (\lambda^2+\rho^2)^r \varphi_\lambda^{(\alpha,\beta)}(x) g(\lambda) d\mu(\lambda)$$

Hence, by the Plancherel equality, we have

$$\|\Delta_h^k D^r_{\alpha,\beta} f\|^2 = \int_0^{\infty} (\lambda^2+\rho^2)^{2r} |1-\varphi_\lambda^{(\alpha,\beta)}(h)|^{2k} |g(\lambda)|^2 d\mu(\lambda)$$

Let $N = [\frac{1}{h}]$, where $0 < h < 1$. Then

$$\|\Delta_h^k D^r_{\alpha,\beta} f\|^2 = \int_0^N + \int_N^{\infty} = I_1 + I_2$$

Estimate the summands I_1 and I_2.

By virtue of (2) in lemma 1, it is easy to see that

$$\begin{aligned}
I_1 &= \int_0^N (\lambda^2+\rho^2)^{2r} |1-\varphi_\lambda^{(\alpha,\beta)}(h)|^{2k} |g(\lambda)|^2 d\mu(\lambda) \\
&= O(h^{4k}) \int_0^N (\lambda^2+\rho^2)^{2r+2k} |g(\lambda)|^2 d\mu(\lambda) \\
&= O(h^{4k}) \sum_{l=0}^{N} \int_l^{l+1} (\lambda^2+\rho^2)^{2r+2k} |g(\lambda)|^2 d\mu(\lambda) \\
&= O(h^{4k}) \sum_{l=0}^{N} (l+1)^{4r+4k} \int_l^{l+1} |g(\lambda)|^2 d\mu(\lambda) \\
&= O(h^{4k}) \left(\sum_{l=0}^{N} (l+1)^{4r+4k}\delta_l^2(f) - \sum_{l=0}^{N} (l+1)^{4r+4k}\delta_{l+1}^2(f)\right)
\end{aligned}$$

$$
\begin{aligned}
&= O(h^{4k})\left(\sum_{l=0}^{N}(l+1)^{4r+4k}\delta_l^2(f)-\sum_{l=0}^{N+1}l^{4r+4k}\delta_l^2(f)\right)\\
&= O(h^{4k})\left(\sum_{l=0}^{N}(l+1)^{4r+4k}\delta_l^2(f)-\sum_{l=0}^{N}l^{4r+4k}\delta_l^2(f)\right)\\
&= O(h^{4k})\sum_{l=0}^{N}((l+1)^{4r+4k}-l^{4r+4k})\delta_l^2(f)\\
&= O(h^{4k})\sum_{l=1}^{N}l^{4r+4k-1}\delta_l^2(f)\\
&= O\left(N^{-4k}\sum_{l=1}^{N}l^{4(k+r)-1}\delta_l^2(f)\right)
\end{aligned}
$$

Now we estimate I_2, by (1) in lemma 1, and obtain

$$
\begin{aligned}
I_2 &= \int_N^{\infty}(\lambda^2+\rho^2)^{2r}|1-\varphi_\lambda^{(\alpha,\beta)}(h)|^{2k}|g(\lambda)|^2 d\mu(\lambda)\\
&= O\left(\int_N^{\infty}(\lambda^2+\rho^2)^{2r}|g(\lambda)|^2 d\mu(\lambda)\right)\\
&= O\left(\sum_{n=1}^{\infty}\int_{2^{n-1}N}^{2^n N}(\lambda^2+\rho^2)^{2r}|g(\lambda)|^2 d\mu(\lambda)\right)\\
&= O\left(\sum_{n=1}^{\infty}(2^n N)^{4r}\int_{2^{n-1}N}^{2^n N}|g(\lambda)|^2 d\mu(\lambda)\right)\\
&= O\left(\sum_{n=1}^{\infty}(2^n N)^{4r}\delta_{2^{n-1}N}^2(f)\right)
\end{aligned}
$$

i.e.

$$
(I_2)^{\frac{1}{2}} = O\left(\sum_{n=1}^{\infty}(2^n N)^{2r}\delta_{2^{n-1}N}(f)\right)
$$

Taking account of the fact that

$$
2^{4r}\sum_{l=2^{n-2}N+1}^{2^{n-1}N} l^{2r-1}\delta_l(f) \geq 2^{4r}(2^{n-2}N)^{(2r-1)}\delta_{2^{n-1}N}(f)2^{n-2}N = (2^n N)^{2r}\delta_{2^{n-1}N}(f)
$$

We obtain the estimate

$$
\begin{aligned}
(I_2)^{\frac{1}{2}} &= O\left(\sum_{n=1}^{\infty}\sum_{l=2^{n-2}N+1}^{2^{n-1}N} l^{2r-1}\delta_l(f)\right)\\
&= O\left(\sum_{l=[\frac{N}{2}]}^{\infty} l^{2r-1}\delta_l(f)\right)
\end{aligned}
$$

combining the estimates for I_1 and I_2 gives

$$\|\Delta_h^k D_{\alpha,\beta}^r f\| = O\left(N^{-4k}\sum_{l=1}^{N} l^{4(k+r)-1}\delta_l^2(f)\right)^{\frac{1}{2}} + O\left(\sum_{l=[\frac{N}{2}]}^{\infty} l^{2r-1}\delta_l(f)\right)$$

which implies

$$\Omega_k(D_{\alpha,\beta}^r f,\delta) = O\left(N^{-4k}\sum_{l=1}^{N} l^{4(k+r)-1}\delta_l^2(f)\right)^{\frac{1}{2}} + O\left(\sum_{l=[\frac{N}{2}]}^{\infty} l^{2r-1}\delta_l(f)\right)$$

Remark. Theorems 1-3 imply that

$$\delta_N(f) = O(N^{-2r-k\nu}) \Longleftrightarrow \Omega_k(D_{\alpha,\beta}^r f,\delta) = O(\delta^{k\nu})$$

where $k = 1, 2, ..., \quad r = 1, 2, ..., \quad 0 < \nu < 2$
This result was proved in [1].

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