Asia Pacific Journal of Mathematics, Vol. 2, No. 1 (2015), 41-53 ISSN 2357-2205

NIKOL'SKII-BESOV SPACES AND THEIR APPROXIMATION CHARACTERISTICS FOR DUNKL HARMONIC ANALYSIS

R. DAHER AND S. EL OUADIH*

Departement of Mathematics,
Faculty of Sciences Ain Chock, University Hassan II, Casablanca, Marocco
*Corresponding author

ABSTRACT. In this paper we define Nikol'skii-Besov spaces for the Dunkl generalized translation and describe them in terms of best approximations.
2010 Mathematics Subject Classification. 43A15.
Key words and phrases. Dunkl transform, Dunkl operator, approximation of functions, generalized continuity modulus.

(1) Introduction

In the present paper, using the direct theorem of Jackson type and Bernstein theorem, we give a complete description of the function spaces $H^r_{p,\alpha}$ of Nikol'skii type and the function spaces $B^r_{p,q,\alpha}$ of Nikol'skii-Besov type in terms of the best approximations by entire functions of exponential type.

(2) The Dunkl transform and its basic properties

Let $L_{p,\alpha}$ the space of functions f defined on $\mathbb{R}$ endowed with the following finite norm

$$\|f\|_{p,\alpha} := \left(\int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x)\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty$$

where the measure $d\mu_\alpha, \alpha > -\frac{1}{2}$, is defined by $d\mu_\alpha(x) = (2^{\alpha+1}\Gamma(\alpha+1))^{-1}|x|^{2\alpha+1}dx$, and $L_{\infty,\alpha}$ denote the space of essentially bounded functions with the finite norm

$$\|f\|_{\infty,\alpha} := ess \sup_{x\in\mathbb{R}} |f(x)|.$$

We denote by $C^k(\mathbb{R})$ the set of k times continuously differentiable functions on $\mathbb{R}$. $\mathcal{S}$ is the Schwartz function space defined on $\mathbb{R}$ with $\mathcal{S}'$ as its dual space.

The Dunkl operator is differential-difference operator D_α which satisfies the condition

$$D_\alpha f(x) = \frac{df}{dx}(x) + (\alpha + \frac{1}{2})\frac{f(x) - f(-x)}{x}, \quad f \in C^1(\mathbb{R})$$

Since for all $\varphi, \psi \in \mathcal{S}$, $\langle D_\alpha \varphi, \psi\rangle = -\langle \varphi, D_\alpha \psi\rangle$, where

$$\langle f, g\rangle := \int_{\mathbb{R}} f(x)g(x)d\mu_\alpha(x)$$

then, $D_\alpha u$ of the distribution u may be defined by the formula $\langle D_\alpha u, \varphi\rangle = -\langle u, D_\alpha \varphi\rangle$, $u \in \mathcal{S}', \varphi \in \mathcal{S}$. It is obvious that the space $L_{p,\alpha}$ is embedded into $\mathcal{S}'$.
Hence D_α is defined on $f \in L_{p,\alpha}$. Generally speaking , $D_\alpha f$ is a distribution.
Let $j_\alpha(z)$ denote the normalized Bessel function of the first kind of order α given by

$$j_\alpha(x) = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+\alpha+1)}(\frac{x}{2})^{2n}$$

We understand a generalized exponential function as the function

$$E_\alpha(x) = j_\alpha(ix) + \frac{x}{2(\alpha+1)}j_{\alpha+1}(ix)$$

The function $y = E_\alpha(x)$ satisfies the equation $D_\alpha y = y$ with the initial data $y(0) = 1$ and it is the unique solution (See [5]). Using the correlation

$$j'_\alpha(x) = -\frac{xj_{\alpha+1}(x)}{2(\alpha+1)}$$

we conclude that the function $E_\alpha(x)$ admits the representation

$$E_\alpha(x) = j_\alpha(ix) + ij'_\alpha(ix)$$

The Dunkl transform of order α for $f \in L_{1,\alpha}$ is defined by

$$\mathcal{F}_\alpha f(x) = \int_{\mathbb{R}} f(y)E_\alpha(-ixy)d\mu_\alpha(y), \quad x \in \mathbb{R}$$

The inverse Dunkl transform is defined by the formula (See [2])

$$f(y) = \int_{\mathbb{R}} \mathcal{F}_\alpha f(x)E_\alpha(ixy)d\mu_\alpha(x)$$

Given $s \in \mathbb{R}$, the generalized translation operator T^s on $L_{2,\alpha}$ is defined by

$$\mathcal{F}_\alpha(T^s f)(x) = E_\alpha(-isx)\mathcal{F}_\alpha f(x)$$

The linear operator T^s can be extended to a continuous operator on $L_{P,\alpha}$ with $\|T^s f\|_{p,\alpha} \leq 3\|f\|_{p,\alpha}$, $1 \leq p \leq \infty$ (See [4]).
Given $f \in L_{p,\alpha}$, we define the differences $\Delta_h^k f$ of order$k(k \in \mathbb{N})$ with the step $h > 0$ and the modulus of smoothness $\omega_k(f, \delta)_{p,\alpha}$ of order k as follows:

$$\Delta_h^k f(t) = \sum_{l=0}^{k}(-1)^l\binom{k}{l}T^{lh}f(t)$$

$$\omega_k(f, \delta)_{p,\alpha} = \sup_{0<h\leq\delta}\|\Delta_h^k f\|_{p,\alpha}, \delta > 0$$

for $\sigma > 0$, denote by $E^{\sigma}_{p,\alpha}$ the space of entire functions of exponential $\leq \sigma$ whose restrictions to $\mathbb{R}$ belong to $L_{p,\alpha}$.Then the functions in the space $E^{\sigma}_{p,\alpha}$ make a natural approximation tool in $L_{p,\alpha}$. the best approximation in $E^{\sigma}_{p,\alpha}$ for $f \in L_{p,\alpha}$ is defined as below:

$$E_{\sigma}(f)_{p,\alpha} := \inf\{\|f - \phi\|_{p,\alpha} : \phi \in E^{\sigma}_{p,\alpha}\}$$

Let $W^{m}_{p,\alpha}$ be the Sobolev space constructed by the differential-difference operator D_{α} as follows:

$$W^{m}_{p,\alpha} = \{f \in L_{p,\alpha} : D^{i}_{\alpha} f \in L_{p,\alpha}, i = 1, ..., m\}$$

Lemma 2.1. For all $f \in W^{r}_{p,\alpha}$, $1 \leq p < \infty$, and $k \geq r, k, r \in \mathbb{N}$, we have

$$\|\Delta^{k}_{h} f\|_{p,\alpha} \leq c_1 h^{r} \|D^{r}_{\alpha} f\|_{p,\alpha}$$

where $c_1 = c(k, r, \alpha)$ is a constant

Proof. (See [1], Lemma 4.2)

Lemma 2.2. For $\alpha > \frac{-1}{2}, k \in \mathbb{N}, \sigma > 0, 1 \leq p < \infty, r \in \mathbb{N}, f \in f \in W^{r}_{p,\alpha}$, then

$$E_{\sigma}(f)_{p,\alpha} \leq \frac{c_2}{\sigma^{r}} \omega_k(D^{r}_{\alpha} f, \frac{1}{\sigma})_{p,\alpha}$$

where c_2 is a positive constant independent of f.

Proof. (See [1], Theorem 2.1)

Lemma 2.3. If $\sigma > 0, r \in \mathbb{N}, f \in E^{\sigma}_{p,\alpha}$ then

$$\|D^{r}_{\alpha} f\|_{p,\alpha} \leq c_3 \sigma^{r} \|f\|_{p,\alpha}$$

where $c_3 = c(r, \alpha)$

Proof. (See [1], Theorem 2.3)

(3) **Main Result**

To prove inverse theorm of approximation theory we use inequalities of Bernstein type inequality in Lemma (2.3) and the following inequality

Lemma 3.1

If $\Phi \in E^{\sigma}_{p,\alpha}, 1 \leq p < \infty$,and $h > 0$, then

(1)
$$\|\Delta^{k}_{h} \Phi\|_{p,\alpha} \leq c_4 (\sigma h)^{k} \|\Phi\|_{p,\alpha}$$

where $c_4 = c(\alpha, k)$ is a constant

proof. If $\Phi \in E^{\sigma}_{p,\alpha}$, then in particular we have $\Phi \in W^{r}_{p,\alpha}$. It follows from Lemma

(2.1) (with $r=1$ and $k=1$) and Lemma (2.3) that

$$\|\Delta_h\Phi\|_{p,\alpha} \leq c_1 h\|D_\alpha\Phi\|_{p,\alpha} \leq c_1c_3h\sigma\|\Phi\|_{p,\alpha}$$

where $\Delta_h f = (I-T^h)f$.In similar way we check the estimate

$$\|\Delta_h^k\Phi\|_{p,\alpha} \leq c_4(\sigma h)^k\|\Phi\|_{p,\alpha}$$

The lemma is proved.

Let $1 \leq p < \infty$,$r > 0$ be real numbers , and let k, m be non-negative integers with $m < r < k+m$, we denote by $H_p^{r,\alpha}$ the set of functions $f \in L_{p,\alpha}$ such that $D_\alpha f, D_\alpha^2 f, ..., D_\alpha^m f \in L_{p,\alpha}$ and

$$\omega_k(D_\alpha^m f, \delta)_{p,\alpha} \leq A_f\delta^{r-m}, \delta > 0$$

for some $A_f > 0$.
For $f \in H_p^{r,\alpha}$ we define the seminorm

$$h_p^r(f) := \sup_{\delta>0} \frac{\omega_k(D_\alpha^m f, \delta)_{p,\alpha}}{\delta^{r-m}}$$

$H_p^{r,\alpha}$ is a Banach space with norm

$$\|f\|_{H_p^{r,\alpha}} := \|f\|_{p,\alpha} + h_p^r(f)$$

In the next theorem we describe the space $H_p^{r,\alpha}$ in terms of the best approximation by functions belonging to $E_{p,\alpha}^\sigma$. in particular, this theorem implies that the $H_p^{r,\alpha}$ does not depend on k or m.

Theorem 3.2.
If $f \in H_p^{r,\alpha}$, then

$$E_\sigma(f)_{p,\alpha} \leq c_2\frac{h_p^r(f)}{\sigma^r}$$

for $\sigma \geq 1$. conversely, if $f \in L_{p,\alpha}$ and

(2) $$E_\sigma(f)_{p,\alpha} \leq \frac{A}{\sigma^r}$$

for $\sigma \geq 1$, where A is a constant that does not depend on σ (but depends on f), then $f \in H_p^{r,\alpha}$ for all non-negative integers k, m such that $m < r < m+k$ and

$$\|f\|_{H_p^{r,\alpha}} \leq C(\|f\|_{p,\alpha} + A)$$

where $C = C(k, m, r, \alpha)$ is a constant

Proof:

If $f \in H_p^{r,\alpha}$, then

$$\omega_k(D_\alpha^m f, \delta)_{p,\alpha} \leq h_p^r(f)\delta^{r-m}$$

and, by Lemma (2.2),

$$\begin{aligned} E_\sigma(f)_{p,\alpha} &\leq c_2 \frac{\omega_k(D_\alpha^m f, \frac{1}{\sigma})_{p,\alpha}}{\sigma^m} \\ &\leq c_2 \frac{h_p^r(f)}{\sigma^r} \end{aligned}$$

We prove the reverse inequality using a conventional technique that goes bak to Bernstein (see [3]). Assume that (2) holds. Consider a sequence of functions $\psi_n \in E_{p,\alpha}^{2^n}$ (n=0,1,2....) such that $\|f - \psi_n\|_{p,\alpha} \leq A2^{-nr}$.

Put $\phi_0 = \psi_0$ and $\phi_n = \psi_n - \psi_{n-1}$ for $n \geq 1$. Then

(3) $$f = \sum_{n=0}^{\infty} \phi_n$$

the series converges in $L_{p,\alpha}$, and $\phi_n \in E_{p,\alpha}^{2^n}$. We obtain upper bounds for the norms of the terms of (3) as follows:

(4) $$\|\phi_0\|_{p,\alpha} = \|\psi_0\|_{p,\alpha} \leq \|\psi_0 - f\|_{p,\alpha} + \|f\|_{p,\alpha} \leq \|f\|_{p,\alpha} + A$$

(5) $$\|\phi_n\|_{p,\alpha} \leq \|f - \psi_n\|_{p,\alpha} + \|f - \psi_{n-1}\|_{p,\alpha} \leq A(1 + 2^r)2^{-nr}$$

Combining (4) and (5), we obtain that

(6) $$\|\phi_n\|_{p,\alpha} \leq c_5 2^{-nr}(\|f\|_{p,\alpha} + A), n = 0, 1, 2...$$

with $c_5 = 1 + 2^r$

Let $s \in \{1, 2, ..., m\}$, Lemma (2.3) implies that

(7) $$\|D_\alpha^s \phi_n\|_{p,\alpha} \leq c_3 (2^n)^s \|\phi_n\|_{p,\alpha}$$

Inequalities (6),(7) and the fact that $r > s$ imply that the series $\sum_{n=0}^{\infty} D_\alpha^s \phi_n$ converges in $L_{p,\alpha}$. That is, there exists a function $\varphi \in L_{p,\alpha}$, such that

$$\varphi = \sum_{n=0}^{\infty} D_\alpha^s \phi_n$$

Denote S_N by

$$S_N = \sum_{n=0}^{N} D_\alpha^s \phi_n$$

For $h \in \mathcal{S}$, we have

$$\begin{aligned}\langle \mathcal{F}_\alpha \varphi, h\rangle &= \langle \varphi, \mathcal{F}_\alpha h\rangle = \lim_{N\to\infty} \langle S_N, \mathcal{F}_\alpha h\rangle \\ &= \lim_{N\to\infty} \langle \mathcal{F}_\alpha S_N, h\rangle = \lim_{N\to\infty} \langle (ix)^s \mathcal{F}_\alpha \psi_N, h\rangle \\ &= \langle (ix)^s \mathcal{F}_\alpha f, h\rangle\end{aligned}$$

and hence $\mathcal{F}_\alpha \varphi = (ix)^s \mathcal{F}_\alpha f$, i.e. $\varphi = D_\alpha^r f$. In particular $D_\alpha f, ..., D_\alpha^{m-1} f$ and $g = D_\alpha^m f$ belong to $L_{p,\alpha}$
let $\Phi_n := D_\alpha^m \phi_n$. Then

(8) $$g = \sum_{n=0}^{\infty} \Phi_n, \quad \Phi_n \in E_{p,\alpha}^{2^n}, \quad \|\Phi_n\|_{p,\alpha} \leq c_6 2^{-n(r-m)} (\|f\|_{p,\alpha} + A)$$

with $c_6 = c_3 c_5$
Take an arbitrary number $h > 0$. Since the difference operator Δ_h^k is continuous in $L_{p,\alpha}$, the equality

$$\Delta_h^k g = \sum_{n=0}^{\infty} \Delta_h^k \Phi_n$$

holds in $L_{p,\alpha}$. Take a non-negative integer N such that

(9) $$2^{-N} \leq h \leq 2^{-(N-1)}$$

(if $h \geq 1$, then (9) contains only the left-hand inequality). we have

(10) $$\Delta_h^k g = \sum_{n=0}^{N-1} \Delta_h^k \Phi_n + \sum_{n=N}^{\infty} \Delta_h^k \Phi_n$$

(if N=0, then (10) contains only the second sum). To estimate the terms in (10), we proceed as follows. for $n \leq N-1$ we use inequalities (1), (8) and (9) and obtain that

$$\|\Delta_h^k \Phi_n\|_{p,\alpha} \leq c_4 (2^n h)^k \|\Phi_n\|_{p,\alpha} \leq c_4 c_6 (\|f\|_{p,\alpha} + A) 2^{n(m+k-r)} 2^{-(N-1)k}$$

Again using (9), we obtain that

$$
\begin{aligned}
\|\sum_{n=0}^{N-1}\Delta_h^k\Phi_n\|_{p,\alpha} &\le \frac{c_4c_6(\|f\|_{p,\alpha}+A)}{2^{k(N-1)}}\sum_{n=0}^{N-1}2^{(m+k-r)n}\\
&= \frac{c_4c_6(\|f\|_{p,\alpha}+A)}{2^{k(N-1)}}\frac{2^{(m+k-r)N}-1}{2^{m+k-r}-1}\\
&\le c_7(\|f\|_{p,\alpha}+A)h^{r-m}
\end{aligned}
\tag{11}
$$

with $c_7=\frac{2^k}{2^{m+k-r}-1}c_4c_6$

For $n\ge N$ we use the obvious inequality $\|\Delta_h^k\Phi_n\|_{p,\alpha}\le 2^k\|\Phi_n\|_{p,\alpha}$. We have

$$
\begin{aligned}
\|\sum_{n=N}^{\infty}\Delta_h^k\Phi_n\|_{p,\alpha} &\le 2^kc_6(\|f\|_{p,\alpha}+A)\sum_{n=N}^{\infty}2^{-(r-m)n}\\
&= 2^kc_6(\|f\|_{p,\alpha}+A)2^{-N(r-m)}(1-2^{m-r})^{-1}\\
&\le c_8(\|f\|_{p,\alpha}+A)h^{r-m}
\end{aligned}
\tag{12}
$$

with $c_8=c_62^k(1-2^{m-r})^{-1}$

It follows from (11) and (12) that $\|\Delta_h^kg\|_{p,\alpha}\le c_9(\|f\|_{p,\alpha}+A)h^{r-m}$, whence

$$\omega_k(g,\delta)_{p,\alpha}\le c_9(\|f\|_{p,\alpha}+A)\delta^{r-m},\quad \delta>0$$

and

$$h_p^r(f)\le c_9(\|f\|_{p,\alpha}+A)$$

with $c_9=c_7+c_8$

Hence, $f\in H_p^{r,\alpha}$ and $\|f\|_{H_p^{r,\alpha}}\le C(\|f\|_{p,\alpha}+A)$

with $C=c_9+1$

The theorem is proved.

Let

$$\widetilde{h}_p^r(f):=\sup_{\sigma\ge1}\sigma^rE_\sigma(f)_{p,\alpha}$$

It follows from Theorem (3.2) that $f\in L_{p,\alpha}$ belongs to $H_{p,\alpha}^r$ if and only if $\widetilde{h}_p^r(f)<\infty$, and the norm in $H_{p,\alpha}^r$ is equivalent to the norm

$$\|f\|_{p,\alpha}+\widetilde{h}_p^r(f)$$

In paticular, if k and m are such that $m<r<k+m$, then the space $H_{p,\alpha}^r$ coincide and their norm are equivalent.

Let $1\le q\le\infty$, $r>0$ and let k,m be non-negative integers such that $m<r<$

$k+m$. As in [3], we say that a function f belongs to the Nikol'skii-Besov class $B^r_{p,q,\alpha}$ if $f, D_\alpha f, D^2_\alpha f, ..., D^m_\alpha f \in L_{p,\alpha}$ and the seminorm

$$b^r_{p,q}(f) = \begin{cases} \left(\int_0^{+\infty} \frac{\omega_k(D^m_\alpha f,\delta)^q_{p,\alpha}}{\delta^{(r-m)q}} \frac{d\delta}{\delta}\right)^{\frac{1}{q}} & \text{if } q<\infty \\ \sup\limits_{\delta>0} \frac{\omega_k(D^m_\alpha f,\delta)_{p,\alpha}}{\delta^{(r-m)}} & \text{if } q=\infty \end{cases}$$

is finite, $B^r_{p,q,\alpha}$ is a banach space with norm

(13)
$$\|f\|_{B^r_{p,q,\alpha}} := \|f\|_{p,\alpha} + b^r_{p,q}(f)$$

Note that $B^r_{p,\infty,\alpha} = H^r_{p,\alpha}$.
The following theorem provides a description of the space $B^r_{q,k,m}$ in terms of the best approximation by the functions in $E^\sigma_{p,\alpha}$.

Theorem 3.3.
Let $a>1$ be an arbitrary number (we can take, for exemple, a=2).
Then $f \in L_{p,\alpha}$ belongs to $B^r_{p,q,\alpha}$ if and only if the seminorm

$$\widetilde{b}^r_{p,q}(f) = \begin{cases} \left(\sum\limits_{n=0}^{\infty} a^{nrq} E_{a^n}(f)^q_{p,\alpha}\right)^{\frac{1}{q}} & \text{if } q<\infty \\ \sup\limits_{n=0,1,2...} a^{nr} E_{a^n}(f)_{p,\alpha} & \text{if } q=\infty \end{cases}$$

is finite, In this case the norm (13) in $B^r_{p,q,\alpha}$ is equivalent to the norm

$$\|f\|_{p,\alpha} + \widetilde{b}^r_{p,q}(f)$$

Proof: This proof is a simple adaptation of the proof in ?5.6 in [3].
If $q=\infty$, Theorem (3.3) follows from Theorem (3.2).
If $1 \le q < \infty$, we shall say that a function $f(t)$ belongs to the space ${}^jB^r_{p,q,\alpha}$, j=1,2,3,4, if $f \in L_{p,\alpha}$ and the seminorm ${}^jb^r_{p,q}(f)$ is finite, where

$${}^1b^r_{p,q}(f) := b^r_{p,q}(f)$$

$${}^2b^r_{p,q}(f) = \left(\int_0^a \frac{\omega_k(D^m_\alpha f,\delta)^q_{p,\alpha}}{\delta^{(r-m)q}} \frac{d\delta}{\delta}\right)^{\frac{1}{q}}$$

$${}^3b^r_{p,q}(f) := \widetilde{b}^r_{p,q}(f)$$

$${}^4b^r_{p,q}(f) = \inf\left(\sum_{n=0}^{\infty} a^{nrq}\|\phi_n\|^q_{p,\alpha}\right)^{\frac{1}{q}}$$

where the infimum is taken over all representations of f in the form of series

$$f = \sum_{n=0}^{\infty} \phi_n, \quad \phi_n \in E_{p,\alpha}^{a^n}$$

convergent in $L_{p,\alpha}$

The ${}^jB_{p,q,\alpha}^r$ are banach spaces with respect to the norms

$$\|f\|_{{}^jB_{p,q,\alpha}^r} := \|f\|_{p,\alpha} +^j b_{p,q}^r(f)$$

For brevity we use the notation ${}^jB := {}^jB_{p,q,\alpha}^r$. the expression $X_1 \hookrightarrow X_2$ means that the Banach space X_1 is embedded in the Banach space X_2. the theorem will be proved if we can prove that

$${}^1B \hookrightarrow {}^2B \hookrightarrow {}^3B \hookrightarrow {}^4B \hookrightarrow {}^1B$$

The embedding ${}^1B \hookrightarrow {}^2B$ is obvious.

Now suppose that $f \in {}^2B$, using the inequalities $E_{a^n}(f)_{p,\alpha} \leq E_{a^t}(f)_{p,\alpha}$ and $a^{nr} \leq a^r a^{tr}$ for all $t \in [n-1, n]$ $(n \in \mathbb{N})$ and Lemma (2.2) we obtain the estimate

$$\begin{aligned}
{}^3b_{p,q}^r(f) &= \left(\sum_{n=0}^{\infty} a^{nrq} E_{a^n}(f)_{p,\alpha}^q \right)^{\frac{1}{q}} \\
&\leq a^r \left(\int_{-1}^{+\infty} a^{trq} E_{a^t}(f)_{p,\alpha}^q dt \right)^{\frac{1}{q}} \\
&\leq a^r c_2 \left(\int_{-1}^{+\infty} \frac{\omega_k(D_\alpha^m f, \frac{1}{a^t})_{p,\alpha}^q}{a^{t(m-r)q}} dt \right)^{\frac{1}{q}} \\
&\leq \frac{a^r c_2}{\ln a} \left(\int_0^a \frac{\omega_k(D_\alpha^m f, \delta)_{p,\alpha}^q}{\delta^{(r-m)q}} \frac{d\delta}{\delta} \right)^{\frac{1}{q}} \\
&\ll {}^2b_{p,q}^r(f)
\end{aligned}$$

Hence

$$\|f\|_{{}^3B} \ll \|f\|_{{}^2B}$$

and we have proved that

$${}^2B \hookrightarrow {}^3B$$

The expression $A \ll B$ has to be understood in the sense $A \leq cB$, where c is a constant

Now suppose that $f \in {}^3B$. let $\psi_n \in E_{p,\alpha}^{a^n}$ such that

$$\|f - \psi_n\|_{p,\alpha} \leq 2E_n \quad n = 0, 1, ...$$

where $E_n := E_{a^n}(f)_{p,\alpha}$
and put

$$\phi_0 = \psi_0, \quad \phi_n = \psi_n - \psi_{n-1} \quad n = 1, 2, ...$$

Then in the sense of $f \in L_{p,\alpha}$

$$f = \sum_{n=0}^{\infty} \phi_n$$

Because it follows from the finiteness of ${}^3b^r_{p,q}(f)$ that $E_{a^n}(f)_{p,\alpha} \to 0$ as $n \to \infty$
Further

$$\begin{aligned} \|\phi_0\|_{p,\alpha} &\leq \|f\|_2 + 2E_0 \\ \|\phi_n\|_{p,\alpha} &\leq 2E_n + 2E_{n-1} \leq 4E_{n-1}, \quad n \geq 1 \end{aligned}$$

Hence, by the inequality $\left(\frac{x+y}{2}\right)^q \leq \frac{1}{2}(x^q + y^q)$ for $x, y \geq 0$, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} a^{nrq}\|\phi_n\|^q_{p,\alpha} &\leq (\|f\|_{p,\alpha} + 2E_0)^q + 2^{2q}\sum_{n=1}^{\infty} a^{nrq}E^q_{n-1} \\ &\leq 2^{q-1}\|f\|^q_{p,\alpha} + 2^{2q-1}E^q_0 + 2^{2q}a^{rq}\sum_{n=0}^{\infty} a^{nrq}E^q_n \\ &\leq 2^{q-1}\|f\|^q_{p,\alpha} + 2^{2q+1}a^{rq}\sum_{n=0}^{\infty} a^{nrq}E^q_n \end{aligned}$$

it follows by the inequality $(x+y)^{\frac{1}{q}} \leq x^{\frac{1}{q}} + y^{\frac{1}{q}}$ that

$$\left(\sum_{n=0}^{\infty} a^{nrq}\|\phi_n\|^q_{p,\alpha}\right)^{\frac{1}{q}} \ll \|f\|_{p,\alpha} + \left(\sum_{n=0}^{\infty} a^{nrq}E^q_n\right)^{\frac{1}{q}}$$

and we have proved that

$${}^3B \hookrightarrow {}^4B$$

Suppose now that $f \in {}^4B$ and $f = \sum_{n=0}^{\infty} \phi_n, \quad \phi_n \in E^{a^n}_{p,\alpha}$
From Lemma (3.1) and the obvious inequality $\|\Delta^k_h D^m_\alpha \phi_n\|_{p,\alpha} \leq 2^k\|D^m_\alpha \phi_n\|_{p,\alpha}$, we

have

$$\begin{aligned}\|\Delta_h^k D_\alpha^m f\|_{p,\alpha} &\leq \sum_{n=0}^{N-1}\|\Delta_h^k D_\alpha^m \phi_n\|_{p,\alpha} + \sum_{n=N}^{\infty}\|\Delta_h^k D_\alpha^m \phi_n\|_{p,\alpha} \\ &\leq c_4 h^k \sum_{n=0}^{N-1} a^{nk}\|D_\alpha^m \phi_n\|_{p,\alpha} + 2^k \sum_{n=N}^{\infty}\|D_\alpha^m \phi_n\|_{p,\alpha} \\ &\leq c_3 c_4 h^k \sum_{n=0}^{N-1} a^{n(k+m)}\|\phi_n\|_{p,\alpha} + c_3 2^k \sum_{n=N}^{\infty} a^{nm}\|\phi_n\|_{p,\alpha}\end{aligned}$$

Hence

$$\begin{aligned}\omega_k(D_\alpha^m f, a^{-N})_{p,\alpha} &= \sup_{0<h\leq a^{-N}}\|\Delta_h^k D_\alpha^m f\|_{p,\alpha} \\ &\leq c_3 c_4 a^{-Nk}\sum_{n=0}^{N-1} a^{n(k+m)}\|\phi_n\|_{p,\alpha} + c_3 2^k \sum_{n=N}^{\infty} a^{nm}\|\phi_n\|_{p,\alpha}\end{aligned}$$

We shall estimate ${}^1b_{p,q}^r(f)$.we have

$$\begin{aligned}\int_0^1 \delta^{-1-q(r-m)}\omega_k(D_\alpha^m f,\delta)_{p,\alpha}^q d\delta &= \ln a\int_0^{+\infty} a^{nq(r-m)}\omega_k(D_\alpha^m f, a^{-n})_{p,\alpha}^q dn \\ &= \ln a\sum_{N=0}^{+\infty}\int_N^{N+1} a^{nq(r-m)}\omega_k(D_\alpha^m f, a^{-n})_{p,\alpha}^q dn \\ &\leq \ln a\sum_{N=0}^{+\infty} a^{q(N+1)(r-m)}\omega_k(D_\alpha^m f, a^{-N})_{p,\alpha}^q \int_N^{N+1} dn \\ &= a^{q(r-m)}\ln a\sum_{N=0}^{+\infty} a^{Nq(r-m)}\omega_k(D_\alpha^m f, a^{-N})_{p,\alpha}^q \\ &\ll I_1 + I_2\end{aligned}$$

Where

$$I_1 = \sum_{N=0}^{+\infty} a^{Nq(r-m-k)}\left(\sum_{n=0}^{N} a^{n(k+m)}\|\phi_n\|_{p,\alpha}\right)^q$$

$$I_2 = \sum_{N=0}^{+\infty} a^{Nq(r-m)}\left(\sum_{n=N}^{\infty} a^{nm}\|\phi_n\|_{p,\alpha}\right)^q$$

The inequalities (see [3, p.221])

$$\sum_{N=0}^{+\infty} x^{-q\beta N}\left(\sum_{n=0}^{N} y_n\right)^q \ll \sum_{N=0}^{+\infty} x^{-q\beta n} y_n^q$$

$$\sum_{N=0}^{+\infty} x^{q\beta N}\left(\sum_{n=N}^{\infty} y_n\right)^q \ll \sum_{n=0}^{+\infty} x^{q\beta n} y_n^q$$

(where $x > 1$ and $y_n > 0$) allow us to obtain

$$I_1 \ll \sum_{n=0}^{+\infty} a^{rqn} \|\phi_n\|_{p,\alpha}^q \quad \text{and} \quad I_2 \ll \sum_{n=0}^{+\infty} a^{rqn} \|\phi_n\|_{p,\alpha}^q$$

We have proved that

$$\int_0^1 \delta^{-1-q(r-m)} \omega_k(D_\alpha^m f, \delta)_{p,\alpha}^q d\delta \ \ll \ \sum_{n=0}^{+\infty} a^{rqn} \|\phi_n\|_{p,\alpha}^q$$

Further , putting $\frac{1}{q'} + \frac{1}{q} = 1$, we get from Lemma (2.1)

$$\begin{aligned} \|D_\alpha^m f\|_{p,\alpha} &\le c_1 \sum_{n=0}^{+\infty} a^{nm} \|\phi_n\|_{p,\alpha} = c_1 \sum_{n=0}^{+\infty} a^{-n(r-m)} a^{nr} \|\phi_n\|_{p,\alpha} \\ &\le c_1 \left(\sum_{n=0}^{+\infty} a^{-n(r-m)q'} \right)^{\frac{1}{q'}} \left(\sum_{n=0}^{+\infty} a^{nrq} \|\phi_n\|_{p,\alpha}^q \right)^{\frac{1}{q}} \\ &\ll \left(\sum_{n=0}^{+\infty} a^{nrq} \|\phi_n\|_{p,\alpha}^q \right)^{\frac{1}{q}} \end{aligned}$$

Then, since the function $\delta^{-1-q(r-m)}$ is integrable on $[1, +\infty[$ and

$$\omega_k(D_\alpha^m f, \delta)_{p,\alpha} \ll \|D_\alpha^m f\|_{p,\alpha}$$

Then

$$\begin{aligned} \int_1^{+\infty} \delta^{-1-q(r-m)} \omega_k(D_\alpha^m f, \delta)_{p,\alpha}^q d\delta &\ll \|D_\alpha^m f\|_{p,\alpha}^q \int_1^{+\infty} \delta^{-1-q(r-m)} d\delta \\ &\ll \sum_{n=0}^{+\infty} a^{nrq} \|\phi_n\|_{p,\alpha}^q \end{aligned}$$

Finally

$$\int_0^{+\infty} \delta^{-1-q(r-m)} \omega_k(D_\alpha^m f, \delta)_{p,\alpha}^q d\delta \ll \sum_{n=0}^{+\infty} a^{nrq} \|\phi_n\|_{p,\alpha}^q$$

and we have proved that

$$^4B \hookrightarrow {}^1B$$

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