Asia Pacific Journal of Mathematics, Vol. 2, No. 1 (2015), 54-68 ISSN 2357-2205

ON SOLVABILITY OF FRACTIONAL COUPLED HYBRID SYSTEMS

YOUSEF GHOLAMI* AND KAZEM GHANBARI

Department of Applied Mathematics,
Sahand University of Technology, P. O. Box: 51335-1996, Tabriz, IRAN
*Corresponding author

Abstract. Of concern is studying solvability results for multiple positive solutions of a certain fractional order coupled hybrid system. To this aim we will apply the generalized hybrid fixed point theorem for simplicity and 2-D Leray-Schauder fixed point theorem for multiplicity results.
2010 Mathematics Subject Classification. 34A08, 34A38, 34B15, 34B18.
Key words and phrases. Fractional derivatives and integrals, Coupled system, Fixed point theorem, Positive solutions.

1. Introduction

Fractional calculus is the theory of integrals and derivatives of arbitrary order, that generalizes the integer order differentiation and multi-fold integrals. Fractional differential calculus is known as a reliable mathematical tool for description of real world phenomena more accurate than ordinary differential calculus specially in combination with other related sciences such as biological, medical and engineering sciences. On the other hand parallel with applicability of fractional differential calculus, one can observe the great theoretical developments that led to the extensive research fields and generalized theory of fractional calculus.
In this paper, we consider fractional order coupled hybrid system

$$\begin{cases} \lambda_1 D^{\alpha}_{0^+}(u(t)-f(t,u(t)))-\lambda_2 u(t)=g_1(t,u(t),v(t)),\ 1<\alpha\leq 2, \\ \\ \mu_1 D^{\beta}_{0^+}(v(t)-f(t,v(t)))-\mu_2 v(t)=g_2(t,u(t),v(t)),\ 1<\beta\leq 2, \\ \\ u(0)-f(0,u(0))=\gamma_1\left(u(1)-\lambda_1 f(1,u(1))\right),\ u'(0)=f'(0,u(0)), \\ v(0)-f(0,v(0))=\gamma_2\left(v(1)-\mu_1 f(1,v(1))\right),\ v'(0)=f'(0,(v(0)), \end{cases} \tag{1.1}$$

where $t \in \mathcal{J} = (0,1)$ and $D^{\alpha}_{0^+}$ denotes the fractional Caputo derivative of order $\alpha > 0$. Additionally assume that $\lambda_i, \mu_i \in \mathbb{R}^+,\ \gamma_i \in (0, 1/2\delta)$ for $i = 1, 2$. Notice that denoted by δ an upper bound for the generalized Mittag-Leffler function $E_{\alpha,1}(t)$ on finite interval $\mathcal{J}$ that will be defined in the next section.

Some general assumptions which will be valid throughout this paper, are as follows:

(C_1) $f \in C^2\left(\mathcal{J} \times \mathbb{R}, \mathbb{R}^+\right)$ such that

$$|f(t,u_1) - f(t,u_2)| \le \frac{L_1|u_1 - u_2|}{2\max\{\lambda_1, \mu_1\}\left(L_2 + |u_1 - u_2|\right)}, \quad t \in \mathcal{J},\ u_1, u_2 \in \mathbb{R},$$

$$0 < L_1 \le L_2,\ \sup f(t,u) = \rho, \quad (t,u) \in \mathcal{J} \times \mathbb{R}.$$

(C_2) $g_1, g_2 \in C\left(\mathcal{J} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}^+\right)$ and

$$\sup g_1(t,u,v) = \theta_1,\ \sup g_2(t,u,v) = \theta_2, \quad (t,u,v) \in \mathcal{J} \times \mathbb{R}^2.$$

2. Technical Overview

In this paper we will use only the left-sided fractional operators. Thus we begin overview of fractional calculus with standard fractional order operators as follows.

Definition 1. [7] The fractional Riemann-Liouville integral of order $\alpha > 0$ for a function $f \in \mathcal{L}^1(\mathcal{J})$ is defined by

$$\left(I^{\alpha}_{0^+} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,\ t \in \mathcal{J}, \tag{2.1}$$

provided that the right hand side is point-wise defined on $\mathcal{J}$.

Definition 2. [7] The fractional Caputo derivative of order $\alpha > 0$ for a function $f \in \mathcal{C}^{(n)}(\mathcal{J})$ is defined by

$$\left(D^{\alpha}_{0^+} f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,\ t \in \mathcal{J}, \tag{2.2}$$

provided that the right hand side is point-wise defined on $\mathcal{J}$ and $n = [\alpha] + 1$.

Because we will follow fixed point techniques for obtaining claimed existence results, so we must transform fractional differential equations in hybrid system (1.1) to the corresponding integral equations. To this aim we shall apply Laplace transforms. So Laplace transform of fractional Caputo derivatives defines as below.

Remark 1.

$$\mathfrak{L}[D^{\alpha}_{0^+} u(t); s] = s^{\alpha}\mathfrak{L}[u(t); s] - \sum_{k=0}^{n-1} s^{\alpha-k-1} u^{(k)}(0),\ n-1 < \alpha \le n, \tag{2.3}$$

where $\mathfrak{L}[f(t); s]$ denotes the Laplace transform of function f.

Definition 3. The generalized Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\quad z\in\mathbb{C},\ \alpha>0,\ \beta>0. \tag{2.4}$$

Lemma 1. [8] *The generalized Mittag-Leffler function $E_{\alpha,\beta}(z)$ defined by (2.4) has the following properties:*

(A_1) $\frac{d^k}{dz^k}\left(z^{\beta-1}E_{\alpha,\beta}(\lambda z^\alpha)\right)=z^{\beta-k-1}E_{\alpha,\beta-k}(\lambda z^\alpha),\ \lambda,z\in\mathbb{C}\quad k=1,2,3,\dots$.

(A_2) $\mathfrak{L}\left[z^{\alpha k+\beta-1}E^{(k)}_{\alpha,\beta}(\pm\lambda z^\alpha);s\right]=\dfrac{k!s^{\alpha-\beta}}{(s^\alpha\mp\lambda)^{k+1}},\quad Re(s)>|\lambda|^{1/\alpha}.$

Now by means of above preparatory results, we can transform fractional differential system (1.1) to the corresponding fractional integral system as follows.

Lemma 2. *Assume that $h(t)\in C(\mathcal{J})$ and condition (C_1) be satisfied. Then fractional boundary value problem*

$$\begin{cases}\lambda_1 D^{\alpha}_{0^+}(u(t)-f(t,u(t)))-\lambda_2 u(t)=h(t),\ t\in\mathcal{J},\\ \\ u(0)-f(0,u(0))=\gamma_1\left(u(1)-\lambda_1 f(1,u(1))\right),\\ \qquad u'(0)=f'(0,u(0)),\end{cases} \tag{2.5}$$

reduces to the Volterra type integral equation

$$u(t)=H_{\lambda_1}(t,u(t))+\int_0^1 G_{\lambda_1,\lambda_2,\gamma_1}(t,s)h(s)ds, \tag{2.6}$$

such that

$$G_{\lambda_1,\lambda_2,\gamma_1}(t,s)=\frac{1}{\lambda_1}\begin{cases}G_{1(\lambda_1,\lambda_2,\gamma_1)}(t,s)\ ;\ 0\le s\le t\le 1,\\ \\ G_{2(\lambda_1,\lambda_2,\gamma_1)}(t,s)\ ;\ 0\le t\le s\le 1,\end{cases} \tag{2.7}$$

where

$$\begin{aligned}G_{1(\lambda_1,\lambda_2,\gamma_1)}(t,s)&=(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\frac{\lambda_2}{\lambda_1}(t-s)^\alpha\right)\\ &\quad+\frac{\gamma_1 E_{\alpha,1}\left(\frac{\lambda_2}{\lambda_1}t^\alpha\right)}{1-\gamma_1 E_{\alpha,1}\left(\frac{\lambda_2}{\lambda_1}\right)}(1-s)^{\alpha-1}E_{\alpha,\alpha}\left(\frac{\lambda_2}{\lambda_1}(1-s)^\alpha\right),\end{aligned} \tag{2.8}$$

and

$$G_{2(\lambda_1,\lambda_2,\gamma_1)}(t,s)=\frac{\gamma_1 E_{\alpha,1}\left(\frac{\lambda_2}{\lambda_1}t^\alpha\right)}{1-\gamma_1 E_{\alpha,1}\left(\frac{\lambda_2}{\lambda_1}\right)}(1-s)^{\alpha-1}E_{\alpha,\alpha}\left(\frac{\lambda_2}{\lambda_1}(1-s)^\alpha\right), \tag{2.9}$$

also

$$H_{\lambda_1}(t,u(t)) = \lambda_1 f(t,u(t)) \tag{2.10}$$

Proof. Taking Laplace transform of fractional differential equation (2.5) gives us

$$\mathfrak{L}[u(t);s] = \lambda_1 \sum_{k=0}^{1} \frac{s^{\alpha-k-1}}{\lambda_1 s^\alpha - \lambda_2}[u(0) - f(0,u(0))]^{(k)} + \frac{\mathfrak{L}[h(t);s]}{\lambda_1 s^\alpha - \lambda_2} + \lambda_1 \mathfrak{L}[f(t,u(t));s].$$

Now by Laplace transform of convolution of two integrable functions f, g that is given by

$$\mathfrak{L}[f(t);s]\mathfrak{L}[g(t);s] = \mathfrak{L}[(f*g)(t);s] = \mathfrak{L}\left[\int_0^t f(t-\tau)g(\tau)d\tau;s\right],$$

and using (A_2) in Lemma 1, we conclude that

$$\begin{aligned} u(t) &= \lambda_1 f(t,u(t)) + [u^{'}(0) - f^{'}(t,u(0))]t^k E_{\alpha,k+1}\left(\frac{\lambda_2}{\lambda_1}t^\alpha\right) \\ &\quad + \frac{1}{\lambda_1}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\frac{\lambda_2}{\lambda_1}(t-s)^\alpha\right)h(s)ds \\ &\quad + [u(0) - f(0,u(0))]E_{\alpha,1}\left(\frac{\lambda_2}{\lambda_1}t^\alpha\right). \end{aligned} \tag{2.11}$$

Implementing the boundary conditions (2.5), we deduce that

$$u(0) - f(0,u(0)) = \frac{\dfrac{\gamma_1}{\lambda_1}\displaystyle\int_0^1 (1-s)^{\alpha-1}E_{\alpha,\alpha}\left(\frac{\lambda_2}{\lambda_1}(1-s)^\alpha\right)h(s)ds}{1 - \gamma_1 E_{\alpha,1}\left(\dfrac{\lambda_2}{\lambda_1}\right)}. \tag{2.12}$$

At last substitution $u(0) - f(0,u(0))$ obtained by (2.12) in (2.11), we find the desirable result as follows:

$$\begin{aligned} u(t) &= \lambda_1 f(t,u(t)) + \frac{1}{\lambda_1}\int_0^t \Bigg\{(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\frac{\lambda_2}{\lambda_1}(t-s)^\alpha\right) \\ &\quad + \frac{\gamma_1(1-s)^{\alpha-1}E_{\alpha,\alpha}\left(\frac{\lambda_2}{\lambda_1}(1-s)^\alpha\right)}{1-\gamma_1 E_{\alpha,1}\left(\frac{\lambda_2}{\lambda_1}\right)}E_{\alpha,1}\left(\frac{\lambda_2}{\lambda_1}t^\alpha\right)\Bigg\}h(s)ds \\ &\quad + \frac{\gamma_1}{\lambda_1}\int_t^1 \frac{(1-s)^{\alpha-1}E_{\alpha,\alpha}\left(\frac{\lambda_2}{\lambda_1}(1-s)^\alpha\right)}{1-\gamma_1 E_{\alpha,1}\left(\frac{\lambda_2}{\lambda_1}\right)}E_{\alpha,1}\left(\frac{\lambda_2}{\lambda_1}t^\alpha\right)h(s)ds \\ &= H_{\lambda_1}(t,u(t)) + \int_0^1 G_{\lambda_1,\lambda_2,\gamma_1}(t,s)h(s)ds. \end{aligned}$$

□

Lemma 3. *The Green's function $G_{\lambda_1,\lambda_2,\gamma_1}(t,s)$ defined by (2.7)-(2.9) and $H_{\lambda_1(t,u(t))}$ defined by (2.10), have the following properties:*

(Q_1) $G_{\lambda_1,\lambda_2,\gamma_1} \in C(\mathcal{J}\times\mathcal{J},\mathbb{R}^+)$ *and* $H_{\lambda_1} \in C(\mathcal{J}\times\mathbb{R},\mathbb{R}^+)$.

(Q_2)
$$G_{\lambda_1,\lambda_2,\gamma_1}(t,s) \le \frac{E_{\alpha,\alpha}\left(\frac{\lambda_2}{\lambda_1}\right)}{\lambda_1\left(1-\gamma_1 E_{\alpha,1}\left(\frac{\lambda_2}{\lambda_1}\right)\right)},$$
where $(t,s)\in\mathcal{J}^2$.

(Q_3) $H_{\lambda_1}(t,u(t)) \le \lambda_1\rho$, *where* $\rho = \sup\{f(t,u)|(t,u)\in\mathcal{J}\times\mathbb{R}\}$.

(Q_4) *There exist constant* $c_1 > 0$ *such that*
$$\min_{t\in\mathcal{J}_p} G_{\lambda_1,\lambda_2,\gamma_1}(t,s) \ge c_1 \sup_{t\in(0,1)} G_{\lambda_1,\lambda_2,\gamma_1}(t,s), \quad \mathcal{J}_p = [p,1-p],\ p\in(0,1/2).$$

Proof. properties $(Q_1)-(Q_3)$ are immediate by conditions $(C_1),(C_2)$. So we only prove the (Q_4).
According to condition (A_1) in Lemma 1 and considering the Green's function $G_{\lambda_1,\lambda_2,\gamma_1}(t,s)$ defined by (2.7)-(2.9), we conclude that both $G_{1(\lambda_1,\lambda_2,\gamma_1)}(t,s)$ and $G_{2(\lambda_1,\lambda_2,\gamma_1)}(t,s)$ are increasing with respect to the first variable t. Thus we have

$$M_1(s) = \sup_{t\in\mathcal{J}} G_{\lambda_1,\lambda_2,\gamma_1}(t,s) = G_{1(\lambda_1,\lambda_2,\gamma_1)}(1,s),\ s\in\mathcal{J}, \tag{2.13}$$

and

$$m_1(s) = \min_{t\in\mathcal{J}_p} G_{\lambda_1,\lambda_2,\gamma_1}(t,s) = G_{2(\lambda_1,\lambda_2,\gamma_1)}(p,s),\ s\in\mathcal{J}. \tag{2.14}$$

Setting

$$c_1 = \frac{m_1(s)}{M_1(s)} = \frac{\min_{t\in\mathcal{J}_p} G_{\lambda_1,\lambda_2,\gamma_1}(t,s)}{\sup_{t\in\mathcal{J}} G_{\lambda_1,\lambda_2,\gamma_1}(t,s)}, \tag{2.15}$$

shows us that

$$c_1 = \gamma_1 E_{\alpha,1}\left(\frac{\lambda_2}{\lambda_1}p^\alpha\right). \tag{2.16}$$

This completes the proof. □

Remark 2. Replacing λ_1 with μ_1 and λ_2 with μ_2 also γ_1 with γ_2, the Green's function $G_{\mu_1,\mu_2,\gamma_2}(t,s)$ corresponding to second differential equation of (1.1) will be obtained such that satisfies in Lemma 3.

Remark 3. Since $\gamma_1\in(0,1/2\delta)$ for $E_{\alpha,1}(\lambda_2/\lambda_1)\le\delta,\ t\in\mathcal{J},\ 1<\alpha<2$, then we conclude that $c_1\in(0,1/2)$.

Remark 4. Similar to the definition c_1 defined by (2.15), real constant c_2 defines as follows:

$$c_2 = \frac{m_2(s)}{M_2(s)} = \frac{\min\limits_{t\in\mathcal{J}_p} G_{\mu_1,\mu_2,\gamma_2}(t,s)}{\sup\limits_{t\in\mathcal{J}} G_{\mu_1,\mu_2,\gamma_2}(t,s)}. \tag{2.17}$$

Thus in the same way we conclude that $c_2 \in (0, 1/2)$.

In order to follow our research we need define the relevant Banach space. Therefore we begin with defining Banach space

$$E = \{u|u(t) \in C(\mathcal{J}, \mathbb{R})\}, \tag{2.18}$$

equipped with the famous sup-norm

$$\|u\|_E = \sup\{u(t)|\ u \in E,\ t \in \mathcal{J}\}. \tag{2.19}$$

Now we can define our desirable Banach space $(\mathfrak{B}, \|.\|_{\mathfrak{B}})$ as below

$$\mathfrak{B} = \{(u, v)|u, v \in E\}, \tag{2.20}$$

with norm

$$\|(u, v)\|_{\mathfrak{B}} = \|u\|_E + \|v\|_E. \tag{2.21}$$

Also the partial order of Banach space $\mathfrak{B}$ is given by

$$(u_1, v_1) \le (u_2, v_2),$$

provided that

$$u_1(t) \le u_2(t),\ v_1(t) \le v_2(t), \quad (u_1, u_2), (v_1, v_2) \in \mathfrak{B},\ t \in \mathcal{J}.$$

Let us define $S \subset \mathfrak{B}$ as

$$S = \left\{ (u, v) \in \mathfrak{B} \;\middle|\; u(t), v(t) \ge 0,\ t \in \mathcal{J},\ \begin{array}{c} \min\limits_{t\in\mathcal{J}_p} u(t) \ge c_1\|u\|_E, \\ \min\limits_{t\in\mathcal{J}_p} v(t) \ge c_2\|v\|_E \end{array} \right\}, \tag{2.22}$$

such that $c_1, c_2,\ \mathcal{J}_p$ defined in property (Q_4) in Lemma 3. Now we define the integral operators $T_1, T_2 : E \to E$ as follows.

Definition 4.

$$\begin{aligned} T_u(t) &= H_{\lambda_1}(t, u(t)) + \int_0^1 G_{\lambda_1,\lambda_2,\gamma_1}(t,s) g_1(s, u(s), v(s)) ds, \\ T_v(t) &= H_{\mu_1}(t, v(t)) + \int_0^1 G_{\mu_1,\mu_2,\gamma_2}(t,s) g_2(s, u(s), v(s)) ds, \end{aligned} \tag{2.23}$$

where $G_{\cdot,\cdot,\cdot}$, $H_{\cdot}$ defined by (2.7)-(2.9) and (2.10) respectively.

Definition 5. Let us define the operator $\mathfrak{T} : \mathfrak{B} \to \mathfrak{B}$ as below

$$\mathfrak{T}(u, v)(t) = (T_u, T_v)(t), \tag{2.24}$$

such that T_u, T_v defined by (2.23).

In the sequel, we are going to represent some definitions and fixed point theorems with their sufficient preparations that will be made our existence and multiplicity results basis.

Definition 6. [2] Let X be a normed vector space. A mapping $T : X \to X$ is said D-Lipschitzian, provided that there exist a continuous and nondecreasing function $\phi_T : \mathbb{R}^+ \to \mathbb{R}^+$ such that for $x, y \in X$

$$\|Tx - Ty\| \leq \phi_T\Big(\|x - y\|\Big), \quad \phi_T(0) = 0.$$

Remark 5. Every Lipschitzian mapping is D-Lipschitzian and if $\phi_T(r) < r$, then T is called nonlinear D-contraction on X with contraction function ϕ_T.

Remark 6. [3] Every nonlinear D-contraction implies D-Lipschitzian but the reverse implication may not hold.

Definition 7. [6] Let X be a normed space and suppose $S \subset X$. A finite set of N balls $B(x_n, \epsilon)$ with $x_n \in X$ and $\epsilon > 0$ is said to be a finite $\epsilon-$covering of S, provided that every element of S lies inside one of the balls $B(x_n, \epsilon)$, i.e.

$$S \subset \bigcup_{n=1}^{N} B(x_n, \epsilon).$$

The set of centers $\{x_n\}$ of a finite ϵ-covering is called a finite $\epsilon-$net for S.

Definition 8. [6] Let X be a normed space. A set $S \subset X$ is said to be a $Totally\ Bounded$ if and only if it has a finite $\epsilon-$covering for every $\epsilon > 0$.

Theorem 1 (Hausdorff compactness criteria)**.** [6] *Assume that X be a normed space. A set $S \subset X$ is compact if and only if it is closed and totally bounded.*

Theorem 2 (Dhage fixed point theorem)**.** [1] *Assume that K be closed convex and bounded subset of Banach space E. Let $A : E \to E$ and $B : K \to E$ be two operators with the following properties:*

(I) *A is nonlinear D-contraction.*
(II) *B is compact and continuous.*

(III) *The operator equation* $x = Ax + By$ *implies that* $x \in K$*, for all* $y \in K$.

Then the operator $A + B$ *has a fixed point in* K.

Theorem 3 (Leray-Schauder fixed point index)**.** [5] *Assume that* S *be a cone in Banach space* E *and* D *be an open bounded subset of* E *with* $D_S = D \cap S \neq \varnothing$ *and* $\overline{D_S} \neq S$*. Let* $T : D_S \to S$ *is a compact mapping such that* $x \neq Tx$*, for* $x \in \partial D_S$*. Then the following properties are satisfied:*

(P_1) *If* $\|Tx\| \leq \|x\|$ *for* $x \in \partial D_S$*, then* $i_S(T, D_S) = 1$.

(P_2) *If there exist* $e \in S \backslash \{0\}$ *such that* $x \neq Tx + \lambda e$ *for* $x \in \partial D_S$ *and* $\lambda > 0$*, then* $i_S(T, D_S) = 0$.

(P_3) *Assume that* D_0 *be an open subset of* E *such that* $\overline{D_0} \subset D_S$*. If* $i_S(T, D_S) = 1,\ i_S(T, D_{0,S}) = 0$*, then* T *has a fixed point in* $D_S \backslash D_{0,S}$*. The same result holds provided that* $i_S(T, D_S) = 0,\ i_S(T, D_{0,S}) = 1$.

In order to apply Theorem 3 we define the sets $S_{\varrho_1,\varrho_2},\ \Omega_{\varrho_1,\varrho_2}$ as follows

$$\begin{aligned}
S_{\varrho_1,\varrho_2} &= \left\{(u,v) \in S \middle| \ \|(u,v)\|_{\mathfrak{B}} \leq \varrho_1 + \varrho_2\right\}, \\
\Omega_{\varrho_1,\varrho_2} &= \left\{(u,v) \in S \middle| \ l_{\min}(u,v) < c(\varrho_1 + \varrho_2)\right\} \\
&= \left\{(u,v) \in S \middle| \ c\|(u,v)\|_{\mathfrak{B}} < l_{\min}(u,v) < c(\varrho_1 + \varrho_2)\right\},
\end{aligned} \tag{2.25}$$

where

$$l_{\min} : S \to [0, +\infty), \quad l_{\min}(u,v) = \min\{(u+v)(t) | t \in \mathcal{J}_p\},$$
$$c = \min\{c_1, c_2\}.$$

Lemma 4. [5] *The set* $\Omega_{\varrho_1,\varrho_2}$ *defined by (2.25) has the following properties:*

(R_1) $\Omega_{\varrho_1,\varrho_2}$ *is open with respect to* S.

(R_2) $S_{c(\varrho_1,\varrho_2)} \subset \Omega_{\varrho_1,\varrho_2} \subset S_{\varrho_1,\varrho_2}$.

(R_3) $(u,v) \in \partial\Omega_{\varrho_1,\varrho_2}$ *if and only if* $l_{\min}(u,v) = c(\varrho_1 + \varrho_2)$.

(R_4) *If* $(u,v) \in \partial\Omega_{\varrho_1,\varrho_2}$*, then* $c(\varrho_1, \varrho_2) \leq (u,v) \leq (\varrho_1, \varrho_2)$ *for* $t \in \mathcal{J}_p$.

Before beginning the last section of our investigation let us define some notation as follow:

$$\begin{aligned}
g_1^{c(\varrho_1,\varrho_2)} &= \min\left\{\frac{g_1(t,u,v)}{c(\varrho_1+\varrho_2)} \middle| t \in \mathcal{J}_p,\ u \in [c\varrho_1, \varrho_1],\ v \in [0,+\infty)\right\}, \\
{}_0g_1^{\varrho_1,\varrho_2} &= \sup\left\{\frac{g_1(t,u,v)}{\varrho_1+\varrho_2-2\xi_1} \middle| t \in \mathcal{J},\ u \in [0, \varrho_1 - \xi_1],\ v \in [0,+\infty)\right\}, \\
g_2^{c(\varrho_1,\varrho_2)} &= \min\left\{\frac{g_2(t,u,v)}{c(\varrho_1+\varrho_2)} \middle| t \in \mathcal{J}_p,\ u \in [0,+\infty),\ v \in [c\varrho_2, \varrho_2]\right\}, \\
{}_0g_2^{\varrho_1,\varrho_2} &= \sup\left\{\frac{g_2(t,u,v)}{\varrho_1+\varrho_2-2\xi_2} \middle| t \in \mathcal{J},\ u \in [0,+\infty),\ v \in [0, \varrho_2 - \xi_2]\right\},
\end{aligned} \tag{2.26}$$

where

$$\xi_1 = \lambda_1 \rho, \quad \xi_2 = \mu_1 \rho. \tag{2.27}$$

Also

$$\begin{aligned} \overline{m_1} &= \left(2\int_0^1 M_1(s)ds\right)^{-1}, \quad \overline{M_1} = \left(c\int_p^{1-p} M_1(s)ds\right)^{-1}, \\ \underline{m_2} &= \left(2\int_0^1 M_2(s)ds\right)^{-1}, \quad \underline{M_2} = \left(c\int_p^{1-p} M_2(s)ds\right)^{-1}, \end{aligned} \tag{2.28}$$

where $M_1(s), M_2(s)$ defined by (2.13),(2.17).

3. Main Results

Theorem 4. *Assume that the conditions $(C_1), (C_2)$ be satisfied. Then the fractional order multi term coupled hybrid system (1.1) has at least one positive solution on S_{ϱ_1,ϱ_2} defined by (2.25).*

Proof. We will divide the proof to the three steps as follows:
$Step.1.$ Considering the Definition 2.23 and Theorem 2, let us define

$$\begin{aligned} A_u(t) &= H_{\lambda_1} u(t), \\ B_u(t) &= \int_0^1 G_{\lambda_1,\lambda_2,\gamma_1}(t,s) g_1(s,u(s),v(s))ds, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} A_v(t) &= H_{\mu_1} v(t), \\ B_v(t) &= \int_0^1 G_{\mu_1,\mu_2,\gamma_2}(t,s) g_2(s,u(s),v(s))ds, \end{aligned} \tag{3.2}$$

According to conditions $C_1,\ C_2$ and Lemma 3, we have the following

$$|A_{u_1}(t) - A_{u_2}(t)| \leq \frac{L_{1A_u}|u_1 - u_2|}{2(L_{2A_u} + |u_1 - u_2|)} < |u_1 - u_2|,\ t \in \mathcal{J}.$$

So taking supremum on t we have

$$\|A_{u_1} - A_{u_2}\|_E \leq \psi_1(\|u_1, u_2\|_E), \quad u_1, u_2 \in E. \tag{3.3}$$

Equivalently A_u is nonlinear D-contraction with corresponding D-function

$$\psi_{A_u}(r) = \frac{L_{1A_u} r}{2(L_{2A_u} + r)}. \tag{3.4}$$

In the similar way we can show that A_v is nonlinear D-contraction with corresponding D-function

$$\psi_{A_v}(r) = \frac{L_{1A_v} r}{2(L_{2A_v} + r)}. \tag{3.5}$$

Thus if we consider the operator $\mathfrak{T}$ defined by (2.24) as

$$\mathfrak{T}(u,v)(t) = \begin{pmatrix} A_u \\ A_v \end{pmatrix}(t) + \begin{pmatrix} B_u \\ B_v \end{pmatrix}(t) = A_{u,v}(t) + B_{u,v}(t). \tag{3.6}$$

Therefore $A_{u,v}$ is nonlinear D-contraction with corresponding D-function

$$\phi_{A_{u,v}}(r) = 2\max\{\psi_{A_u}(r), \psi_{A_v}(r)\}.$$

This is end of $Step.1$.

$Step.2.$ In this step we must prove that $B_{u,v}$ defined by (3.1),(3.2) and (3.6) is completely continuous. To this goal we will use the Hausdorff compactness criterion in Theorem 1.

Obviously $S \subset \mathfrak{B}$ is a nonempty closed convex and bounded in $\mathfrak{B}$. Let us define

$$\begin{aligned} E_u &= \{u \in E | u(t) \geq 0, t \in \mathcal{J}\}, \\ E_v &= \{v \in E | v(t) \geq 0, t \in \mathcal{J}\}. \end{aligned} \tag{3.7}$$

Clearly E_u, E_v are closed. So both of them are Banach spaces with the norm of E. Also $u(t), v(t)$ are equicontinuous on $\mathcal{J}$. Then by means of $Arzela - Ascoli$ theorem we conclude that E_u, E_v are compact. Hence Theorem 1 ensures that E_u, E_v are totally bounded. Thus Definition 7 implies that there exist two finite $\epsilon-$coverings as

$$\mathfrak{U}_\epsilon(u_i),\ \mathfrak{U}_\epsilon(v_j), \quad i = 1,2,3,...,l_1,\ j = 1,2,3,...,l_2,$$

such that

$$\begin{aligned} E_u &\subset \bigcup_{i=1}^{l_1} \mathfrak{U}_\epsilon(u_i), \\ E_v &\subset \bigcup_{j=1}^{l_2} \mathfrak{U}_\epsilon(v_j), \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} \mathfrak{U}_\epsilon(u_i) &= \{u \in E_u |\ \|u - u_i\|_E < \epsilon\}, \\ \mathfrak{U}_\epsilon(v_j) &= \{v \in E_v |\ \|v - v_j\|_E < \epsilon\}. \end{aligned} \tag{3.9}$$

Define

$$E_{ij} = \{(u,v) \in E_u \times E_v | u \in \mathfrak{U}_\epsilon(u_i), v \in \mathfrak{U}_\epsilon(v_j)\}.$$

It is easy to see that $S \subset E_u \times E_v \subset \bigcup_{i,j} E_{ij},\ 1 \leq i \leq l_1,\ 1 \leq j \leq l_2$.

In fact if we take $(u_{ij}, v_{ij}) \in E_{ij}$, then $E_u \times E_v$ can be covered by finite $4\epsilon-$covering

$$\mathfrak{U}_{4\epsilon}(u_{ij}, v_{ij}) = \left\{(u,v) \in E_u \times E_v \middle|\ \|(u,v) - (u_{ij}, v_{ij})\|_{\mathfrak{B}} < 4\epsilon\right\}.$$

In other means for every $(u,v)\in E_u\times E_v$, there exist indices i,j such that

$$u\in\mathfrak{U}_\epsilon(u_i),\ v\in\mathfrak{U}_\epsilon(v_j).$$

Therefore

$$\begin{aligned}|u-u_{ij}|&\le|u-u_i|+|u_i-u_{ij}|<\epsilon+\epsilon=2\epsilon,\\|v-v_{ij}|&\le|v-v_i|+|v_i-v_{ij}|<\epsilon+\epsilon=2\epsilon.\end{aligned}\tag{3.10}$$

(3.10) implies that $\|(u,v)-(u_{ij},v_{ij})\|_{\mathfrak{B}}<4\epsilon$. So our claim has been proved that S has a finite $4\epsilon-$covering. Hence using Theorem 1 we conclude that S is compact. Since S_{ϱ_1,ϱ_2} is a closed subset of S, then S_{ϱ_1,ϱ_2} also is compact.
Now let us return to definition of $B_{u,v}(t)$ in (3.6). According to conditions $(C_1)-(C_3)$ and Lemma 3, we know that $B_{u,v}$ is continuous on S_{ϱ_1,ϱ_2}. Thus $B_{u,v}(S_{\varrho_1,\varrho_2})$ is relatively compact and consequently $B_{u,v}$ is completely continuous on S_{ϱ_1,ϱ_2}. This completes the $Step.2$.
$Step.3$. We begin with showing that $T_u\le\varrho_1, T_v\le\varrho_2$, for $u,v\in E$. Using Definition 2.23 and Lemma 3 also conditions $(C_1)-(C_3)$, we have

$$\begin{aligned}T_u(t)&=H_{\lambda_1}(t,u(t))+\int_0^1G_{\lambda_1,\lambda_2,\gamma_1}(t,s)g_1(s,u(s),v(s))ds,\\&\le\lambda_1\rho+\frac{\theta_1}{\lambda_1\left(1-\gamma_1E_{\alpha,1}\left(\frac{\lambda_2}{\lambda_1}\right)\right)}=\varrho_1.\end{aligned}\tag{3.11}$$

Similarly we can prove that

$$\begin{aligned}T_v(t)&=H_{\mu_1}(t,v(t))+\int_0^1G_{\mu_1,\mu_2,\gamma_2}(t,s)g_2(s,u(s),v(s))ds,\\&\le\mu_1\rho+\frac{\theta_2}{\mu_1\left(1-\gamma_2E_{\beta,1}\left(\frac{\mu_2}{\mu_1}\right)\right)}=\varrho_2.\end{aligned}\tag{3.12}$$

So considering (3.11) and (3.12) we conclude that

$$\|T_u\|_E\le\varrho_1,\ \|T_v\|_E\le\varrho_2,\quad u,v\in E.\tag{3.13}$$

Finally

$$\|\mathfrak{T}(u,v)\|_{\mathfrak{B}}=\|T_u\|_E+\|T_v\|_E\le\varrho_1+\varrho_2,\quad (u,v)\in S.\tag{3.14}$$

Hence we have been proved that $\mathfrak{T}(S_{\varrho_1,\varrho_2})\subset S_{\varrho_1,\varrho_2}$. Indeed we have been showed that $\mathfrak{T}$ is compact on S_{ϱ_1,ϱ_2}. In other words if we consider

$$\begin{aligned}x_1&=A_ux_1+B_uy_1,\\x_2&=A_vx_2+B_vy_2,\end{aligned}\tag{3.15}$$

then (3.14) enures that $(x_1, x_2) \in S_{\varrho_1,\varrho_2}$ for all $(y_1, y_2) \in S_{\varrho_1,\varrho_2}$. The $Step.3.$ completed. Since all conditions of Theorem 2 are satisfied, then operator $\mathfrak{T}$ defined by (2.24) or equivalently by (3.6) has a fixed point in S_{ϱ_1,ϱ_2}. In other means the fractional order coupled system (1.1) has one positive solution in S_{ϱ_1,ϱ_2}. The proof is complete. □

In the following, we state and prove two technical lemmas that will support the 2-D $Leray - Schauder$ base theorem for the multiplicity of positive solutions of coupled system (1.1).

Lemma 5. *Assume that conditions $(C_1), (C_2)$ and the conditions*

$$g_1^{c(\varrho_1,\varrho_2)} \geq \overline{M_1},\ u \neq T_u, \quad (u, v) \in \partial\Omega_{\varrho_1,\varrho_2},$$

(3.16)

$$g_2^{c(\varrho_1,\varrho_2)} \geq \underline{M_2},\ v \neq T_v, \quad (u, v) \in \partial\Omega_{\varrho_1,\varrho_2},$$

hold. Then $i_S(\mathfrak{T}, \Omega_{\varrho_1,\varrho_2}) = 0$.

Proof. Suppose that $e(t) = (1, 1)$ for $t \in \mathcal{J}$. Thus obviously $e \in S$. We claim that

$$(u, v) \neq \mathfrak{T}(u, v) + \eta e, \quad (u, v) \in \partial\Omega_{\varrho_1,\varrho_2},\ \eta > 0.$$

Otherwise there exist a $(u_0, v_0) \in \partial\Omega_{\varrho_1,\varrho_2}$ and $\eta > 0$ such that $u_0 = T_{u_0}(t) + \eta$ and $v_0 = T_{v_0}(t) + \eta$. Using condition (3.16) and (R_3) in Lemma 4 and $(Q_1), (Q_4)$ in Lemma 3 and (2.28), we conclude that for $t \in \mathcal{J}_p$,

$$u_0(t) = H_{\lambda_1}(t, u_0(t)) + \int_0^1 G_{\lambda_1,\lambda_2,\gamma_1}(t, s) g_1(s, u_0(s), v(s))ds + \eta.$$

Consequently we have

$$\begin{aligned}\min u_0(t) &> \int_p^{1-p} \min_{t\in\mathcal{J}_p} G_{\lambda_1,\lambda_2,\gamma_1}(t, s) g_1(s, u_0(s), v(s))ds + \eta \\ &\geq c(\varrho_1 + \varrho_2)\overline{M_1} \left(c \int_p^{1-p} M_1(s)ds \right) + \eta = c(\varrho_1 + \varrho_2) + \eta.\end{aligned}$$

Similarly we can prove that $v_0(t) > c(\varrho_1 + \varrho_2) + \eta$. This implies that

$$l_{\min}(u_0, v_0) = \min\{(u_0 + v_0)(t)|\ t \in \mathcal{J}_p\} > c(\varrho_1 + \varrho_2),$$

which is contradiction with (R_3) in Lemma 4. Hence by means of (P_2) in Theorem 3, we deduce that $i_S(\mathfrak{T}, \Omega_{\varrho_1,\varrho_2}) = 0$. The proof is complete. □

Lemma 6. *Let conditions* $(C_1),(C_2)$ *and the conditions*

$$ {}_0g_1^{\varrho_1,\varrho_2} \leq \overline{m_1},\ u \neq T_u, \quad (u,v) \in \partial S_{\varrho_1,\varrho_2}, $$

(3.17)

$$ {}_0g_2^{\varrho_1,\varrho_2} \leq \underline{m_2},\ u \neq T_v, \quad (u,v) \in \partial S_{\varrho_1,\varrho_2}, $$

be satisfied. Then $i_S(\mathfrak{T}, S_{\varrho_1,\varrho_2}) = 1$.

Proof. According to (2.26),(2.28) and conditions (3.17), for $(u,v) \in \partial S_{\varrho_1,\varrho_2}$ we find that

$$ \begin{aligned} T_u(t) &= H_{\lambda_1}(t,u(t)) + \int_0^1 G_{\lambda_1,\lambda_2,\gamma_1}(t,s) g_1(s,u(s),v(s))ds \\ &\leq \xi_1 + \int_0^1 M_1(s) g_1(s,u(s),v(s))ds \\ &\leq \xi_1 + \frac{\varrho_1+\varrho_2-2\xi_1}{2}\overline{m_1}\left(2\int_0^1 M_1(s)ds\right) = \frac{\varrho_1+\varrho_2}{2} = \frac{1}{2}\|u\|_E. \end{aligned} $$

This implies that $\|T_u\|_E \leq \frac{\|u\|_E}{2}$. With the similar manner we can show that $\|T_v\|_E \leq \frac{\|v\|_E}{2}$. Equivalently we have proved that $\|\mathfrak{T}(u,v)\|_{\mathfrak{B}} \leq \|(u,v)\|_{\mathfrak{B}}$. Thus by means of (P_1) in Theorem 3, we conclude that $i_S(\mathfrak{T}, S_{\varrho_1,\varrho_2}) = 1$. This completes the proof. $\square$

Now according to the recent two technical lemmas we are ready to prove the main theorem of Leray-Schauder method for multiplicity results of positive solutions for coupled system (1.1) as below.

Theorem 5. [4] *Assume that conditions* $(C_1),(C_2)$ *be satisfied. Also assume that either condition*

(I) *There exist positive constants* $\varrho_1,\nu_1,\omega_1,\varrho_2,\nu_2,\omega_2$ *with*

$$ \begin{aligned} &(\varrho_1,\varrho_2) < c(\nu_1,\nu_2),\ (\nu_1,\nu_2) < (\omega_1,\omega_2), \\ &{}_0g_1^{(\varrho_1,\varrho_2)} \leq \overline{m_1},\ g_1^{c(\nu_1,\nu_2)} \geq \overline{M_1},\ u \neq T_u,\ (u,v) \in \partial\Omega_{\nu_1,\nu_2},\ {}_0g_1^{\omega_1,\omega_2} \leq \overline{m_1}, \\ &{}_0g_2^{(\varrho_1,\varrho_2)} \leq \underline{m_2},\ g_2^{c(\nu_1,\nu_2)} \geq \underline{M_2},\ v \neq T_v,\ (u,v) \in \partial\Omega_{\nu_1,\nu_2},\ {}_0g_2^{\omega_1,\omega_2} \leq \underline{m_2}, \end{aligned} $$

or

(II) *There exist positive constants* $\varrho_1,\nu_1,\omega_1,\varrho_2,\nu_2,\omega_2$ *with*

$$ \begin{aligned} &(\varrho_1,\varrho_2) < (\nu_1,\nu_2) < c(\omega_1,\omega_2), \\ &g_1^{c(\varrho_1,\varrho_2)} \geq \overline{M_1},\ {}_0g_1^{(\nu_1,\nu_2)} \leq \overline{m_1},\ u \neq T_u,\ (u,v) \in \partial S_{\nu_1,\nu_2},\ g_1^{c(\omega_1,\omega_2)} \geq \overline{M_1}, \\ &g_2^{c(\varrho_1,\varrho_2)} \geq \underline{M_2},\ {}_0g_2^{(\nu_1,\nu_2)} \leq \underline{m_2},\ v \neq T_v,\ (u,v) \in \partial S_{\nu_1,\nu_2},\ g_2^{c(\omega_1,\omega_2)} \geq \underline{M_2}, \end{aligned} $$

be satisfied. Then the fractional coupled system (1.1) has two positive solutions in S.

Proof. Assume that the condition (II) satisfies. We will prove that the operator $\mathfrak{T}$ defined by (2.24) has two fixed point as $(u_*, v_*), (u^*, v^*)$ in $S_{\nu_1,\nu_2}\backslash\overline{\Omega_{\varrho_1,\varrho_2}}$ and $\Omega_{\omega_1,\omega_2}\backslash\overline{S_{\nu_1,\nu_2}}$. By means of Lemma 5, Lemma 6, we find that

$$i_S(\mathfrak{T}, S_{\nu_1,\nu_2}) = 1,\ i_S(\mathfrak{T}, \Omega_{\varrho_1,\varrho_2}) = 0,\ i_S(\mathfrak{T}, \Omega_{\omega_1,\omega_2}) = 0.$$

Also according to (R_2) in Lemma 4, we conclude that

$$S_{c(\varrho_1,\varrho_2)} \subset \Omega_{\varrho_1,\varrho_2} \subset S_{\varrho_1,\varrho_2} \subset S_{\nu_1,\nu_2} \subset S_{c(\omega_1,\omega_2)} \subset \Omega_{\omega_1,\omega_2} \subset S_{\omega_1,\omega_2}.$$

Hence we have

$$\begin{aligned} i_S(\mathfrak{T}, S_{\nu_1,\nu_2}\backslash\overline{\Omega_{\varrho_1,\varrho_2}}) &= i_S(\mathfrak{T}, S_{\nu_1,\nu_2}) - i_S(\mathfrak{T}, \Omega_{\varrho_1,\varrho_2}) = 1 - 0 = 1,\\ i_S(\mathfrak{T}, \Omega_{\omega_1,\omega_2}\backslash\overline{S_{\nu_1,\nu_2}}) &= i_S(\mathfrak{T}, \Omega_{\omega_1,\omega_2}) - i_S(\mathfrak{T}, S_{\nu_1,\nu_2}) = 0 - 1 = -1. \end{aligned}$$

Therefore using (P_3) in Theorem 3, we conclude that operator $\mathfrak{T}$ has two fixed point as $(u_*, v_*), (u^*, v^*)$ in $S_{\nu_1,\nu_2}\backslash\overline{\Omega_{\varrho_1,\varrho_2}}$ and $\Omega_{\omega_1,\omega_2}\backslash\overline{S_{\nu_1,\nu_2}}$. In Equivalent manner, have proved that the fractional coupled hybrid system (1.1) has two positive solutions in S. In the similar way we can prove the same result when the condition (I) be satisfied. This completes the proof. □

Remark 7. If in (II) in the Theorem 5 we replace $g_1^{c(\varrho_1,\varrho_2)} \geq \overline{M_1}$ with $g_1^{c(\varrho_1,\varrho_2)} > \overline{M_1}$ and $g_2^{c(\varrho_1,\varrho_2)} \geq \underline{M_2}$ with $g_2^{c(\varrho_1,\varrho_2)} > \underline{M_2}$, then we can prove that the fractional coupled system (1.1) has a third positive solution $(\widehat{u}, \widehat{v})$ in S. For more details, we refer the eager researcher to the (Theorem 2.11, [5]).

References

[1] B. C. Dhange, N. jadhav; Basic results in the theory of hybrid differential equations with linear perturbations of second type, *Tamkang J. Math.*, 44 (2013), 171-186.

[2] B. C. Dhage; On a fixed point theorem in Banach algebras with applications, *Appl. Math. Lett.*, 18 (2005), 273-280.

[3] B. C. Dhage; On some nonlinear alternatives of Leray-Schauder type and functional integral equations, *Arch. Math.*, 42 (2006), 11-23.

[4] Kazem Ghanbari, Yousef Gholami; Existence and multiplicity of positive solutions for a coupled system of perturbed nonlinear fractional differential equations, *Comput. Methods Differ. Equ.*, 1 (2013), 39-54.

[5] K.Q. Lan, Multiple positive solutions of semilinear differential equations with singularities, *J. Lond. Math. Soc.* 63 (2001), 690-704.

[6] L. P. Lebedev, I. I. Vorovich, G. M. L. Gladwell; Functional Analysis (Applications in Mechanics and Inverse Problems, 2^{nd} Edition), *Kluwer Academic Publishers* , (2002).

[7] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo; Theory and Applications of fractional Differential Equations, *North-Holland Mathematics Studies, Elsevier Science*, 204 (2006).

[8] I. Podlubny; Fractional Differential Equations, *Mathematics in Science and Applications, Academic Press, New York,* 19 (1999).