Asia Pacific Journal of Mathematics, Vol. 2, No. 1 (2015), 69-75 ISSN 2357-2205

STAR COVERING PROPERTIES IN PARACOMPACT AND PARALINDELÖF SPACES

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Abstract. For a topological property P, we say that a topological space X is star P if for every open cover $\mathscr{U}$ of the space, there exists $Y \subseteq X$ such that Y has the property P and $st(Y, \mathscr{U}) = X$. We study the relationship of star paracompact and star paralindelöf properties with other Lindelöf type properties and investicate topological properties of star paracompact and star paralindelöf spaces.

2010 Mathematics Subject Classification. 54A40.

Key words and phrases. Star paracompact space, star paralindelöf space, star countable, strongly feebly Lindelöf space.

1. Introduction

If X is a topological space and $\mathscr{U}$ is a family of subsets of X, then the star of a subset $A \subseteq X$ is the set $st(A, \mathscr{U}) = \cup\{U \in \mathscr{U} : U \cap A \neq \phi\}$. Given a topological property P, a space X is called star P if for an arbitrary open cover $\mathscr{U}$ of the space X, there exists a set $Y \subseteq X$ such that Y has the property P and $st(Y, \mathscr{U}) = X$. The set Y will be called a star kernal of the cover $\mathscr{U}$. The classes of star P spaces were first defined(under another name) by Ikenaga in his paper [6] where he studied the cases of star countable, star Lindelöf and star σ-compact spaces. Star P properties were also introduced and studied systematically in the survey of Matveev [7].

One of the motivations to study star P properties is a Folklore fact that every space is star discrete and hence star metrizable. In fact, given any open cover $\mathscr{U}$, there is a closed and discrete kernal for $\mathscr{U}$ Star P properties were studied in several papers. Some of them are [1] and [2], [3], [4], [8], [9].

The second section of this paper discusses the properties of star paracompact and star paralindelöf spaces.We study the relationship of these spaces with other Lindelöf type properties in section 3. Last section of this article concerns with the extend of a space.

For any space X, the extend of X is the supremum of cardinalities of closed discrete subsets of X. A space X is paracompact(paralindel$\ddot{o}$f) if every open cover of X has a locally finite(countable) open refinement which covers X. All the spaces in this paper are assumed to be T_1.

2. Properties

Theorem 2.1. *Closed subspace of a star open paracompact space is star paracompact.*

Proof. Let A be a closed subset of a star open paracompact space X. Let $\mathscr{U}$ be any open cover for A. Then the collection $\mathscr{U}' = \mathscr{U} \cup \{X \backslash A\}$ is an open cover for X. Let Y be an open paracompact subset of X such that $st(Y, \mathscr{U}') = X$. Now we claim $Y' = Y \cap A$ is paracompact subset of A. Let $\mathscr{V} = \{U \cap A : U \in \tau_X\}$ be any open cover for Y' in A. Then the collection $\mathscr{V}_1 = \mathscr{V}' \cup \{Y \backslash A\}$ covers Y, where $\mathscr{V}' = \{U : U \cap A \in \mathscr{V}\}$. Let $\mathscr{W}$ be a locally finite open refinement of $\mathscr{V}_1$ such that $\mathscr{W}$ covers Y. Then $\mathscr{W}_1 = \{W \cap A : W \in \mathscr{W}, W \subset U$ and $U \in \mathscr{V}_1\}$ is a locally finite open refinement of $\mathscr{V}$ which covers Y'. Therefore Y' is a paracompact subset of A such that $st(Y', \mathscr{U}) = A$. □

Theorem 2.2. *Closed subspace of a star open paralindel$\ddot{o}$f space is star paralindel$\ddot{o}$f.*

Proof. Proof is similar to theorem2.1. □

Let $f : X \to Y$ be a mapping. f is a closed mapping if the image of the closed set under f is closed and it is a compact mapping if $f^{-1}(y)$ is compact, for all $y \in Y$. f is a perfect mapping if f is a closed, continuous, compact and onto mapping.

Theorem 2.3. *Let f be a closed and continuous mapping from a star paracompact space X to a space Y. Then Y is a star paracompact space.*

Proof. Let $\mathscr{U}$ be any open cover of Y. The collection $\mathscr{V} = \{f^{-1}(U) : U \in \mathscr{U}\}$ is an open cover for X. Let P be a paracompact subset of X such that $st(P, \mathscr{V}) = X$. Since f is closed and continuous, $f(P)$ is a paracompact subset of Y(See: [[5]]).

Let $y \in Y$. Then $f^{-1}(y) \subseteq X = st(P, \mathscr{V}) = \cup\{f^{-1}(U) : f^{-1}(U) \cap P \neq \phi\}$, implies that $f^{-1}(y) \cap f^{-1}(U) \neq \phi$, where $f^{-1}(U) \cap P \neq \phi$ and $U \in \mathscr{U}$. Then $f(f^{-1}(y)) \cap f(f^{-1}(U)) \supseteq f(f^{-1}(y) \cap f^{-1}(U)) \neq \phi$ implies that $y \cap U \neq \phi$. That is $y \in U$. Note that $f(f^{-1}(U) \cap P) \subseteq f(f^{-1}(U)) \cap f(P) \neq \phi, U \cap f(P) \neq \phi$. Thus $y \in U$ and $U \cap f(P) \neq \phi$, for some $U \in \mathscr{U}$ and hence $st(f(P), \mathscr{U}) = Y$. □

Theorem 2.4. *Let f be a perfect map from a star paralindel$\ddot{o}$f space X to a space Y. Then Y is a paralindel$\ddot{o}$f space.*

Proof. Image of a paralindel$\ddot{o}$f space under perfect mapping is a paralindel$\ddot{o}$f space and the proof is similar to previous theorem 2.3. □

Theorem 2.5. *Let f be an open perfect mapping from a space X to a star paracompact space Y. Then X is a star paracompact space.*

Proof. Let $\mathscr{U}$ be any open cover of X and let $y \in Y$. Since $f^{-1}(y)$ is compact, there exists a finite subcollection $\mathscr{U}_y$ of $\mathscr{U}$ such that $\mathscr{U}_y$ covers $f^{-1}(y)$ and $U \cap f^{-1}(y) \neq \phi$, for each $U \in \mathscr{U}_y$. Since f is open, we can find an open set V_y of y in Y such that $V_y \subset \cap\{f(U) : U \in \mathscr{U}_y\}$ and

$$f^{-1}(V_y) \subseteq \cup\{U : U \in \mathscr{U}_y\} \tag{1}$$

. Now the collection $\mathscr{V} = \{V_y : y \in Y\}$ is a cover for Y. Let P be a paracompact subset of Y such that $st(P, \mathscr{V}) = Y$. Since f is a perfect mapping, $f^{-1}(P)$ is a paracompact subset of X.

Let $x \in X$. Then there exists $y \in Y$ such that $f(x) \in V_y$ and $V_y \cap P \neq \phi$. Since $x \in f^{-1}(V_y)$ and $f^{-1}(V_y) \subseteq \cup\{U : U \in \mathscr{U}_y\}$, we can find $U \in \mathscr{U}_y$ such that $x \in U$. Since by 1 $V_y \subseteq f(U)$, $f(U) \cap P \neq \phi$ and hence $U \cap f^{-1}(P) \neq \phi$. Thus $x \in st(f^{-1}(P), \mathscr{U})$. □

3. Star paracompact spaces

Theorem 3.1. *Every regular star Lindelöf space is star paracompact.*

Proof. For any open cover $\mathscr{U}$ of X, there is a Lindelöf subset A such that $st(A, \mathscr{U}) = X$. Since A is regular Lindelöf, A is paracompact and $st(A, \mathscr{U}) = X$. □

An open set U in a space X is called regular open if $U = int(clA)$. The complement of the regular open set is a regular closed set. A space X is called semi-regular if the regular open sets of X is a open base for the open sets of X.

Theorem 3.2. *Let X be a semi-regular space. Then X is star compact if and only if X is a feebly compact, star closed paracompact space.*

Proof. Each star compact space is star paracompact space. Let $\{U_\alpha : \alpha \in I\}$ be an infinite locally finite family of open subsets of X. For each $\alpha \in I$, choose $x_\alpha \in U_\alpha$ and let $A = \{x_\alpha : \alpha \in I\}$. Then A is closed. The collection $\mathscr{U} = \{X \backslash A\} \cup \{U_\alpha : \alpha \in I\}$ covers X. Since X is star compact, there is a compact subset A such that $st(A, \mathscr{U}) = X$. Now $\mathscr{U}' = \{U \in \mathscr{U} : U \cap A \neq \phi\}$ is an open cover for A. Let $\{U_i : i = 1, 2, \cdots\cdots, n\}$ be a finite subcollection covers A. Since each U_α contains atmost finite number of $x_\alpha^{\prime s}$, we have $\{U_\alpha : \alpha \in I\}$ is finite. Thus X is feebly Lindelöf.

If $\mathscr{U}$ is an open cover of X, then there is a closed and paracompact subset A of X such that $st(A, \mathscr{U}) = X$. Since X is semi-regular, A is a intersection of regular closed subsets of X. Since X is feebly compact, every regular closed subset of feebly compact space is feebly compact and hence A is a compact and $st(A, \mathscr{U}) = X$. This completes the proof. $\square$

In the above theorem 3.3, closedness of paracompactness and semi-regularity are used only in the converse part of the theorem.

Theorem 3.3. *If X is star Lindelöf space, then X is feebly Lindelöf, star paralindelöf space.*

Proof. Proof is similar to previous theorem. $\square$

Theorem 3.4. *Star countable, paralindelöf spaces are Lindelöf.*

Proof. Let $\mathscr{U}$ be any open cover of star countable, paralindelöf space X. Let $\mathscr{V}$ be a locally countable open refinement of $\mathscr{U}$ covering X. Since X is star countable, there is a countable subset $A \subseteq X$ such that $st(A, \mathscr{V}) = X$. Since every point of x lies atmost countable number of elements of $\mathscr{V}$, $\{V \in \mathscr{V} : V \cap A \neq \phi\}$ is countable and covers X. Since $\mathscr{V}$ refines $\mathscr{U}$, for each $V \in \mathscr{V}$, there is an $U_V \in \mathscr{U}$ such that $V \subseteq U_V$. The family $\{U_V \in \mathscr{U} : V \in \{V \in \mathscr{V} : V \cap A \neq \phi\}\}$ is a countable subcover for $\mathscr{U}$. $\square$

Remark 3.5. *In the above theorem 3.4, we can replace metalindelöf space insteadof paralindelöf space.*

Theorem 3.6. *A space X is compact if and only if X is star feebly compact, paracompact space.*

Proof. For any open cover $\mathscr{U}$ of X, let $\mathscr{U}'$ be a locally finite open refinement of $\mathscr{U}$ covering X. Then there exists a feebly compact set $A \subseteq X$ such that $st(A, \mathscr{U}') = X$. Let $\mathscr{V} = \{U' \cap A : U' \in \mathscr{U}'$ and $U' \cap A \neq \phi\}$. Since A is feebly compact, $\mathscr{V}$ is finite. For each $U' \cap A \in \mathscr{V}$, there exists $U \in \mathscr{U}$ such that $U' \subseteq U$. Let $x \in X = st(A, \mathscr{U}')$. Then there exists $U' \in \mathscr{U}'$ such that $x \in U' \cap A$ where $U' \cap A \in \mathscr{V}$. This implies that $x \in U' \subseteq U$ and hence $x \in \cup\{U \in \mathscr{U} : U' \subseteq U$ and $U' \cap A = V \in \mathscr{V}\}$. Since $\mathscr{V}$ is finite, the collection $\mathscr{W} = \{U \in \mathscr{U} : U' \subseteq U$ and $U' \cap A = V \in \mathscr{V}\}$ is finite and covers X. Thus X is compact.

If X is compact, then X is paracompact and feebly compact and hence it is paracompact star feebly compact. $\square$

Definition 3.7. *A space is said to be strongly feebly Lindelöf space if every locally countable collection of open subsets is countable.*

Theorem 3.8. *If X is star Lindelöf, then X is strongly feebly Lindelöf and star paralindelof.*

Proof. Proof is similar to theorem 3.2. □

Theorem 3.9. *X is Lindelöf if and only if X is star strongly feebly Lindelöf and paralindelöf.*

Proof. Proof is similar to theorem 3.6. □

Theorem 3.10. *If X is strongly feebly Lindelöf and Y is seperable, then $X \times Y$ is strongly feebly Lindelöf.*

Proof. Suppose $X \times Y$ is not a strongly feebly Lindelöf space and that $\mathscr{U}$ is an locally countable family of open sets in $X \times Y$. Let $\mathscr{U} = \{U_\alpha \times V_\alpha : \alpha \in I\}$ and each element of $\mathscr{U}$ is a basic open set in the product set. Let A be a countable dense subset of Y. For each $a \in A$, the set $\{U_\alpha : \alpha \in I$ and $a \in V_\alpha\}$ is locally countable in X.
For,

If $x \in X$, then $(x, a) \in X \times Y$. Let $U \times V$ be an open set containing (x, a) such that $(U \times V) \cap (U_\alpha \times V_\alpha) \neq \phi$ for $\alpha \in F, F$ is a countable subset of I. That is $(U \cap U_\alpha) \times (V \cap V_\alpha) \neq \phi, \alpha \in F \subseteq I$. Then $U \cap U_\alpha \neq \phi$ for countable number of α'^s. Since X is strongly feebly Lindelöf, $\{U_\alpha : \alpha \in I$ and $a \in V_\alpha\}$ is countable. Since A is countable, we have $\mathscr{U}$ is countable. □

Definition 3.11. *A topological property P is said to be paracompactly productive(compactly productive) if whenever X has P and Y is paracompact(compact), then $X \times Y$ has P.*

It is easy to see that if P is paracompactly productive, then it is compactly productive. Converse need not be true. For, Compactness is compactly productive, but it is not paracompactly productive.

Paracompactness is not paracompactly productive, because the product of two paracompact spaces need not be paracompact. Similarly, paralindelofness is also not a paracompactly productive. Sorgenfreyplane is an example for Lindelöfness is not paracompactly productive.

4. Countable Extend

Definition 4.1. *If X is a space and $A \subseteq X$, we say that a family $\mathscr{U}$ is an open expansion of A if $\mathscr{U} = \{U_a : a \in A\}$ and $U_a \in \tau(a, X)$ for any $a \in A$.*

Definition 4.2. *Given an infinite cardinal, we say that a space X is weakly k-paracompact(weakly k-metacompact) if any closed discrete set $D \subseteq X$ with $|D| = k$, we can find a set $D' \subseteq D$ such that $|D'| = k$ and D' has a locally finite(point finite) open expansion.*

Definition 4.3. *Given an infinite cardinal, we say that a space X is weakly k-paralindelöf(weakly k-metalindelöf) if any closed discrete set $D \subseteq X$ with $|D| = k$, we can find a set $D' \subseteq D$ such that $|D'| = k$ and D' has a locally countable(point countable) open expansion.*

Definition 4.4. *If k is an infinite cardinal, we call a space X is weakly k-collectionwise Hausdorff if any closed discrete set $D \subseteq X$ with $|D| = k$ we can find a set $D' \subseteq D$ such that $|D'| = k$ and D' has a disjoint open expansion.*

Lemma 4.5. *Suppose that a space X has a countably infinite closed discrete subspace D such that some countably infinite set $E \subseteq D$ has a locally finite open expansion. Then X is not star feebly compact.*

Proof. Assume that X is star feebly compact and take a locally finite open expansion $\mathscr{U} = \{U_x : x \in E\}$ of the countably infinite set E such that $U_x \cap E = \{x\}$, for any $x \in E$. The family of open sets $\mathscr{V} = \mathscr{U} \cup \{X \backslash E\}$ is an open cover of the space X. Let A be a feebly compact subset of X such that $st(A, \mathscr{V}) = X$. Let $B = \{x \in E : U_x \cap A \neq \phi\}$. Since $\mathscr{V}$ is locally finite in X, $\mathscr{V}' = \{V \cap A : V \in \mathscr{V}\}$ is a locally finite family of open sets in A. Since A is feebly compact, $\mathscr{V}'$ is finite and hence $\mathscr{U}' = \{U_x \cap A : U_x \in \mathscr{U}\}$ is finite. Therefore B is a finite set. Then for any $x \in E \backslash B, U_x \cap A = \phi$ implies that $x \notin st(A, \mathscr{V}) = X$, which is a contradiction. □

Lemma 4.6. *Suppose that a space X has an uncountable closed discrete subspace D such that some uncountable set $E \subseteq D$ has a locally finite open expansion. Then X is not star feebly Lindelöf.*

Proof. Proof is similar to previous lemma 4.5. □

Corollary 4.7. *For any weakly ω-paracompact space, $ext(X)$ is finite if X is star feebly compact.*

Corollary 4.8. *Let X be a weakly ω_1-paracompact space. Then $ext(X)$ is countable if X is star feebly Lindelöf.*

Lemma 4.9. *Suppose that a space X has an uncountable closed discrete subspace D such that some uncountable set $E \subseteq D$ has a locally countable open expansion. Then X is not star strongly feebly Lindelöf.*

Proof. Proof is similar to lemma 4.5. □

Corollary 4.10. *For any weakly ω_1-paralindelöf space X, $ext(X)$ is countable if X is star strongly feebly Lindelöf.*

Corollary 4.11. (1) *For a weakly ω-collectionwise Hausdorff space X, $ext(X)$ is finite if X is star feebly compact.*

(2) *For a weakly ω_1-collectionwise Hausdorff space X, $ext(X)$ is countable if X is star feebly Lindelöf or star strongly feebly Lindelöf.*

Corollary 4.12. *For a regular P-space X, $ext(X)$ is countable if X is star feebly Lindelöf or star strongly feebly Lindelöf.*

Theorem 4.13. *Let X be a normal P-space. Then $ext(X)$ is countable if X is strongly feebly Lindelöf.*

Proof. Every strongly feebly Lindelöf space is feebly Lindelöf. Then by the theorem 3.2 in [1], we have $ext(X)$ is countable. $\square$

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