ON STARLIKE AND CONVEX FUNCTIONS OF COMPLEX ORDER WITH FIXED SECOND COEFFICIENT

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Abstract. Let \( F_p(b, M) \) denote the class of functions \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \) which are analytic in the open unit disc \( U = \{ z : |z| < 1 \} \) and satisfy the inequality

\[
\left| \frac{b - 1 + \frac{zf'(z)}{f(z)}}{b} - M \right| < M
\]

for \( b \neq 0 \), complex, \( M > \frac{1}{2}, |a_2| = 2p, 0 \leq p \leq \left( \frac{1+m}{2} \right) |b| , \)

\( m = 1 - \frac{1}{M} \) and for all \( z \in U \). Further \( f(z) \) is in the class \( G_p(b, M) \) if \( zf'(z) \) is in the class \( F_p(b, M) \). In the present paper, we obtain lower bounds for the classes introduced above and apply them to determine \( \gamma \)-spiral radius for functions of the class \( F_p(b, M) \) and \( \gamma \)-convex radius for functions of the class \( G_p(b, M) \).

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1. Introduction.

Let \( A \) denote the class of functions of the form

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k , \quad (k \in N = \{1, 2, \ldots\} ) ,
\]

which are analytic in the open unit disc \( U = \{ z \in C : |z| < 1 \} \), and let \( S \) denote the subclass of \( A \). In [17] Nasr and Aouf denote the class of bounded starlike functions of complex order \( F(b, M)(b \neq 0, \) complex, \( M \) fixed , \( M > \frac{1}{2} \) and \( \frac{f(z)}{z} \neq 0 ) \) by

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Also in [18] Nasr and Aouf denote and introduced the class of bounded convex function of complex order $G(b, M)$ ($b \neq 0$, complex, $M$ fixed and $M > \frac{1}{2}$) by

\[
\left| \frac{b - 1 + \frac{zf'(z)}{f(z)}}{b} - M \right| < M, \quad (z \in U)
\]

From equations (1.2) and (1.3), we get

\[
\left| \frac{b + \frac{zf''(z)}{f'(z)}}{b} - M \right| < M, \quad (z \in U).
\]

(1.4) $f(z) \in G(b, M)$ if and only if $zf'(z) \in F(b, M)$.

By specializing $b$ and $M$, we obtain the following subclasses studied by earlier authors:

1. $F(b, \infty) = S(1 - b)$ \hspace{1cm} (Nasr and Aouf [19]);
2. $G(b, \infty) = C(b)$ \hspace{1cm} (Wiatrowski [29], Nasr and Aouf [16], and Aouf [2]);
3. $F((1 - \alpha)e^{-i\lambda}\cos \lambda, \infty) = S^\lambda(\alpha) \quad (|\lambda| < \frac{\pi}{2}, \ 0 \leq \alpha < 1)$ \hspace{1cm} (Libera [11] and Patil and Thakare [21]);
4. $G((1 - \alpha)e^{-i\lambda}\cos \lambda, \infty) = C^\lambda(\alpha) \quad (|\lambda| < \frac{\pi}{2}, \ 0 \leq \alpha < 1)$ \hspace{1cm} (Chichra [7] and Sizuk [28]);
5. $F(1 - \alpha, 1) = S_\alpha(0 \leq \alpha < 1)$ \hspace{1cm} (Wright [30] and MaCarty [13]);
6. $F(1, 1) = S^*$ \hspace{1cm} (Singh [26]);
7. $F(1, M), \quad (M > \frac{1}{2})$ \hspace{1cm} (Singh and Singh [27]);
8. $F(e^{-i\lambda}\cos \lambda, (\cos \lambda)^{-1}) = H(\lambda)(|\lambda| < \frac{\pi}{2})$ \hspace{1cm} (Geol [9]);
9. $F(e^{-i\lambda}\cos \lambda, M) = F_{\lambda,M}$ and ; $G(e^{-i\lambda}\cos \lambda, M) = G_{\lambda,M} \quad (|\lambda| < \frac{\pi}{2}, \ M > \frac{1}{2})$ \hspace{1cm} (Kulshrestha [10]);
10. $F((1 - \alpha)e^{-i\lambda}\cos \lambda, M) = F_M(\lambda, \alpha)$, and $G((1 - \alpha)e^{-i\lambda}\cos \lambda, M) = G_M(\lambda, \alpha) \quad (|\lambda| < \frac{\pi}{2}, \ M > \frac{1}{2}, \ 0 \leq \alpha < 1)$ \hspace{1cm} (Aouf [3]);
(11) \( F(1-a-d, \frac{d}{a+d-1}) = S(a,d), \quad (a+d \geq 1, \quad d \leq a < d+1) \) (Silverman [23]);

(12) \( F((1-a-d)e^{-i\lambda} \cos \lambda, \frac{d}{a+d-1}) = S(\lambda, a,d) \quad (|\lambda| < \frac{\pi}{2}, \quad a+d \geq 1, \quad d \leq a \leq d+1) \) (Silvia [25]);

**Definition 1.** A function \( f(z) \) defined by (1.1) is said to be in the class \( F_p(b,M) \) \((b \neq 0, \text{complex}, M > \frac{1}{2}, |a_2| = 2p, 0 \leq p \leq \frac{1+m}{2} |b|, \ m = 1 - \frac{1}{M}) \) if it satisfies (1.2).

**Definition 2.** A function \( f(z) \) defined by (1.1) is said to be in the class \( G_p(b,M) \) \((b \neq 0, \text{complex}, M > \frac{1}{2}, |a_2| = p, \ 0 \leq p \leq \frac{1+m}{2} |b|, \ m = 1 - \frac{1}{M}) \) if it satisfies (1.3).

In [11] Libera (see also [14]) introduced the concept of ”\( \gamma \)-spiral radius” for the classes of univalent functions as follows

**Definition 3.** If \( f \in S \) and \( |\gamma| < \frac{\pi}{2} \), then \( \gamma \)-spiral radius of \( f \) given by

\[
g - s.r.\{f\} = \sup \{r : \text{Re}(e^{i\gamma}zf'(z)) > 0, \quad |z| < r\},
\]

and if \( F \subset S \), then \( \gamma \)-spiral radius of \( F \) is

\[
g - s.r.F = \inf_{f \in F} \{g - s.r.\{f\}\}.
\]

In this paper, we obtain convolution conditions for the classes \( K(A,B,p,\alpha), S^*(A,B,p,\alpha), K_\lambda(A,B,p,\alpha) \) and \( S^*_\lambda(A,B,p,\alpha) \) as described below:

**Definition 4.** If \( f \in A \) and \( |\gamma| < \frac{\pi}{2} \), then the \( \gamma \)-convex radius of \( f \)

\[
g - c.r.\{f\} = \sup \{r : \text{Re}\{e^{i\gamma}(1 + zf''(z))\} > 0, \ z \in U\}.
\]

**Definition 5.** If \( E \subset A \), and \( |\gamma| < \frac{\pi}{2} \), then the \( \gamma \)-convex radius of \( E \) is

\[
g - c.r.E = \inf_{f \in E} \{g - c.r.\{f\}\}.
\]

Results in terms of a fixed second coefficient have been obtained for various subclasses of \( S \). Finkelstein [8] investigated the classes \( F_p(1, \infty) \) and \( G_p(1, \infty) \) the starlike
and convex functions with pre-assigned second coefficient. Extensions of these results can be found in ([1], [4], [5], [6], [14], [23] and [34]).

In the present paper, we obtain lower bounds for the classes introduced above and apply them to determine $\gamma$-spiral radii for functions of the class $F_p(b,M)$ and $\gamma$-convex radius for functions of the class $G_p(b,M)$.

2. Growth estimates

To prove our results, we need the following lemma.

Lemma 1. Let $w(z) = d_1 z + d_2 z^2 + \ldots$, be an analytic map of the unit disc into itself. Then

$$|d_1| \leq 1$$

and

$$|w(z)| \leq \frac{r(r + |d_1|)}{(1 + |d_1|r)} \quad (|z| = r).$$

Equality holds at some point $z(z \neq 0)$ if and only if

$$w(z) = \frac{e^{-ut}z + d_1 e^{it}}{1 + d_1 e^{-ut}z} \quad (t \geq 0).$$

This lemma is an iterated form of Schwarz’s Lemma [20] and is due to Lowner [12].

To obtain growth estimates for the classes $F_p(b,M)$ and $G_p(b,M)$ it is useful to consider the following class.

Definition 6. A function $h(z) = 1 + 2a_2 z + \ldots$, analytic in the unit disc $U$, is in the class $H_p(b,M)$ ($b \neq 0$, complex, $M > \frac{1}{2}, |a_2| = p, 0 \leq p \leq (\frac{1 + m}{2}) |b|, m = 1 - \frac{1}{M}$) if the inequality

$$|b - 1 + h(z) - M| < M$$

holds for $b \neq 0$, complex, $M > \frac{1}{2}$, and for all $z \in U$.

Observe that $f(z) \in F_p(b,M)$ if and only if $\frac{zf'(z)}{f(z)} \in H_p(b,M)$ and $f(z) \in G_p(b,M)$ if and only if $1 + \frac{zf''(z)}{f'(z)} \in H_p(b,M)$.

Theorem 2. Suppose $h(z) \in H_p(b,M), |\gamma| < \frac{\pi}{2}$ and
\[ u = pr + \left( \frac{1+m}{2} \right) |b|, \quad v = \left( \frac{1+m}{2} \right) |b|r + p. \]

Then, for \(|z| = r < 1\) and for all \(b, M, p (b \neq 0, \text{ complex}), M > \frac{1}{2}, 0 \leq p \leq \left( \frac{1+m}{2} \right) |b|, m = 1 - \frac{1}{M}\),

\[
\text{Re}\{e^{i\gamma}h(z)\} \geq \frac{1}{u^2 - m^2 v^2 r^2} \left\{ u^2 \cos \gamma - (1+m) |b| w r \right. \\
+ m v^2 [(m+1) (\text{Re}\{b\} \cos \gamma - \text{Im}\{b\} \sin \gamma) - m \cos \gamma] r^2 \}.
\]

The result is sharp.

**Proof.** Since \(h(z) \in H_p(b, M)\), an application of Schwarz’s Lemma [20] gives

\[ h(z) = \frac{1 + [(1+m)b - m]w(z)}{1 - mw(z)}, \quad m = 1 - \frac{1}{M}, \]

where \(w(z) = d_1 z + ... (|d_1| = \frac{2p}{(1+m)|b|})\) satisfies the hypotheses of the lemma. Thus

\[ |w(z)| \leq \frac{v}{u}r, \]

where \(u\) and \(v\) are given by (2.2) and \(|z| = r\). Let \(B(z) = e^{-i\gamma}h(z)\) and \(|\gamma| < \frac{\pi}{2}\). Then (2.4) may be written as

\[ w(z) = -\frac{e^{i\gamma} - B(z)}{mB(z) + e^{i\gamma}[(1+m)b - m]}. \]

From (2.5) and (2.6), we have

\[ \left| \frac{e^{i\gamma} - B(z)}{mB(z) + e^{i\gamma}[(1+m)b - m]} \right| \leq \frac{v}{u} r. \]

Setting \(B(z) = \zeta + i\eta\) and simplifying, (2.7) gives

\[ \left| \zeta + i\eta - e^{i\gamma}[1 + \frac{m(1+m)bv^2r^2}{u^2 - m^2 v^2 r^2}] \right| \leq \frac{(1+m)|b| w r}{u^2 - m^2 v^2 r^2}. \]

From (2.8) it follows that

\[ \text{Re}\{B(z)\} \geq \text{Re}\{e^{i\gamma}[1 + \frac{m(1+m)bv^2r^2}{u^2 - m^2 v^2 r^2}] - \frac{(1+m)|b| w r}{u^2 - m^2 v^2 r^2} \}. \]

Now the result follows immediately from (2.9).
The bound in (2.3) is sharp for the function

\[ h(z) = \begin{cases} 
  \frac{u + [m - (1 + m)b]vz}{u + mvz}, & m \neq 0, \\
  \frac{1}{1 - \frac{b}{u}z}, & m = 0
\end{cases} \tag{2.10} \]

and

\[ \zeta = \begin{cases} 
  \frac{r[mvr - \sqrt{b} e^{-i\gamma} u]}{u - m \sqrt{b} e^{-i\gamma} vr}, & m \neq 0, \\
  r\sqrt{\frac{b}{b} e^{-i\gamma}}, & m = 0.
\end{cases} \tag{2.11} \]

3. The \( \gamma \)-spiral and \( \gamma \)-convex radius.

**Theorem 3.** \( \gamma \)-s.r.\( F_p(b, M) \) is the smallest positive root \( r_0 \) of the equation

\[ u^2 \cos \gamma - (1 + m) |b| uvr + mv^2[(1 + m)(\text{Re}\{b\} \cos \gamma - \text{Im}\{b\} \sin \gamma) - m \cos \gamma]r^2 = 0, \tag{3.1} \]

where \( u \) and \( v \) are given by (2.2). The result is sharp for all admissible values of \( b \), \( M \) and \( p \).

**Proof.** Setting \( h(z) = \frac{zf'(z)}{f(z)} \), in Theorem 1, we get

\[ \text{Re}\{e^{\gamma}z\frac{f'(z)}{f(z)}\} > \frac{1}{u^2 - m^2v^2r^2}\{u^2 \cos \gamma - (1 + m) |b| uvr + mv^2[(1 + m)(\text{Re}\{b\} \cos \gamma - \text{Im}\{b\} \sin \gamma) - m \cos \gamma]r^2\}. \tag{3.2} \]

Thus, from (1.5) and the inequality (3.2), \( f(z) \) is \( \gamma \)-spiral in \( |z| < r_0 \), where \( r_0 \) is the smallest positive root of the equation (3.1). Hence the theorem.

The result is sharp for the function \( f(z) \) given by

\[ f(z) = \begin{cases} 
  z(u + mvz)^{(1 + m)b}, & m \neq 0, \\
  z \exp(-b\frac{z}{u}z), & m = 0.
\end{cases} \tag{3.3} \]

where \( \zeta \) is given by (2.11).

**Corollary 4.** \( \gamma \)-s.r.\( S_p(1-b) \)is the smallest positive root \( r_0 \) of the equation
where $u = pr + |b|$ and $v = |b| r + p$. The result is sharp.

The above result is obtained by fixing $m = 1$ in Theorem 2. Further, taking $p = |b|$ in Corollary 1, we get the following result.

**Corollary 5.** $\gamma - s.r.S(1 - b)$ is the smallest positive root $r_0$ of the equation

\[ u^2 \cos \gamma - 2 |b| uvr + v^2 [2(\text{Re}\{b\}) \cos \gamma - \text{Im}\{b\} \sin \gamma] - \cos \gamma |r|^2 = 0, \]

The result is sharp.

**Corollary 6.** $\gamma - s.r.S_p^\lambda(\alpha)$ is the smallest positive root $r_0$ of the equation

\[ u^2 \cos \gamma - 2 |b| r + 2(\text{Re}\{b\}) \cos \gamma - \text{Im}\{b\} \sin \gamma - \cos \gamma |r|^2 = 0. \]

The result is sharp.

**Corollary 7.** $\gamma - s.r.F_{M,p}(\lambda, \alpha)$ is the smallest positive root $r_0$ of the equation

\[ u^2 \cos \gamma - (1 + m) uvr (1 - \alpha) \cos \lambda + \]

\[ mv^2 [(1 + m) (1 - \alpha) \cos \lambda \cos (\gamma - \lambda) - m \cos \gamma] |r|^2 = 0, \]

where $u = pr + \left(1 + \frac{m}{2}\right)(1 - \alpha) \cos \lambda$ and $v = \left(1 + \frac{m}{2}\right) r (1 - \alpha) \cos \lambda + p$. The result is sharp.

The above result is obtained by choosing $b = (1 - \alpha) e^{-i\lambda} \cos \lambda, |\lambda| < \frac{\pi}{2}$ and $0 \leq \alpha < 1$, in Theorem 3.1.

Further, taking $p = \left(1 + \frac{m}{2}\right)(1 - \alpha) \cos \lambda$ in Corollary 3.4, we get the result obtained by Mogra [14, Corollary 1]. Further, taking $p = (1 - \alpha) \cos \lambda$ in Corollary 3.5, we get the result obtained by Mogra and Ahuja [15] and Mogra [14].
\[ (3.8) \quad \cos \gamma - (1 + m) r (1 - \alpha) \cos \lambda + m [(1 + m) (1 - \alpha) \cos \lambda \cos (\gamma - \lambda) - m \cos \gamma] r^2 = 0 \]

The result is sharp

**Remark 1.** (1) Putting \( \alpha = 0 \) in Corollary 3.5, we get the corresponding result for the class \( F_{\lambda, M, p}(\lambda, 0) = F_{\lambda, M} \).

(2) Putting \( \alpha = 0 \) in Corollary 3.6, we get the result obtained by Mogra and Ahuja [15, Corollary 6] for the class \( F_M(\lambda, 0) = F_{\lambda, M} \).

**Corollary 9.** \( \gamma - s.r.H_p(\lambda) \) is the smallest positive root \( r_0 \) of the equation

\[ u^2 \cos \gamma - (2 - \cos \lambda) r u v \cos \lambda + (1 - \cos \lambda) v^2 [(2 - \cos \lambda) \cos \lambda \cos (\gamma - \lambda)] \]

\[ -(1 - \cos \lambda) \cos \gamma] r^2 = 0, \]

where \( u = pr + \left( \frac{2 - \cos \lambda}{2} \right) \cos \lambda \) and \( v = \left( \frac{2 - \cos \lambda}{2} \right) r \cos \lambda + p \). The result is sharp.

The above result is obtained by choosing \( b = e^{i\lambda} \cos \lambda \) and \( M = (\cos \lambda)^{-1} (|\lambda| < \frac{\pi}{2}) \) in Theorem 3.1. Further, taking \( p = \left( \frac{2 - \cos \lambda}{2} \right) \cos \lambda \) in Corollary 7, we get the result obtained by Mogra and Ahuja [15, Corollary 7].

**Corollary 10.** \( \gamma - s.r.S_p(\lambda, a, d) \) is the smallest positive root \( r_0 \) of the equation

\[ u^2 \cos \gamma - \left( \frac{\rho}{d} u v \cos \lambda \right) r - \left( \frac{1 - a}{d} \right) v^2 \left[ \frac{\rho}{d} \cos \lambda \cos (\gamma - \lambda) \right. \]

\[ \left. + \left( \frac{1 - a}{d} \right) \cos \gamma \right] r^2 = 0, \]

where \( \rho = d^2 - (1 - a)^2 \), \( u = pr + (a + d - 1) \cos \lambda \) and \( v = (a + d - 1) r \cos \lambda + p \). The result is sharp.

The above result is obtained by choosing

\( b = (1 - a - d) e^{-i\lambda} \cos \lambda \) and \( M = \frac{d}{a + d - 1} (|\lambda| < \frac{\pi}{2}, a + d \geq 1, d \leq a < d + 1) \) in Theorem 3.1. Further, taking \( p = (a + d - 1) \cos \lambda \) in Corollary 3.7 get the following result:

**Corollary 11.** \( \gamma - s.r.S(\lambda, a, d) \) is the smallest positive root \( r_0 \) of the equation

\[ \cos \gamma - \left( \frac{\rho}{d} \cos \lambda \right) r - \left( \frac{1 - a}{d} \right) \left[ \frac{\rho}{d} \cos \lambda \cos (\gamma - \lambda) + \left( \frac{1 - a}{d} \right) \cos \gamma \right] r^2 = 0, \]

The result is sharp.
**Remark 2.** (1) Putting $\gamma = 0$ in Corollary 3.9, we get the result obtained by Silvia [25, Theorem 3].

(2) Putting $\lambda = 0$ in Corollary 3.8 and Corollary 3.9, respectively, we get the corresponding results for the classes $S_p(a, d)$ and $S(a, d)$, respectively.

Setting $h(z) = 1 + \frac{zf''(z)}{f'(z)}$ in Theorem 2.3 and using Definition 1.4, we get the following result for the class $G_p(b, M)$.

**Theorem 12.** $\gamma - c.r. G_p(b, M)$ is the smallest positive root $r_0$ of the equation (3.1), where $u$ and $v$ are given by (2.2). The result is sharp for the functions $f(z)$ given by

\[
(3.12) \quad f'(z) = \begin{cases} \frac{1 + m}{m} b, & m \neq 0, \\ \exp(-b \frac{v}{u} z), & m = 0 \end{cases}
\]

where $\zeta$ is given by (2.11).

**Remark 3.** On taking the appropriate values of $b$ and $M$ the above theorem can give $\gamma - c.r.$ for the functions in the different classes.

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