Asia Pacific Journal of Mathematics, Vol. 3, No. 1 (2016), 10-23 ISSN 2357-2205

SOME UNIFIED AND GENERALIZED KUMMER'S FIRST SUMMATION THEOREMS WITH APPLICATIONS IN LAPLACE TRANSFORM TECHNIQUE

M. I. QURESHI[1] AND M. S. BABOO[2,*]

[1]Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Millia Islamia (A Central University), New Delhi -110025, India

[2]School of Basic Sciences and Research, Sharda University, Greater Noida, Uttar Pradesh, 201306, India

*Corresponding author

ABSTRACT. Some significant hypergeometric summation theorems with suitable convergence conditions are obtained in the present study, which are analogous to Kummer's summation theorem ${}_2F_1(-1)$ recorded by Prudnikov *et al.* and derived by Choi, Kim *et al.*, Rakha-Rathie and Rathie-Kim. By means of these summation theorems we also find the Laplace transforms of Kummer's confluent hypergeometric function ${}_1F_1$ in closed form.

2010 Mathematics Subject Classification. 33B15, 33C05, 44A20.

Key words and phrases. beta and gamma functions; Gauss and Kummer hypergeometric functions; Legendre duplication formula; principle of analytic continuation; Laplace transforms.

1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

In the usual notation, let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. Also let

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\} , \qquad \mathbb{N} = \{1, 2, 3, \dots\} = \mathbb{N}_0 \backslash \{0\} ,$$

$$\mathbb{Z}_0^- = \{0, -1, -2, \dots\} = \mathbb{Z}^- \cup \{0\} , \qquad \mathbb{Z}^- = \{-1, -2, -3, \dots\}$$

and $\mathbb{Z} = \mathbb{Z}_0^- \cup \mathbb{N}$ being the sets of integers.

Here, in our present investigation, we propose to explore several summation and other related formulas for the Gauss and Kummer hypergeometric functions which are, respectively, in the cases

$$p - 1 = q = 1 \quad \text{and} \quad p = q = 1.$$

Here *generalized hypergeometric function* ${}_pF_q$ with p numerator parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ and q denominator parameters $\beta_1, \beta_2, \ldots, \beta_q$, is defined by (see, for example, [15, p. 41 et.seq.]; see also [14, pp. 71–72]):

$$ {}_pF_q\left[\begin{array}{c}\alpha_1, \alpha_2, \ldots, \alpha_p; \\ \\ \beta_1, \beta_2, \ldots, \beta_q;\end{array} z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\alpha_j)_n}{\prod_{j=1}^{q}(\beta_j)_n} \frac{z^n}{n!} $$

$$ \Big(p, q \in \mathbb{N}_0;\ p \leqq q+1\ ;\ p \leqq q \text{ and } |z| < \infty\ ;\ p = q+1 \text{ and } |z| < 1; $$
$$ p = q+1, |z| = 1 \text{ and } \Re(\omega) > 0\ ;\ p = q+1, |z| = 1, z \neq 1 \text{ and } 0 \geq \Re(\omega) > -1\Big), $$

where

$$ \omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j $$

$$ \Big(\alpha_j \in \mathbb{C}\ (j = 1, 2, \ldots, p);\ \beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-(j = 1, 2, \ldots, q)\Big). $$

In terms of Gamma function $\Gamma(z)$, the widely-used Pochhammer symbol $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ is defined, in general, by

$$ (\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}). \end{cases} \tag{1.1} $$

It is understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ quotient exists (see, for details, [15]).

$$ \int_0^\infty e^{-st} t^{\alpha-1} \mathrm{d}t = \frac{\Gamma(\alpha)}{s^\alpha} \tag{1.2} $$

$$ \Big(\Re(s) > 0,\ 0 < \Re(\alpha) < \infty \qquad \text{or} \qquad \Re(s) = 0,\ 0 < \Re(\alpha) < 1\Big). $$

Most of the elementary functions as well as special functions of mathematical physics and other areas of applied sciences are special or limited cases of the Gauss hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; z)$. For a detailed history of this function, especially about the origin of term *hypergeometric* for it, by Kummer [8], see (among other places) [16, p. 281].

Euler's Beta-type integral representation for the Gauss hypergeometric function ${}_2F_1$ is given by [14, p. 65, Equation 1.5(11)]

$$(1.3)\qquad {}_2F_1\left[\begin{array}{c}\alpha,\beta;\\ \gamma;\end{array} z\right]=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\mathrm{d}t$$

$$\Big(|\arg(1-z)|\leqq\pi-\epsilon\ (0<\epsilon<\pi);\ \Re(\gamma)>\Re(\beta)>0;\ \alpha\in\mathbb{C}\Big),$$

or, equivalently [14, p. 65, Equation 1.5(10)],

$$(1.4)\qquad {}_2F_1\left[\begin{array}{c}\alpha,\beta;\\ \gamma;\end{array} z\right]=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-zt)^{-\beta}\mathrm{d}t$$

$$\Big(|\arg(1-z)|\leqq\pi-\epsilon\ (0<\epsilon<\pi);\ \Re(\gamma)>\Re(\alpha)>0;\ \beta\in\mathbb{C}\Big).$$

The familiar Beta function $B(\alpha,\beta)$ [14, p.8, Equation 1.1(43)] is defined by

$$(1.5)\qquad B(\alpha,\beta)=\begin{cases}\displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t & (\min\{\Re(\alpha),\Re(\beta)\}>0)\\[2ex] \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\beta\in\mathbb{C}\backslash\mathbb{Z}_0^-)\end{cases}$$

and Legendre's duplication formula [15, p. 23, Equation 1.1(25)] is given by

$$(1.6)\qquad \sqrt{(\pi)}\Gamma(2z)=2^{2z-1}\Gamma(z)\Gamma\Big(z+\frac{1}{2}\Big)\qquad \big(2z\in\mathbb{C}\backslash\mathbb{Z}_0^-\big).$$

In addition to the Gauss summation theorem [15, p. 30, Equation 1.2(7)] for ${}_2F_1(1)$, there are numerous closed-form summation theorems for ${}_2F_1(z)$ for different values of the argument z, see [1, 2, 3, 10, 13]. Here, for the purpose of our present investigation, we choose to recall the following summation theorem, which is due to Kummer [8, p. 134, Entry 1] .

Kummer's first summation theorem [6, p. 852, Equation (1.3)]:

$$(1.7)\qquad {}_2F_1\left[\begin{array}{c}a,b;\\ a-b+1;\end{array} -1\right]=\frac{2^{-a}\sqrt{\pi}\,\Gamma(a-b+1)}{\Gamma(\frac{1+a}{2})\Gamma(\frac{a}{2}-b+1)}=\frac{\Gamma(1+a-b)\Gamma(1+\frac{a}{2})}{\Gamma(1+\frac{a}{2}-b)\Gamma(1+a)}$$

$$\Big(a-b\in\mathbb{C}\backslash\mathbb{Z}^-;\ \Re(b)<1\Big).$$

2. Known Results Analogous to Kummer's First Summation Theorem (1.7)

In the year 1990, following summation theorems were recorded by Prudnikov *et al.* [9, p. 489, Entries (7.3.6.7), (7.3.6.8)]

$$(2.1)\qquad {}_2F_1\left[\begin{matrix} 1, \quad a; \\ -a-m; \end{matrix}\; -1\right] = \frac{2^{-m-2a-2}\Gamma(1-a)\Gamma(-m-a)}{\Gamma(-m-2a)} + \frac{1}{2}\sum_{r=0}^{m+1}\left\{\frac{(-1)^r (a)_r}{(-a-m)_r}\right\}$$

$$\left(\Re(a) < \left(\tfrac{-m}{2}\right);\ -m-2a,\ 1-a,\ -a-m,\ a \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}_0 \cup \{-1\}\right),$$

$$(2.2)\qquad {}_2F_1\left[\begin{matrix} 1, \quad a; \\ -a+m; \end{matrix}\; -1\right] = \frac{2^{m-2a-2}\Gamma(1-a)\Gamma(m-a)}{\Gamma(m-2a)} - \frac{1}{2}\sum_{r=1}^{m-2}\left\{\frac{(-1)^r (1+a-m)_r}{(1-a)_r}\right\}$$

$$\left(\Re(a) < \left(\tfrac{m}{2}\right);\ m-2a,\ 1-a,\ m-a,\ 1+a-m \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}\backslash\{1,2\}\right).$$

In the year 2007, the following summation theorems were given by Choi-Rathie and Malani [5, pp. 1523–1524, Equations (2.2), (2.3)]

$$(2.3)\qquad {}_2F_1\left[\begin{matrix} a, \quad b; \\ 1+a-b-m; \end{matrix}\; -1\right] = \frac{\Gamma(1+a-b-m)}{2\ \Gamma(a)}\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{\Gamma(\frac{r+a}{2})}{\Gamma(\frac{r+a}{2}+1-b-m)}\right\}$$

$$\left(\Re(b) < \left(\tfrac{2-m}{2}\right);\ a,\ 1+a-b-m \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}_0\right),$$

$$(2.4)\qquad {}_2F_1\left[\begin{matrix} a, \quad b; \\ 1+a-b+m; \end{matrix}\; -1\right] = \frac{\Gamma(1+a-b+m)}{2\ \Gamma(a)(1-b)_m}\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(-1)^r\Gamma(\frac{r+a}{2})}{\Gamma(\frac{r+a}{2}+1-b)}\right\}$$

$$\left(\Re(b) < \left(\tfrac{m+2}{2}\right);\ a,\ 1-b,\ 1+a-b+m \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}_0\right).$$

In the year 2011, the following summation theorems were given by Rakha-Rathie [11, p. 828, Theorems (4), (3)]

(2.5)

$${}_2F_1\left[\begin{matrix} a, \quad b; \\ 1+a-b-m; \end{matrix}\; -1\right] = \frac{2^{-a}\Gamma(\frac{1}{2})\Gamma(1+a-b-m)}{\Gamma(\frac{1+a-m}{2}-b)\Gamma(\frac{a-m}{2}+1-b)}\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{\Gamma(\frac{a-m+r+1}{2}-b)}{\Gamma(\frac{1+a+r-m}{2})}\right\}$$

$$\left(\Re(b) < \left(\tfrac{2-m}{2}\right);\ 1+a-b-m,\ 1+a-m-2b \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}_0\right),$$

(2.6)

$${}_2F_1\left[\begin{matrix} a, \quad b; \\ 1+a-b+m; \end{matrix}\; -1\right] = \frac{2^{-a}\Gamma(\frac{1}{2})\Gamma(b-m)\Gamma(1+a-b+m)}{\Gamma(b)\Gamma(\frac{a+m+1}{2}-b)\Gamma(\frac{a+m}{2}-b+1)}\sum_{r=0}^{m}\left\{(-1)^r\binom{m}{r}\frac{\Gamma(\frac{a+m+r+1}{2}-b)}{\Gamma(\frac{1+a+r-m}{2})}\right\}$$

$$\left(\Re(b) < \left(\tfrac{2+m}{2}\right);\ b,\ b-m,\ 1+a-b+m, 1+a+m-2b \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}_0\right).$$

3. New Results Analogous to Kummer's First Summation Theorem (1.7)

Any values of parameters and variables leading to the result which do not make sense are tacitly excluded, then we have

(3.1)

$$ {}_2F_1\left[\begin{matrix} a, b; \\ a-b-m; \end{matrix}\ -1\right] = \frac{\Gamma(a-b-m)}{2\,\Gamma(a)}\sum_{r=0}^{m}\left\{\binom{m}{r}\left[\frac{\Gamma(\frac{a+r}{2})}{\Gamma(\frac{a+r-2b-2m}{2})}+\frac{\Gamma(\frac{a+r+1}{2})}{\Gamma(\frac{a+r+1-2b-2m}{2})}\right]\right\}$$

$$\left(\Re(b) < \left(\tfrac{1-m}{2}\right);\ a,\ a-b-m \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}_0\right),$$

(3.2)

$$ {}_2F_1\left[\begin{matrix} a, b; \\ a-b+m; \end{matrix}\ -1\right] = \frac{\Gamma(a-b+m)}{2\Gamma(a)(-b)_m}\sum_{r=0}^{m}\left\{\binom{m}{r}\left[\frac{(-1)^r\Gamma(\frac{a+r}{2})}{\Gamma(\frac{a+r-2b}{2})}+\frac{(-1)^r\Gamma(\frac{a+r+1}{2})}{\Gamma(\frac{a+r+1-2b}{2})}\right]\right\}$$

$$\left(\Re(b) < \left(\tfrac{m+1}{2}\right);\ a,\ -b,\ a-b+m \in \mathbb{C}\backslash\mathbb{Z}_0^-\ ; m \in \mathbb{N}_0\right),$$

(3.3)
$$ {}_2F_1\left[\begin{matrix} n, \quad a; \\ -a-m; \end{matrix}\ -1\right] = \frac{\Gamma(-m-a)}{2\Gamma(n)}\sum_{r=0}^{m+n+1}\left\{\frac{(-1)^r(-m-n-1)_r\Gamma(\frac{r+n}{2})}{r!\Gamma(\frac{r-n-2a-2m}{2})}\right\}$$

$$\left(\Re(a) < \left(\tfrac{1-m-n}{2}\right);\ n,\ -m-a \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m+n \in \mathbb{N}_0\cup\{-1\}\right),$$

(3.4)
$$ {}_2F_1\left[\begin{matrix} n, \quad a; \\ -a+m; \end{matrix}\ -1\right] = \frac{\Gamma(1-a)\Gamma(m-a)}{2\Gamma(n)\Gamma(m-a-n)}\sum_{r=0}^{m-n-1}\left\{\frac{(1+n-m)_r\Gamma(\frac{r+n}{2})}{r!\Gamma(\frac{n+r+2-2a}{2})}\right\}$$

$$\left(\Re(a) < \left(\tfrac{m+1-n}{2}\right);\ n,\ m-a-n,\ 1-a,\ m-a \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m-n \in \mathbb{N}\right),$$

(3.5)
$$ {}_2F_1\left[\begin{matrix} n, \quad a; \\ -a+m; \end{matrix}\ -1\right] = \frac{\Gamma(m-a)}{2\Gamma(n)}\sum_{r=0}^{n+1-m}\left\{\frac{(-1)^r(m-n-1)_r\Gamma(\frac{r+n}{2})}{r!\Gamma(\frac{r-2a+2m-n}{2})}\right\}$$

$$\left(\Re(a) < \left(\tfrac{m+1-n}{2}\right);\ n,\ m-a \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ n-m \in \mathbb{N}_0\cup\{-1\}\right).$$

Remark 1: If we put $n=1$ in summation theorems (3.3) and (3.4), we get the results analogous to known summation theorems (2.1) and (2.2) respectively.

$$(3.6)\qquad {}_2F_1\left[\begin{matrix}1, \quad a;\\ -a-m;\end{matrix}\; -1\right] = \frac{\Gamma(-m-a)}{2}\sum_{r=0}^{m+2}\left\{\frac{(-1)^r(-m-2)_r\Gamma(\frac{r+1}{2})}{r!\Gamma(\frac{r-1-2a-2m}{2})}\right\}$$

$$\left(\Re(a) < \left(\tfrac{-m}{2}\right);\ -m-a \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}_0 \cup \{-1,-2\}\right),$$

$$(3.7)\qquad {}_2F_1\left[\begin{matrix}1, \quad a;\\ -a+m;\end{matrix}\; -1\right] = \frac{(m-a-1)\Gamma(1-a)}{2}\sum_{r=0}^{m-2}\left\{\frac{(2-m)_r\Gamma(\frac{r+1}{2})}{r!\Gamma(\frac{r+3-2a}{2})}\right\}$$

$$\left(\Re(a) < \left(\tfrac{m}{2}\right);\ 1-a,\ m-a-1 \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}\backslash\{1\}\right).$$

Remark 2: The summation theorems (2.3, 2.4, 2.5, 2.6, 3.1 and 3.2) are the unifications and generalizations of summation theorems (given in terms of gamma function) recorded by Prudnikov *et al.* [9, p. 489, Entries (7.3.6.1), (7.3.6.2), (7.3.6.3) and (7.3.6.4)] and other summation theorems given by Choi [4, p.4, Table 1, p. 5, Table 2 ; m= 0, 1, 2, ..., 9.], Kim *et al.* [7, p.1070, Table 3 ; m= 0, 1, 2, ..., 5.] and Rathie-Kim [12, pp. 994–995, Table 1 ; m= 0, 1, 2, ..., 5.], are written in terms of greatest integer function, absolute valued function and gamma function. The coefficients associated with summation theorems discussed in the references [4, 7 and 12], are not the functions of m. For particular values of m, coefficients are calculated and are arranged in tabular form. The summation theorems of the references [4, 7 and 12] are not general in nature.

4. Proof of Summation Theorems

In order to evaluate ${}_2F_1\left[\begin{matrix}a, b;\\ a-b-m;\end{matrix}\; -1\right]$, apply Euler-Beta type integral representation (1.3) for ${}_2F_1$, we get

$$ {}_2F_1\left[\begin{matrix}a, b;\\ a-b-m;\end{matrix}\; -1\right] = \frac{\Gamma(a-b-m)}{\Gamma(a)\Gamma(-b-m)}\int_0^1 t^{a-1}(1-t)^{-b-m-1}(1+t)^{-b}\mathrm{d}t$$

$$= \frac{\Gamma(a-b-m)}{\Gamma(a)\Gamma(-b-m)}\int_0^1 t^{a-1}(1-t^2)^{-b-m-1}(1+t)^{m+1}\mathrm{d}t$$

$$= \frac{\Gamma(a-b-m)}{\Gamma(a)\Gamma(-b-m)}\int_0^1 t^{a-1}(1-t^2)^{-b-m-1}(1+t)\left\{\sum_{r=0}^{m}\frac{(-m)_r(-1)^r t^r}{r!}\right\}\mathrm{d}t$$

$$= \frac{\Gamma(a-b-m)}{\Gamma(a)\Gamma(-b-m)}\sum_{r=0}^{m}\frac{(-m)_r(-1)^r}{r!}\int_0^1 t^{r+a-1}(1-t^2)^{-b-m-1}(1+t)\mathrm{d}t$$

$$= \frac{\Gamma(a-b-m)}{2\Gamma(a)\Gamma(-b-m)} \sum_{r=0}^{m} \binom{m}{r} \left[\int_0^1 t^{r+a-1}(1-t^2)^{-b-m-1} + \int_0^1 t^{r+a}(1-t^2)^{-b-m-1} \right] \mathbf{d}y$$

$$= \frac{\Gamma(a-b-m)}{2\Gamma(a)\Gamma(-b-m)} \sum_{r=0}^{m} \binom{m}{r} \left[\int_0^1 y^{\frac{r+a}{2}-1}(1-y)^{-b-m-1} + \int_0^1 y^{\frac{r+a+1}{2}-1}(1-y)^{-b-m-1} \right] \mathbf{d}y$$

which, in view of Beta function definition (1.5), yields the desired formula (3.1) by appealing also to the principle of analytic continuation. Similarly we can derive other results of section 3.

5. Applications of Summation Theorems

For the classical Laplace transform defined by

$$\mathfrak{L}\{f(t):s\} = \int_0^\infty e^{-st} f(t)dt = \mathcal{F}(s),$$

whenever the integral exists in the Lebesgue sense, it is easily seen for Kummer's confluent hypergeometric function ${}_1F_1$ that (see, for example, [15, p. 219, Equation 4.1(6)])

$$\mathfrak{L}\left\{ t^{\lambda-1} {}_1F_1 \left[\begin{matrix} \mu; \\ \nu; \end{matrix} \ zt \right] : s \right\} = \int_0^\infty e^{-st} t^{\lambda-1} {}_1F_1 \left[\begin{matrix} \mu; \\ \nu; \end{matrix} \ zt \right] \mathbf{d}t$$

(5.1) $$= \frac{\Gamma(\lambda)}{s^\lambda} {}_2F_1 \left[\begin{matrix} \lambda, \mu; \\ \nu; \end{matrix} \ \frac{z}{s} \right],$$

$$\Big(|s| > |z| \ ; \ |s| = |z|, \Re(\nu-\mu-\lambda) > 0 \ ; \ |s| = |z| \ , \ s \neq z \ , 0 \geq \Re(\nu-\mu-\lambda) > -1 \ ;$$

$$\Re(\lambda) > 0 \ ; \ \nu \in \mathbb{C} \backslash \mathbb{Z}_0^- \text{ and } \Re(s) > \max\{\Re(z), 0\} \Big).$$

In this section, we apply the summation formulas (2.3), (2.4), (2.5), (2.6), (3.1), (3.2), (3.3), (3.4) and (3.5) in order to derive several closed-form expressions for the Laplace transforms of Kummer's confluent hypergeometric function ${}_1F_1$ with suitable convergence conditions for validity of the results. In each of the following results, any exceptional values of the parameters and variables, which would make the results invalid, are tacitly excluded.

If we assume $\lambda = b, \mu = a, \nu = 1 + a - b - m$ and $z = -s$ in equation (5.1) and use summation theorem (2.3) we obtain

$$\mathfrak{L}\left\{t^{b-1}{}_1F_1\left[\begin{array}{c} a; \\ 1+a-b-m; \end{array} -st\right] : s\right\} = \int_0^\infty e^{-st}t^{b-1}{}_1F_1\left[\begin{array}{c} a; \\ 1+a-b-m; \end{array} -st\right] \mathrm{d}t$$

(5.2)
$$= \frac{\Gamma(b)}{s^b}\frac{\Gamma(1+a-b-m)}{2\ \Gamma(a)}\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{\Gamma(\frac{r+a}{2})}{\Gamma(\frac{r+a}{2}+1-b-m)}\right\}$$

$$\left(\Re(b) < \left(\tfrac{2-m}{2}\right);\ a,\ 1+a-b-m \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}_0\ ;\ \Re(b) > 0, \Re(s) > 0\right).$$

If we choose $\lambda = a, \mu = b, \nu = 1+a-b-m$ and $z = -s$ in equation (5.1) and apply summation theorem (2.3) we deduce

$$\mathfrak{L}\left\{t^{a-1}{}_1F_1\left[\begin{array}{c} b; \\ 1+a-b-m; \end{array} -st\right] : s\right\} = \int_0^\infty e^{-st}t^{a-1}{}_1F_1\left[\begin{array}{c} b; \\ 1+a-b-m; \end{array} -st\right] \mathrm{d}t$$

(5.3)
$$= \frac{\Gamma(1+a-b-m)}{2\ s^a}\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{\Gamma(\frac{r+a}{2})}{\Gamma(\frac{r+a}{2}+1-b-m)}\right\}$$

$$\left(\Re(b) < \left(\tfrac{2-m}{2}\right);\ a,\ 1+a-b-m \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}_0\ ;\ \Re(a) > 0, \Re(s) > 0\right).$$

If we let $\lambda = b, \mu = a, \nu = 1+a-b+m$ and $z = -s$ in equation (5.1) and use summation theorem (2.4) we obtain

$$\mathfrak{L}\left\{t^{b-1}{}_1F_1\left[\begin{array}{c} a; \\ 1+a-b+m; \end{array} -st\right] : s\right\} = \int_0^\infty e^{-st}t^{b-1}{}_1F_1\left[\begin{array}{c} a; \\ 1+a-b+m; \end{array} -st\right] \mathrm{d}t$$

(5.4)
$$= \frac{\Gamma(b)}{s^b}\frac{\Gamma(1+a-b+m)}{2\ \Gamma(a)(1-b)_m}\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(-1)^r\Gamma(\frac{r+a}{2})}{\Gamma(\frac{r+a}{2}+1-b)}\right\}$$

$$\left(\Re(b) < \left(\tfrac{m+2}{2}\right);\ a,\ 1-b,\ 1+a-b+m \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}_0\ ;\ \Re(b) > 0, \Re(s) > 0\right).$$

If we choose $\lambda = a, \mu = b, \nu = 1+a-b+m$ and $z = -s$ in equation (5.1) and apply summation theorem (2.4) we get

$$\mathfrak{L}\left\{t^{a-1}{}_1F_1\left[\begin{array}{c} b; \\ 1+a-b+m; \end{array} -st\right] : s\right\} = \int_0^\infty e^{-st}t^{a-1}{}_1F_1\left[\begin{array}{c} b; \\ 1+a-b+m; \end{array} -st\right] \mathrm{d}t$$

(5.5)
$$= \frac{\Gamma(1+a-b+m)}{2\ s^a(1-b)_m}\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{(-1)^r\Gamma(\frac{r+a}{2})}{\Gamma(\frac{r+a}{2}+1-b)}\right\}$$

$$\left(\Re(b) < \left(\tfrac{m+2}{2}\right);\ a,\ 1-b,\ 1+a-b+m \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}_0\ ;\ \Re(a) > 0, \Re(s) > 0\right).$$

If we set $\lambda = a, \mu = b, \nu = 1+a-b-m$ and $z = -s$ in equation (5.1) and use summation theorem (2.5) we obtain

$$\mathfrak{L}\left\{t^{a-1}{}_1F_1\left[\begin{matrix} b; \\ 1+a-b-m; \end{matrix} -st\right] : s\right\} = \int_0^\infty e^{-st}t^{a-1}{}_1F_1\left[\begin{matrix} b; \\ 1+a-b-m; \end{matrix} -st\right]\mathrm{d}t$$

$$(5.6)\qquad = \frac{\Gamma(a)}{s^a}\frac{2^{-a}\Gamma(\frac{1}{2})\Gamma(1+a-b-m)}{\Gamma(\frac{1+a-m}{2}-b)\Gamma(\frac{a-m}{2}+1-b)}\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{\Gamma(\frac{a-m+r+1}{2}-b)}{\Gamma(\frac{1+a+r-m}{2})}\right\}$$

$$\left(\Re(b) < \left(\tfrac{2-m}{2}\right);\ 1+a-b-m,\ 1+a-m-2b \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}_0\ ;\ \Re(a) > 0, \Re(s) > 0\right).$$

If we assume $\lambda = b, \mu = a, \nu = 1+a-b-m$ and $z = -s$ in equation (5.1) and apply summation theorem (2.5) we obtain

$$\mathfrak{L}\left\{t^{b-1}{}_1F_1\left[\begin{matrix} a; \\ 1+a-b-m; \end{matrix} -st\right] : s\right\} = \int_0^\infty e^{-st}t^{b-1}{}_1F_1\left[\begin{matrix} a; \\ 1+a-b-m; \end{matrix} -st\right]\mathrm{d}t$$

$$(5.7)\qquad = \frac{\Gamma(b)}{s^b}\frac{2^{-a}\Gamma(\frac{1}{2})\Gamma(1+a-b-m)}{\Gamma(\frac{1+a-m}{2}-b)\Gamma(\frac{a-m}{2}+1-b)}\sum_{r=0}^{m}\left\{\binom{m}{r}\frac{\Gamma(\frac{a-m+r+1}{2}-b)}{\Gamma(\frac{1+a+r-m}{2})}\right\}$$

$$\left(\Re(b) < \left(\tfrac{2-m}{2}\right);\ 1+a-b-m,\ 1+a-m-2b \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}_0\ ;\ \Re(b) > 0, \Re(s) > 0\right).$$

If we let $\lambda = a, \mu = b, \nu = 1+a-b+m$ and $z = -s$ in equation (5.1) and use summation theorem (2.6) we find

$$\mathfrak{L}\left\{t^{a-1}{}_1F_1\left[\begin{matrix} b; \\ 1+a-b+m; \end{matrix} -st\right] : s\right\} = \int_0^\infty e^{-st}t^{a-1}{}_1F_1\left[\begin{matrix} b; \\ 1+a-b+m; \end{matrix} -st\right]\mathrm{d}t$$

$$(5.8)\qquad = \frac{\Gamma(a)}{s^a}\frac{2^{-a}\Gamma(\frac{1}{2})\Gamma(b-m)\Gamma(1+a-b+m)}{\Gamma(b)\Gamma(\frac{a+m+1}{2}-b)\Gamma(\frac{a+m}{2}-b+1)}\sum_{r=0}^{m}\left\{(-1)^r\binom{m}{r}\frac{\Gamma(\frac{a+m+r+1}{2}-b)}{\Gamma(\frac{1+a+r-m}{2})}\right\}$$

$$\left(\Re(b) < \left(\tfrac{2+m}{2}\right);\ b,\ b-m,\ 1+a-b+m, 1+a+m-2b \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}_0\ ;\ \Re(a) > 0, \Re(s) > 0\right).$$

If we select $\lambda = b, \mu = a, \nu = 1+a-b+m$ and $z = -s$ in equation (5.1) and apply summation theorem (2.6) we get

$$\mathfrak{L}\left\{t^{b-1}{}_1F_1\left[\begin{array}{c} a; \\ 1+a-b+m; \end{array} -st\right] : s\right\} = \int_0^\infty e^{-st}t^{b-1}{}_1F_1\left[\begin{array}{c} a; \\ 1+a-b+m; \end{array} -st\right]\mathrm{d}t$$

(5.9)
$$= \frac{2^{-a}}{s^b}\frac{\Gamma(\frac{1}{2})\Gamma(b-m)\Gamma(1+a-b+m)}{\Gamma(\frac{a+m+1}{2}-b)\Gamma(\frac{a+m}{2}-b+1)}\sum_{r=0}^{m}\left\{(-1)^r\binom{m}{r}\frac{\Gamma(\frac{a+m+r+1}{2}-b)}{\Gamma(\frac{1+a+r-m}{2})}\right\}$$

$$\left(\Re(b) < \left(\tfrac{2+m}{2}\right);\ b,\ b-m,\ 1+a-b+m, 1+a+m-2b \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}_0\ ; \right.$$
$$\left.\Re(b) > 0, \Re(s) > 0\right).$$

If we choose $\lambda = b, \mu = a, \nu = a-b-m$ and $z = -s$ in equation (5.1) and use summation theorem (3.1) we find

$$\mathfrak{L}\left\{t^{b-1}{}_1F_1\left[\begin{array}{c} a; \\ a-b-m; \end{array} -st\right] : s\right\} = \int_0^\infty e^{-st}t^{b-1}{}_1F_1\left[\begin{array}{c} a; \\ a-b-m; \end{array} -st\right]\mathrm{d}t$$

(5.10)
$$= \frac{\Gamma(b)}{s^b}\frac{\Gamma(a-b-m)}{2\ \Gamma(a)}\sum_{r=0}^{m}\left\{\binom{m}{r}\left[\frac{\Gamma(\frac{a+r}{2})}{\Gamma(\frac{a+r-2b-2m}{2})} + \frac{\Gamma(\frac{a+r+1}{2})}{\Gamma(\frac{a+r+1-2b-2m}{2})}\right]\right\}$$

$$\left(\Re(b) < \left(\tfrac{1-m}{2}\right);\ a,\ a-b-m \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}_0\ ;\ \Re(b) > 0, \Re(s) > 0\right).$$

If we let $\lambda = a, \mu = b, \nu = a-b-m$ and $z = -s$ in equation (5.1) and apply summation theorem (3.1) we find

$$\mathfrak{L}\left\{t^{a-1}{}_1F_1\left[\begin{array}{c} b; \\ a-b-m; \end{array} -st\right] : s\right\} = \int_0^\infty e^{-st}t^{a-1}{}_1F_1\left[\begin{array}{c} b; \\ a-b-m; \end{array} -st\right]\mathrm{d}t$$

(5.11)
$$= \frac{\Gamma(a-b-m)}{2\ s^a}\sum_{r=0}^{m}\left\{\binom{m}{r}\left[\frac{\Gamma(\frac{a+r}{2})}{\Gamma(\frac{a+r-2b-2m}{2})} + \frac{\Gamma(\frac{a+r+1}{2})}{\Gamma(\frac{a+r+1-2b-2m}{2})}\right]\right\}$$

$$\left(\Re(b) < \left(\tfrac{1-m}{2}\right);\ a,\ a-b-m \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m \in \mathbb{N}_0\ ;\ \Re(a) > 0, \Re(s) > 0\right).$$

If we let $\lambda = b, \mu = a, \nu = a-b+m$ and $z = -s$ in equation (5.1) and use summation theorem (3.2) we obtain

$$\mathfrak{L}\left\{t^{b-1}{}_1F_1\left[\begin{matrix} a; \\ a-b+m; \end{matrix}\ -st\right]:s\right\} = \int_0^\infty e^{-st}t^{b-1}{}_1F_1\left[\begin{matrix} a; \\ a-b+m; \end{matrix}\ -st\right]\mathrm{d}t$$

$$= \frac{\Gamma(b)}{s^b}\frac{\Gamma(a-b+m)}{2\,\Gamma(a)(-b)_m}\sum_{r=0}^{m}\left\{\binom{m}{r}\left[\frac{(-1)^r\Gamma(\frac{a+r}{2})}{\Gamma(\frac{a+r-2b}{2})}+\frac{(-1)^r\Gamma(\frac{a+r+1}{2})}{\Gamma(\frac{a+r+1-2b}{2})}\right]\right\} \tag{5.12}$$

$$\left(\Re(b) < \left(\tfrac{m+1}{2}\right);\ a,\ -b,\ a-b+m \in \mathbb{C}\backslash\mathbb{Z}_0^-\ ; m\in\mathbb{N}_0\ ;\ \Re(b)>0, \Re(s)>0\right).$$

If we choose $\lambda = a, \mu = b, \nu = a-b+m$ and $z = -s$ in equation (5.1) and apply summation theorem (3.2) we obtain

$$\mathfrak{L}\left\{t^{a-1}{}_1F_1\left[\begin{matrix} b; \\ a-b+m; \end{matrix}\ -st\right]:s\right\} = \int_0^\infty e^{-st}t^{a-1}{}_1F_1\left[\begin{matrix} b; \\ a-b+m; \end{matrix}\ -st\right]\mathrm{d}t$$

$$= \frac{\Gamma(a-b+m)}{2\,s^a(-b)_m}\sum_{r=0}^{m}\left\{\binom{m}{r}\left[\frac{(-1)^r\Gamma(\frac{a+r}{2})}{\Gamma(\frac{a+r-2b}{2})}+\frac{(-1)^r\Gamma(\frac{a+r+1}{2})}{\Gamma(\frac{a+r+1-2b}{2})}\right]\right\} \tag{5.13}$$

$$\left(\Re(b) < \left(\tfrac{m+1}{2}\right);\ a,\ -b,\ a-b+m \in \mathbb{C}\backslash\mathbb{Z}_0^-\ ; m\in\mathbb{N}_0\ ;\ \Re(a)>0, \Re(s)>0\right).$$

If we let $\lambda = a, \mu = n, \nu = -a-m$ and $z = -s$ in equation (5.1) and use summation theorem (3.3) we get

$$\mathfrak{L}\left\{t^{a-1}{}_1F_1\left[\begin{matrix} n; \\ -a-m; \end{matrix}\ -st\right]:s\right\} = \int_0^\infty e^{-st}t^{a-1}{}_1F_1\left[\begin{matrix} n; \\ -a-m; \end{matrix}\ -st\right]\mathrm{d}t$$

$$= \frac{\Gamma(a)}{s^a}\frac{\Gamma(-m-a)}{2\,\Gamma(n)}\sum_{r=0}^{m+n+1}\left\{\frac{(-1)^r(-m-n-1)_r\Gamma(\frac{r+n}{2})}{r!\Gamma(\frac{r-n-2a-2m}{2})}\right\} \tag{5.14}$$

$$\left(\Re(a) < \left(\tfrac{1-m-n}{2}\right);\ n,\ -m-a \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m+n\in\mathbb{N}_0\cup\{-1\}\ ;\ \Re(a)>0, \Re(s)>0\right).$$

If we assume $\lambda = n, \mu = a, \nu = -a-m$ and $z = -s$ in equation (5.1) and apply summation theorem (3.3) we obtain

$$\mathfrak{L}\left\{t^{n-1}{}_1F_1\left[\begin{matrix} a; \\ -a-m; \end{matrix}\ -st\right]:s\right\} = \int_0^\infty e^{-st}t^{n-1}{}_1F_1\left[\begin{matrix} a; \\ -a-m; \end{matrix}\ -st\right]\mathrm{d}t$$

$$= \frac{\Gamma(-m-a)}{2\,s^n}\sum_{r=0}^{m+n+1}\left\{\frac{(-1)^r(-m-n-1)_r\Gamma(\frac{r+n}{2})}{r!\Gamma(\frac{r-n-2a-2m}{2})}\right\} \tag{5.15}$$

$$\left(\Re(a) < \left(\tfrac{1-m-n}{2}\right);\ -m-a \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m+n\in\mathbb{N}_0\cup\{-1\}\ ;\ \Re(s)>0\right).$$

If we let $\lambda = a, \mu = n, \nu = -a+m$ and $z = -s$ in equation (5.1) and use summation theorem (3.4) we get

$$\mathfrak{L}\left\{t^{a-1}{}_1F_1\left[\begin{array}{r}n;\\-a+m;\end{array}\ -st\right]:s\right\} = \int_0^\infty e^{-st}t^{a-1}{}_1F_1\left[\begin{array}{r}n;\\-a+m;\end{array}\ -st\right]\mathrm{d}t$$

$$= \frac{\Gamma(a)}{s^a}\frac{\Gamma(m-a)\Gamma(1-a)}{2\,\Gamma(n)\Gamma(m-a-n)}\sum_{r=0}^{m-n-1}\left\{\frac{(n+1-m)_r\Gamma(\frac{r+n}{2})}{r!\Gamma(\frac{n+r+2-2a}{2})}\right\} \tag{5.16}$$

$$\left(\Re(a) < \left(\tfrac{m+1-n}{2}\right);\ n,\ m-a-n,\ 1-a, m-a \in \mathbb{C}\backslash\mathbb{Z}_0^-\ ;\ m-n\in\mathbb{N}\ ;\ \Re(a)>0, \Re(s)>0\right).$$

If we choose $\lambda = n, \mu = a, \nu = -a+m$ and $z = -s$ in equation (5.1) and apply summation theorem (3.4) we find

$$\mathfrak{L}\left\{t^{n-1}{}_1F_1\left[\begin{array}{r}a;\\-a+m;\end{array}\ -st\right]:s\right\} = \int_0^\infty e^{-st}t^{n-1}{}_1F_1\left[\begin{array}{r}a;\\-a+m;\end{array}\ -st\right]\mathrm{d}t$$

$$= \frac{\Gamma(m-a)\Gamma(1-a)}{2\,s^n\,\Gamma(m-a-n)}\sum_{r=0}^{m-n-1}\left\{\frac{(n+1-m)_r\Gamma(\frac{r+n}{2})}{r!\Gamma(\frac{n+r+2-2a}{2})}\right\} \tag{5.17}$$

$$\left(\Re(a) < \left(\tfrac{m+1-n}{2}\right);\ m-a-n,\ 1-a,\ m-a \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ m-n\in\mathbb{N}\ ;\ \Re(s)>0\right).$$

If we let $\lambda = a, \mu = n, \nu = -a+m$ and $z = -s$ in equation (5.1) and use summation theorem (3.5) we get

$$\mathfrak{L}\left\{t^{a-1}{}_1F_1\left[\begin{array}{r}n;\\-a+m;\end{array}\ -st\right]:s\right\} = \int_0^\infty e^{-st}t^{a-1}{}_1F_1\left[\begin{array}{r}n;\\-a+m;\end{array}\ -st\right]\mathrm{d}t$$

$$= \frac{\Gamma(a)}{s^a}\frac{\Gamma(-a+m)}{2\,\Gamma(n)}\sum_{r=0}^{n+1-m}\left\{\frac{(-1)^r(m-n-1)_r\Gamma(\frac{r+n}{2})}{r!\Gamma(\frac{r-2a+2m-n}{2})}\right\} \tag{5.18}$$

$$\left(\Re(a) < \left(\tfrac{m+1-n}{2}\right);\ n,\ m-a \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ n-m\in\mathbb{N}_0\cup\{-1\}\ ;\ \Re(a)>0, \Re(s)>0\right).$$

If we select $\lambda = n, \mu = a, \nu = -a+m$ and $z = -s$ in equation (5.1) and apply summation theorem (3.5) we deduce

$$\mathfrak{L}\left\{t^{n-1}{}_1F_1\left[\begin{array}{r}a;\\-a+m;\end{array}\ -st\right]:s\right\} = \int_0^\infty e^{-st}t^{n-1}{}_1F_1\left[\begin{array}{r}a;\\-a+m;\end{array}\ -st\right]\mathrm{d}t$$

$$= \frac{\Gamma(-a+m)}{2\,s^n}\sum_{r=0}^{n+1-m}\left\{\frac{(-1)^r(m-n-1)_r\Gamma(\frac{r+n}{2})}{r!\Gamma(\frac{r-2a+2m-n}{2})}\right\} \tag{5.19}$$

$$\left(\Re(a) < \left(\tfrac{m+1-n}{2}\right);\ m-a \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ n-m \in \mathbb{N}_0 \cup \{-1\}\ ;\ \Re(s) > 0\right).$$

We conclude our present investigation by observing that several other corollaries and consequences of the remaining summation formulas (2.1), (2.2), (3.6), (3.7) of sections 2, 3 and its applications in Laplace transforms of Kummer's confluent hypergeometric functions, can also be deduced in an analogous manner.

References

[1] Andrews, G. E., Askey, R. and Roy, R. ; *Special Function, Encyclopedia of Mathematics and its Applications*, Vol. 71, Cambridge University Press, Cambridge, 1999.

[2] Bailey, W. N. ; *Generalized Hypergeometric Series*, Cambridge Math. Tract No. 32, Cambridge University Press, Cambridge, 1935; Reprinted by Stechert-Hafner, New York, 1964.

[3] Carlson, B. C. ; *Special Functions of Applied Mathematics,* Academic Press, New York, San Francisco and London, 1977.

[4] Choi, J. ; Contiguous Extensions of Dixon's Theorem on the Sum of a ${}_3F_2$, *Journal of Inequalities and Applications,* Art. ID 589618, (2010), 1–17.

[5] Choi, J., Rathie, A. K. and Malani, S. ; Kummer's Theorem and its Contiguous Identities, *Taiwanese Journal of Mathematics* 11 (5) (2007), 1521–1527.

[6] Choi, J., Rathie, A. K. and Srivastava, H. M. ; A Generalization of a Formula Due to Kummer, *Integral transforms and Special Functions,* 22 (11) (2011), 851–859.

[7] Kim, Y. S. , Rathie, A. K. and Cvijović, D. ; New Laplace Transforms of Kummer's Confluent Hypergeometric Functions, *Mathematical and Computer Modelling,* 55 (2012), 1068–1071.

[8] Kummer, E. E. ; Über die hypergeometrische Reihe

$$1+\frac{\alpha.\beta}{1.\gamma}x+\frac{\alpha(\alpha+1).\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2+\frac{\alpha(\alpha+1)(\alpha+2).\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3+\ldots,$$

J. Reine Angew. Math. **15** (1836), 39–83 and 127–172; see also *Collected papers*, Vol. II: *Function Theory, Geometry and Miscellaneous* (Edited and with a Foreword by André Weil), Springer-Verlag, Berlin, Heidelberg and New York, 1975.

[9] Prudnikov, A. P., Brychkov, Yu. A. and Marichev, O. I. ; *Integrals and Series, Volume III : More Special Functions* Nauka, Moscow, 1986 (In Russian); (Translated from the Russian by G.G.Gould) Gordon and Breach Science Publishers, New York, 1990.

[10] Rainville, E. D. ; *Special Functions*, The Macmillan Company, New York, 1960 ; Reprinted by Chelsea Publ. Co., Bronx, New York, 1971.

[11] Rakha, M. A. and Rathie, A. K. ; Generalizations of Classical Summation Theorems for the Series ${}_2F_1$ and ${}_3F_2$ with Applications, *Integral Transforms and Special Functions,* 22 (11) (2011), 823–840.

[12] Rathie, A. K. and Kim, Y. S. ; Further Results on Srivastava's Triple Hypergeometric Series H_A and H_C, *Indian J. Pure Appl. Math.* 35 (8) (2004), 991–1002.

[13] Slater, L. J. ; *Generalized Hypergeometric Functions,* Cambridge University Press, Cambridge, 1966.

[14] Srivastava, H. M. and Choi, J. ; *Zeta and q-Zeta Functions and Associated Series and Integrals,* Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[15] Srivastava, H. M. and Manocha, H. L. ; *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

[16] Whittaker, E. T. and Watson, G. N. ; *A Course of Modern Analysis*, Fourth Edition, Cambridge University Press, Cambridge, London, and New York, 1927.