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SOME FIXED POINT RESULTS ON $(\varphi, L, \mathfrak{m})$-WEAK CONTRACTION IN CONE 2-METRIC SPACES

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Abstract. The purpose of this paper is to introduce the notion of $(\varphi, L, \mathfrak{m})$-weak contractions in cone 2-metric spaces over Banach algebra and to prove some fixed point results for such mappings. Our results generalize and extend several known results of literature into cone 2-metric spaces. Also, we provide some illustrative examples to show the significance of the results proved herein.
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1. Introduction and Preliminaries

In the year 1906, Fréchet [9] introduced the notion of metric spaces. The metric function is a real-valued function and takes values in the interval $[0, \infty)$. There are several generalizations of the concept of metric spaces. In 1934, Kurepa [2] (see also, [3]) introduced some abstract metric spaces in which the metric function takes values in an ordered vector space. Since then, several attempts has been made towards such generalizations. In 2007, Huang and Zhang [7] reintroduced the metric spaces in which metric takes values in an ordered Banach space and named such spaces as cone metric spaces. They went further and defined the Cauchy sequences and the convergence of sequence in terms of the interior points of a subset of ordered Banach space called cone. They also proved some fixed point results in such spaces with exploiting the notion of normality of the underlying cone. Recently, some papers appeared which tell that the fixed point results in cone metric spaces are a consequence of their corresponding metric versions (see, [4, 22, 23, 24]), therefore are equivalent to those in metric spaces. In view of such equivalence, Li and Xu [5] reconsidered the notion of cone metric and define the cone metric spaces over Banach algebra. They also improved the contractive conditions on self-mappings on cone metric spaces by using a vector contractive constant instead a real number. An example of Li and Xu [5] verifies their claims.

Another generalization of metric spaces is 2-metric space which was considered by Gähler in a series of papers [10, 11, 12]. Let X be a nonempty set, then the metric function is defined from the cartesian product $X \times X$ into the nonnegative real numbers. Gähler's 2-metric was defined from the product $X \times X \times X$ into the nonnegative real numbers. A function $d\colon X \times X \times X \to [0, \infty)$ is called a 2-metric and the pair (X, d) is called a 2-metric space if the following conditions are satisfied:

(1) for every $x, y \in X$ with $x \neq y$ there exists $z \in X$ such that $d(x, y, z) \neq 0$;
(2) if at least two of $x, y, z \in X$ are equal, then $d(x, y, z) = 0$;
(3) $d(x, y, z) = d(p(x, y, z))$ for all $x, y, z \in X$, where $p(x, y, z)$ denotes all the permutations of x, y, z;
(4) $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z, w \in X$.

Singh et al. [1] combined the concepts of cone metric and 2-metric and introduced a new type of spaces called cone 2-metric spaces. A fixed point result for contractive mapping in cone 2-metric spaces was also proved in [1]. Very recently, Wang et al. [15] improved the result of Singh et al. [1] by introducing the cone 2-metric spaces over Banach algebra and by using improved contractive conditions with vector contractive constants.

Banach contraction plays a very important role in several branches of applied mathematics. Due to simplicity and usefulness this principle is generalized by several researchers. In 2004, Berinde [16, 17, 18] introduced the notion of weak contractions and generalized Banach contraction and several other contractions. Further, he also investigate the notion of weak φ-contractions and obtained its fixed points (see, [19]).

In this paper, we give a minor modification to the definitions of cone 2-metric used in the papers [1, 15], so that, the new definition is more consistent with the definition of Gähler. Some fixed point results for $(\varphi, L, \mathfrak{m})$-weak contractions in cone 2-metric spaces over Banach algebra are proved. Our results generalize and extend several known results of literature into cone 2-metric spaces. Several examples are provided which illustrate and verify the significance of the new notions and the results proved herein.

First, we recall some definitions and properties which will be used in the sequel.

In the further discussion, we always suppose that A is a Banach algebra with multiplicative unit e, that is, $ex = xe = x$ for all $x \in A$. The following proposition can be found, e.g., in [21].

Proposition 1.1. *Let A be a Banach algebra with the unit e and $x \in A$. If the spectral radius $\rho(x)$ of x is less than 1, that is,*

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\geq 1} \|x^n\|^{1/n} < 1$$

then $e - x$ is invertible. Actually,

$$(e - x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

A subset P of A is called a cone if:

(1) P is nonempty closed and $\{\theta, e\} \subset P$;

(2) $\alpha P + \beta P \subset P$ for all nonnegative real numbers α, β;

(3) $P^2 = PP \subset P$;

(4) $P \cap (-P) = \{\theta\}$

where θ and e are respectively the zero vector and the unit of A.

Given a cone $P \subset A$, we define a partial ordering $\preceq$ in A with respect to P by $x \preceq y$ (or equivalently $y \succeq x$) if and only if $y - x \in P$. We shall write $x \prec y$ (or equivalently $y \succ x$) to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ (or equivalently $y \gg x$) will stand for $y - x \in \text{int}P$, where int$P$ denotes the interior of P.

The cone is called normal if there exists a number $K > 0$ such that for all $x, y \in P$

$$x \preceq y \implies \|x\| \leq K\|y\|.$$

The least number K satisfying the above inequality is called the normal constant of P. The cone P is called solid if int$P \neq \emptyset$.

In the following, we always assume that the cone P is solid cone in Banach algebra A and $\preceq$ is partial ordering with respect to P.

Definition 1.2 ([1, 15])**.** Let X be a nonempty set. Suppose the mapping $d : X \times X \times X \to P$ satisfies:

(1) for every $x, y \in X$ with $x \neq y$ there exists $z \in X$ such that $d(x, y, z) \neq \theta$;

(2) $\theta \preceq d(x, y, z)$ for all $x, y, z \in X$, and $d(x, y, z) = \theta$ if and only if at least two of x, y, z are equal;

(3) $d(x, y, z) = d(p(x, y, z))$ for all $x, y, z \in X$, where $p(x, y, z)$ denotes all the permutations of x, y, z;

(4) $d(x, y, z) \preceq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z, w \in X$.

Then d is called a cone 2-metric on X, and (X, d) will be called a cone 2-metric space over Banach algebra A. Cone 2-metric space will be called normal, if the cone P is normal.

In this paper, we use the following definition of cone 2-metric spaces which is more consistent with the definition of Gähler [10, 11, 12].

Definition 1.3. Let X be a nonempty set. Suppose the mapping $d\colon X \times X \times X \to P$ satisfies:

(1) for every $x, y \in X$ with $x \neq y$ there exists $z \in X$ such that $d(x, y, z) \neq \theta$;
(2) if at least two of $x, y, z \in X$ are equal, then $d(x, y, z) = \theta$;
(3) $d(x, y, z) = d(p(x, y, z))$ for all $x, y, z \in X$, where $p(x, y, z)$ denotes all the permutations of x, y, z;
(4) $d(x, y, z) \preceq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z, w \in X$.

Then d is called a cone 2-metric on X, and (X, d) will be called a cone 2-metric space over Banach algebra A.

Example 1.4 (A projective vector-area function)**.** Let $X = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\}$ and $A = C^1_{\mathbb{R}}[0,1]$ be the Banach algebra with the norm $\|x(t)\| = \|x(t)\|_\infty + \|x'(t)\|_\infty$, the point-wise multiplication and the unit $e(t) = 1$ for all $t \in [0,1]$. Let $P = \{\psi \in C^1_{\mathbb{R}}[0,1] \colon \psi(t) \geq 0 \text{ for all } t \in [0,1]\}$ be the solid cone in A. Define the function $d\colon X \times X \times X \to P$ by

$$d\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}\right) = \left|\det\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}\right| \cdot e^t.$$

Then, it is easy to see that (X, d) is a cone 2-metric space over the Banach algebra A in the sense of Definition 1.3, but not in the sense of Definition 1.2.

Example 1.5. Let $X = \left\{0, \frac{1}{n} \colon n \in \mathbb{N}\right\}$ and $A = C^1_{\mathbb{R}}[0,1]$ be the Banach algebra with the norm $\|x(t)\| = \|x(t)\|_\infty + \|x'(t)\|_\infty$, the point-wise multiplication and the unit $e(t) = 1$ for all $t \in [0,1]$. Let $P = \{\psi \in C^1_{\mathbb{R}}[0,1] \colon \psi(t) \geq 0 \text{ for all } t \in [0,1]\}$ be the solid cone in A. Let $\psi \in C^1_{\mathbb{R}}[0,1]$ be such that $\psi(t) > 0$ for all $t \in [0,1]$ and define the function $d\colon X \times X \times X \to P$ by

$$d(x, y, z) = \begin{cases} \psi(t), & t \in [0,1] \text{ if } x, y, z \text{ are distinct, } \left\{\frac{1}{n}, \frac{1}{n+1}\right\} \subset \{x, y, z\},\ n \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}$$

Then, it is easy to see that (X, d) be a cone 2-metric space over Banach algebra A in the sense of Definition 1.3, but not in the sense of Definition 1.2.

Definition 1.6 ([1]). Let (X, d) be a cone 2-metric space over Banach algebra A. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in A$ with $\theta \ll c$ (that is, $c \in \text{int}P$) there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x, a) \ll c$ for all $a \in X$ and for all $n > n_0$. Then $\{x_n\}$ is said to be convergent and converges to x. We denote it by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Remark 1.7 ([20]). The limit of a convergent sequence in a cone 2-metric space over Banach algebra A is unique.

Definition 1.8 ([1]). Let (X, d) be a cone 2-metric space over Banach algebra A. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in A$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m, a) \ll c$ for all $a \in X$ and for all $n, m > n_0$, then $\{x_n\}$ is said to be a Cauchy sequence.

Definition 1.9 ([1]). Let (X, d) be a cone 2-metric space over Banach algebra A. If every Cauchy sequence in X is convergent in X, then X is said to be a complete cone 2-metric space.

Definition 1.10 ([8, 25]). Let P be a solid cone in a Banach algebra A. A sequence $\{u_n\} \subset P$ is a c-sequence if for each $c \in A$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for $n > n_0$.

Proposition 1.11 ([14]). *Let P be a solid cone in a Banach algebra A and let $\{u_n\}$ be a sequence in P. Suppose that $k \in P$ is an arbitrarily given vector and $\{u_n\}$ is a c-sequence in P. Then $\{ku_n\}$ is a c-sequence.*

Proposition 1.12 ([14]). *Let A be a Banach algebra with a unit e, P be a cone in A. Then, for any $a, b \in A$, $c \in P$ with $a \preceq b$ we have $ac \preceq bc$.*

Lemma 1.13 ([13, 26]). *Let A be a Banach algebra with a solid cone P. Then:*

(a) If $a \preceq \lambda a$ with $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.

(b) If $\theta \preceq u \ll c$ for each $\theta \ll c$, then $u = \theta$.

(c) If $\|x_n\| \to 0$ as $n \to \infty$, then for any $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that, $x_n \ll c$ for all $n < n_0$.

2. Main Results

In this section, we state some new definitions and our main results.

Definition 2.1. Let A be a Banach algebra and P be a cone in A. A mapping $\mathfrak{m}\colon P \times P \to P$ is called a min function if

$$\mathfrak{m}(a, b) \preceq a, \ \mathfrak{m}(a, b) \preceq b \text{ for all } a, b \in P.$$

We denote the set of all min functions on P by $\mathfrak{M}_P$.

Example 2.2. If $A = \mathbb{R}$ and $P = [0, \infty)$. Define the function $\mathfrak{m}\colon [0,\infty) \times [0,\infty)$ by $\mathfrak{m}(a, b) = \min\{a, b\}$ for all $a, b \in [0, \infty)$, then $\mathfrak{m} \in \mathfrak{M}_{[0,\infty)}$.

Example 2.3. Let $A = \mathbb{R}^2$ and $P = \{(x, y)\colon x, y \geq 0\}$. Define a function $\mathfrak{m}\colon P \times P \to P$ by $\mathfrak{m}((a_1, a_2), (b_1, b_2)) = (\min\{a_1, b_1\}, \min\{a_2, b_2\})$ for all $(a_1, a_2), (b_1, b_2) \in [0, \infty)$, then $\mathfrak{m} \in \mathfrak{M}_P$.

Definition 2.4. Let A be a Banach algebra and P be a solid cone in A. A function $\varphi\colon P \to P$ is called a c-function if the following conditions are satisfied:

($c1$) φ is nondecreasing with respect to the partial order induced by P;

($c2$) the series $\sum\limits_{i=1}^{\infty} \varphi^i(a)$ converges in P for all $a \in P$;

($c3$) if $\{a_n\}$ be a c-sequence in P, then $\{\varphi(a_n)\}$ is a c-sequence.

By $\mathfrak{C}$ we denote the set of all c-functions defined on P.

Example 2.5. Let A be an arbitrary Banach algebra, P a solid cone in A and $k \in P$ be such that $\rho(k) < 1$. Then, the function $\varphi\colon P \to P$ defined by $\varphi(a) = ka$ for all $a \in P$, is a c-function.

Proof. The condition (c1) follows from Proposition 1.12. For (c2), note that if $a \in P$ we have

$$\begin{aligned} \sum_{i=1}^{\infty} \varphi^i(a) &= \sum_{i=1}^{\infty} k^i a \\ &= ka + k^2 a + \cdots = (k + k^2 + \cdots)a \\ &= [(e-k)^{-1} - e]a \in P. \end{aligned}$$

Therefore, (c2) is satisfied. Now, the condition (c3) follows from Proposition 1.11. □

Example 2.6. A function $\phi\colon [0, \infty) \to [0, \infty)$ is called a c-comparison function (see, [6, 18]) if (i) ϕ is monotonic increasing; (ii) $\sum\limits_{i=0}^{\infty} \phi^i(t)$ converges for all $t > 0$. Let $A = \mathbb{R}^2$ be the Banach algebra with the norm $\|(x_1, x_2)\| = |x_1| + |x_2|$ for all $(x_1, x_2) \in A$, the multiplication defined by $(x_1, x_2)(y_1, y_2) = (x_1, y_1, x_1 y_2 + x_2 y_1)$ and $P = \{(x_1, x_2) \in \mathbb{R}^2\colon x_1, x_2 \geq 0\}$ be the cone in A. Define a function $\varphi\colon P \to P$ by

$$\varphi(x_1, x_2) = (\phi_1(x_1), \phi_2(x_2))$$

for all $(x_1, x_2) \in P$, where ϕ_1, ϕ_2 are two c-comparison functions. Then φ is a c-function.

Remark 2.7. If $\varphi \in \mathfrak{C}$, then:

(i) if $a \preceq \varphi(a)$ for some $a \in P$, then $a = \theta$.
(ii) $\varphi(\theta) = \theta$.

Proof. (i). Let $a \in P$ and $a \preceq \varphi(a)$. By (c1) we have

$$a \preceq \varphi(a) \preceq \varphi^2(a) \cdots \preceq \varphi^n(a).$$

By (c2) we have $\varphi^n(a) \to \theta$ as $n \to \infty$. Therefore, it follows from the above inequality that $a = \theta$.

(ii). It follows from (i). □

Definition 2.8. Let (X, d) be a cone 2-metric space over Banach algebra A. A mapping $T\colon X \to X$ is said to be a Banach contraction if there exist $k \in P$ such that $\rho(k) < 1$ and the following condition is satisfied:

$$d(Tx, Ty, z) \preceq kd(x, y, z)$$

for all $x, y, z \in X$. The mapping $T\colon X \to X$ is said to be a φ-contraction if there exists $\varphi \in \mathfrak{C}$ such that the following condition is satisfied:

$$d(Tx, Ty, z) \preceq \varphi(d(x, y, z))$$

for all $x, y, z \in X$.

Definition 2.9. Let (X, d) be a cone 2-metric space over Banach algebra A. A mapping $T\colon X \to X$ is said to be a $(\varphi, L, \mathfrak{m})$-weak contraction if there exist $\varphi \in \mathfrak{C}$, $L \in P$ and $\mathfrak{m} \in \mathfrak{M}_P$ such that the following condition is satisfied:

$$d(Tx, Ty, z) \preceq \varphi(d(x, y, z)) + L\mathfrak{m}(d(y, Tx, z), d(x, Ty, z)) \tag{2.1}$$

for all $x, y, z \in X$.

It is obvious that every Banach contraction is a φ-contraction and every φ-contraction is a $(\varphi, L, \mathfrak{m})$-weak contraction. But the converse is not true in general, that is, there may exists a $(\varphi, L, \mathfrak{m})$-weak contraction which is neither a Banach contraction nor a φ-contraction in cone 2-metric space (see Example 2.12 of this paper).

Remark 2.10. Let (X, d) be a cone 2-metric space over Banach algebra A and $T\colon X \to X$ be a $(\varphi, L, \mathfrak{m})$-weak contraction on X. If $\{x_n\}$ is a Picard sequence in X with initial value $x_0 \in X$, that is, $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$, then for $k > j$ we have $d(x_k, x_{k-1}, x_j) = \theta$.

Proof. Let $k > j$, then we have

$$\begin{aligned} d(x_k, x_{k-1}, x_j) &= d(Tx_{k-1}, Tx_{k-2}, x_j) \\ &\preceq \varphi(d(x_{k-1}, x_{k-2}, x_j)) + L\mathfrak{m}(d(x_{k-2}, Tx_{k-1}, x_j), d(x_{k-1}, Tx_{k-2}, x_j)) \\ &= \varphi(d(x_{k-1}, x_{k-2}, x_j)) + L\mathfrak{m}(d(x_{k-2}, Tx_{k-1}, x_j), \theta) \\ &= \varphi(d(x_{k-1}, x_{k-2}, x_j)). \end{aligned}$$

Using (c1) and repeating this process $k - j - 1$ times we obtain:

$$d(x_k, x_{k-1}, x_j) \preceq \varphi^{k-j}(d(x_{j+1}, x_j, x_j)) = \varphi^{k-j}(\theta)$$

which with Remark 2.7 yields $d(x_k, x_{k-1}, x_j) = \theta$. □

Next theorem is an existence result for fixed point of a $(\varphi, L, \mathfrak{m})$-weak contraction on a cone 2-metric space.

Theorem 2.11. *Let (X, d) be a complete cone 2-metric space over Banach algebra A and $T\colon X \to X$ be a $(\varphi, L, \mathfrak{m})$-weak contraction on X. Then T has at least one fixed point.*

Proof. Let $x_0 \in X$ and define a sequence $\{x_n\}$ by

$$x_n = Tx_{n-1} \quad \text{for all} \quad n \in \mathbb{N}.$$

Since T is a $(\varphi, L, \mathfrak{m})$-weak contraction on X, for $n \in \mathbb{N}$ and $z \in X$ we have

$$\begin{aligned} d(x_n, x_{n+1}, z) &= d(Tx_{n-1}, Tx_n, z) \\ &\preceq \varphi(d(x_{n-1}, x_n, z)) + L\mathfrak{m}(d(x_n, Tx_{n-1}, z), d(x_{n-1}, Tx_n, z)) \\ &= \varphi(d(x_{n-1}, x_n, z)) + L\mathfrak{m}(d(x_n, x_n, z), d(x_{n-1}, x_{n+1}, z)) \\ &= \varphi(d(x_{n-1}, x_n, z)) + L\mathfrak{m}(\theta, d(x_{n-1}, x_{n+1}, z)) \\ &= \varphi(d(x_{n-1}, x_n, z)). \end{aligned}$$

Repeating this process and using the properties of φ we obtain

$$d(x_n, x_{n+1}, z) \preceq \varphi^n(d(x_0, x_1, z)). \tag{2.2}$$

Now, for $n, m \in \mathbb{N}$, $n > m$ and $z \in X$ from (2.2) and Remark 2.10 we obtain

$$\begin{aligned}
d(x_n, x_m, z) &\preceq d(x_n, x_m, x_{n-1}) + d(x_n, x_{n-1}, z) + d(x_{n-1}, x_m, z) \\
&\preceq \varphi^{n-1}(d(x_0, x_1, z)) + d(x_{n-1}, x_m, z) \\
&\preceq \varphi^{n-1}(d(x_0, x_1, z)) + d(x_{n-1}, x_m, x_{n-2}) + d(x_{n-1}, x_{n-2}, z) \\
&\quad + d(x_{n-2}, x_m, z) \\
&\preceq \varphi^{n-1}(d(x_0, x_1, z)) + \varphi^{n-2}(d(x_0, x_1, z)) + d(x_{n-2}, x_m, z) \\
&\vdots \\
&\preceq \varphi^{n-1}(d(x_0, x_1, z)) + \varphi^{n-2}(d(x_0, x_1, z)) \\
&\quad + \cdots + \varphi^{m}(d(x_0, x_1, z)) \\
&= \sum_{i=m}^{n-1} \varphi^{i}(d(x_0, x_1, z)).
\end{aligned}$$

Since the series $\sum_{i=1}^{\infty} \varphi^i(d(x_0, x_1, z))$ is convergent, the sequence of partial sums of this series will be a Cauchy sequence in A. Therefore, for given $c \gg \theta$, we can choose $n_0 \in \mathbb{N}$ and $\delta > 0$ such that $c + \{u \in E : \|u\| < \delta\} \subset \text{int}P$, $\left\|\sum_{i=m}^{n-1} \varphi^i(d(x_0, x_1, z))\right\| < \delta$ and $\sum_{i=m}^{n-1} \varphi^i(d(x_0, x_1, z)) \ll c$ for all $n, m > n_0, z \in X$. Therefore, the above inequality yields

$$d(x_n, x_m, z) \ll c \quad \text{for all} \quad n, m > n_0, z \in X.$$

Thus, we have proved that the sequence $\{x_n\}$ is a Cauchy sequence in X. By completeness of the space (X, d), there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. We shall show that x^* is a fixed point of T.

For every $n \in \mathbb{N}$, $z \in X$ we have

$$\begin{aligned}
d(x^*, Tx^*, z) &\preceq d(x^*, Tx^*, x_{n+1}) + d(x^*, x_{n+1}, z) + d(x_{n+1}, Tx^*, z) \\
&= d(Tx_n, Tx^*, x^*) + d(x^*, x_{n+1}, z) + d(Tx_n, Tx^*, z) \\
&\preceq \varphi(d(x_n, x^*, x^*)) + L\mathfrak{m}(d(x^*, Tx_n, x^*), d(x_n, Tx^*, x^*)) + d(x^*, x_{n+1}, z) \\
&\quad + \varphi(d(x_n, x^*, z)) + L\mathfrak{m}(d(x^*, Tx_n, z), d(x_n, Tx^*, z)) \\
&= \varphi(\theta) + L\mathfrak{m}(\theta, d(x_n, Tx^*, x^*)) + d(x^*, x_{n+1}, z) \\
&\quad + \varphi(d(x_n, x^*, z)) + L\mathfrak{m}(d(x^*, Tx_n, z), d(x_n, Tx^*, z)) \\
&= d(x^*, x_{n+1}, z) + \varphi(d(x_n, x^*, z)) + L\mathfrak{m}(d(x^*, x_{n+1}, z), d(x_n, Tx^*, z)).
\end{aligned}$$

Suppose, $\theta \ll c \in A$, then for every $i \in \mathbb{N}$ we have $\theta \ll \frac{c}{i} \in A$. Now since $\lim_{n\to\infty} x_n = x^*$ and by (c3) there exists $n_0 \in \mathbb{N}$ such that $d(x^*, x_{n+1}, z) \ll \frac{c}{i}$ and $\varphi(d(x_n, x^*, z)) \ll \frac{c}{i}$ for all $n > n_0$. Therefore, for sufficient large i, the above inequality with the definition of $\mathfrak{m}$ yields

$$d(x^*, Tx^*, z) \ll (2e + L)\frac{c}{i}$$

that is, $(2e + L)\frac{c}{i} - d(x^*, Tx^*, z) \in \text{int}P$. Since P is closed, letting $i \to \infty$ we obtain $-d(x^*, Tx^*, z) \in P$, that is, $d(x^*, Tx^*, z) = \theta$ for all $z \in X$, that is, $Tx^* = x^*$. Thus, x^* is a fixed point of T. □

Example 2.12. Let $A = \mathbb{R}^2$ and $P = [0, \infty) \times [0, \infty)$ be a cone in A. Then A is a Banach algebra with multiplication defined by

$$(x_1, x_2)(y_1, y_2) = (x_1x_2, x_1y_2 + x_2y_1)$$

for all $(x_1, x_2), (y_1, y_2) \in A$ with unit $e = (1, 0)$ and norm $\|(x_1, x_2)\| = |x_1| + |x_2|$. Let $X = [0, 1] \times [0, \infty)$ and for $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in X$ define a function $d\colon X \times X \times X \to P$ by

$$d(x, y, z) = \begin{cases} (x_1y_1 + y_1z_1 + z_1x_1, x_2y_2 + y_2z_2 + z_2x_2), & \text{if } x, y, z \text{ are distinct;} \\ (0, 0), & \text{otherwise.} \end{cases}$$

Then (X, d) is a complete cone 2-metric space over Banach algebra A. Define a mapping $T\colon X \to X$ by

$$T(x_1, x_2) = \begin{cases} \left(\frac{x_1}{2}, x_2\right), & \text{if } x_1 \leq \frac{1}{2}; \\ (1, x_2), & \text{otherwise.} \end{cases}$$

Define $\varphi\colon P \to P$ by $\varphi(x_1, x_2) = \frac{1}{2}(x_1, x_2)$ for all $(x_1, x_2) \in P$. Let $L = (2, 1) \in P$, then T is a $(\varphi, L, \mathfrak{m})$-weak contraction on X. Therefore, by Theorem 2.11 we can conclude the existence of fixed point of T in X. Indeed, the set of fixed points of T;

$$\textbf{Fix}(T) = \{(0, x), (1, x) \colon x \in [0, \infty)\}.$$

On the other hand, it is easy to see that T is not a φ-contraction (and so, not a Banach contraction) in the cone 2-metric space (X, d).

Remark 2.13. In the above example, we see that the fixed point a $(\varphi, L, \mathfrak{m})$-weak contraction on a complete cone 2-metric space may not be unique.

Next, we consider an additional contractive condition on T to ensure the uniqueness of the fixed point.

Theorem 2.14. *Let (X, d) be a complete cone 2-metric space over Banach algebra A and $T\colon X \to X$ be a $(\varphi, L, \mathfrak{m})$-weak contraction on X. Suppose in addition that there exist $\varphi \in \mathfrak{C}$ and $L \in P$ such that*

$$d(Tx, Ty, z) \preceq \varphi(d(x, y, z)) + L\mathfrak{m}(d(x, Tx, z), d(y, Ty, z)) \tag{2.3}$$

for all $x, y, z \in X$. Then T has a unique fixed point.

Proof. The existence of fixed point of T follows from Theorem 2.11. To prove uniqueness, suppose on the contrary that T has two distinct fixed points say x^* and y^*, that is, $Tx^* = x^*$ and $Ty^* = y^*$. Now using the contractive condition (2.3) we obtain

$$\begin{aligned} d(T^n x^*, T^n y^*, z) &= d(TT^{n-1}x^*, TT^{n-1}y^*, z) \\ &\preceq \varphi(d(T^{n-1}x^*, T^{n-1}y^*, z)) \\ &\quad + L\mathfrak{m}(d(T^{n-1}x^*, TT^{n-1}x^*, z), d(T^{n-1}y^*, TT^{n-1}y^*, z)) \\ &\preceq \varphi(d(T^{n-1}x^*, T^{n-1}y^*, z)) + L\mathfrak{m}(d(x^*, x^*, z), d(y^*, y^*, z)) \\ &= \varphi(d(T^{n-1}x^*, T^{n-1}y^*, z)) + L\mathfrak{m}(\theta, \theta) \\ &= \varphi(d(T^{n-1}x^*, T^{n-1}y^*, z)). \end{aligned}$$

By repeating the above process we obtain

$$d(x^*, y^*, z) = d(T^n x^*, T^n y^*, z) \preceq \varphi^n(d(x^*, y^*, z)).$$

The above inequality with the condition (c2) yields $d(x^*, y^*, z) = \theta$ for all $z \in X$, that is, $x^* = y^*$. This contradiction proves the uniqueness. □

Remark 2.15. Note that, in Example 2.12 the mapping T does not satisfy the condition (2.3). For example, one can see that the condition (2.3) is not satisfied at points $x = (1, 1), y = (1, 0), z = (0, 3)$.

Corollary 2.16. *Let (X, d) be a complete cone 2-metric space over Banach algebra A and $T\colon X \to X$ be a mapping. Suppose, there exist $k, L \in P$ with $\rho(k) < 1$ and $\mathfrak{m} \in \mathfrak{M}_P$ such that*

$$d(Tx, Ty, z) \preceq kd(x, y, z) + L\mathfrak{m}(d(y, Tx, z), d(x, Ty, z))$$

for all $x, y, z \in X$. Then T has at least one fixed point.

Proof. Define $\varphi\colon P \to P$ by $\varphi(a) = ka$ for all $a \in P$ and using Example 2.5 and Theorem 2.11 we obtain the required result. □

Corollary 2.17. *Let (X, d) be a complete cone 2-metric space over Banach algebra A and $T\colon X \to X$ be a mapping. Suppose, there exists $k \in P$ with $\rho(k) < 1$ such that*

$$d(Tx, Ty, z) \preceq kd(x, y, z)$$

for all $x, y, z \in X$. Then T has at a unique fixed point.

Proof. By taking $L = \theta$, $\varphi(a) = ka$ for all $a \in P$, in the Theorem 2.14 we obtain the existence and uniqueness of fixed point. □

In the next theorem, we show that the completeness of X can be replaces by another condition on T.

Theorem 2.18. *Let (X,d) be a cone 2-metric space over Banach algebra A and $T\colon X\to X$ be a $(\varphi, L, \mathfrak{m})$-weak contraction on X. Suppose that there exists $u\in X$ such that*

$$d(u,Tu,z)\preceq d(x,Tx,z) \tag{2.4}$$

for all $x,z\in X$. Then T has a fixed point. More precisely, u is a fixed point of T.

Proof. Suppose, $F_z(x)=d(x,Tx,z)$ for all $x,z\in X$. Then by assumption we have

$$F_z(u)\preceq F_z(x) \quad \text{for all} \quad x,z\in X. \tag{2.5}$$

Now since $T\colon X\to X$ by (2.1) we have

$$\begin{aligned} F_z(Tu) &= d(Tu,TTu,z)\\ &\preceq \varphi(d(u,Tu,z))+L\mathfrak{m}(d(Tu,Tu,z),d(u,TTu,z))\\ &= \varphi(d(u,Tu,z))+L\mathfrak{m}(\theta,d(u,TTu,z))\\ &= \varphi(d(u,Tu,z))=\varphi(F_z(u)). \end{aligned}$$

By (c1) and (2.5) we have $\varphi(F_z(u))\preceq\varphi(F_z(Tu))$. Thus, $F_z(Tu)\preceq\varphi(F_z(Tu))$, and so, by Remark 2.7 we have $F_z(Tu)=\theta$. Therefore, from (2.5) we have $F_z(u)=d(u,Tu,z)=\theta$ for all $z\in X$, that is, $Tu=u$. Thus, u is a fixed point of X. □

Example 2.19. Let $\mathbb{Q}_{[0,1]}=\mathbb{Q}\cap[0,1]$ and $X=\{(x,0),(0,x)\colon x\in\mathbb{Q}_{[0,1]}\}$ and $A=\mathbb{R}^2$ be the Banach algebra with the norm $\|(x_1,x_2)\|=|x_1|+|x_2|$ and the multiplication defined by $(x_1,x_2)(y_1,y_2)=(x_1x_2,x_2y_1+y_2x_1)$ with unit $e=(1,0)$. Let $P=\{(x_1,x_2)\in\mathbb{R}^2\colon x_1,x_2\geq 0\}$ be the cone in A. Define a mapping $d: X\times X\times X\to P$ by: $d(\alpha_1,\alpha_2,\alpha_3)=d_1(\beta_1,\beta_2)$ for all $\alpha_1,\alpha_2,\alpha_3\in X$, where $\beta_1,\beta_2\in\{\alpha_1,\alpha_2,\alpha_3\}$ are such that $||\beta_1-\beta_2||=\min\{||\alpha_1-\alpha_2||,||\alpha_2-\alpha_3||,||\alpha_3-\alpha_1||\}$ and

$$\begin{aligned} d_1((x,0),(y,0)) &= \left(\tfrac{5}{4}|x-y|,|x-y|\right)\\ d_1((0,x),(0,y)) &= \left(|x-y|,\tfrac{3}{4}|x-y|\right)\\ d_1((x,0),(0,y)) &= d_1((0,y),(x,0))=\left(\tfrac{5}{4}x+y,x+\tfrac{3}{4}y\right). \end{aligned}$$

Then (X,d) is a cone 2-metric space which is not complete. Define the mapping $T\colon X\to X$ by

$$T(x,0)=\left(0,\frac{6x}{5}\right),\ T(0,x)=\left(\frac{2x}{3},0\right).$$

Then it is easy to see that T is $(\varphi, L, \mathfrak{m})$-weak contraction on X with $\varphi(a) = \frac{24}{25}a$ for all $a \in P$, $\mathfrak{m}((x_1, x_2), (y_1, y_2)) = (\min\{x_1, y_1\}, \min\{x_2, y_2\})$ and arbitrary $L \in P$. Also, we have

$$d((0,0), T(0,0), Z) \preceq d((x,0), T(x,0), Z); \; d((0,0), T(0,0), Z) \preceq d((0,x), T(0,x), Z)$$

for all $Z \in X, x \in \mathbb{Q} \cap [0,1]$. Therefore, all the conditions of Theorem 2.18 are satisfied and $(0,0)$ is the fixed point of T.

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