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THE DOUBLE ALMOST $(\lambda_m \mu_n)$ CONVERGENCE IN Γ^2-RIESZ SPACE DEFINED BY A MUSIELAK-ORLICZ FUNCTION

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Abstract. In this paper we introduce a new concept for almost $(\lambda_m \mu_n)$ convergence in Γ^2-Riesz spaces strong $P-$ convergent to zero with respect to an Musielak Orlicz functions and examine some properties of the resulting sequence spaces. We also introduce and study statistical convergence of almost $(\lambda_m \mu_n)$ convergence in Γ^2-Riesz space and also some inclusion theorems are discussed.
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1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^2 for the set of all complex double sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [1]. Later on it was investigated by Hardy [3]; Moricz [7]; Moricz and Rhoades [8]; Basarir and Solankan [2]; Tripathy et al. [12-16]; Turkmenoglu [17]; Raj [9-11] and many others.

Let (x_{mn}) be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ give one space is said to be convergent if and only if the double sequence $(S_{mn})$is convergent, where

$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n = 1, 2, 3, ...) .$$

A sequence $x = (x_{mn})$is said to be double analytic if

$$sup_{m,n} |x_{mn}|^{\frac{1}{m+n}} < \infty.$$

The vector space of all double analytic sequences are usually denoted by Λ^2. A sequence $x = (x_{mn})$ is called double entire sequence if

$$|x_{mn}|^{\frac{1}{m+n}} \to 0 \text{ as } m, n \to \infty.$$

The vector space of all double entire sequences are usually denoted by Γ^2. Let the set of sequences with this property be denoted by Λ^2 and Γ^2 is a metric space with the metric

$$d(x,y) = sup_{m,n} \left\{ |x_{mn} - y_{mn}|^{\frac{1}{m+n}} : m,n : 1,2,3,... \right\}, \tag{1.1}$$

forall $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ $in \, \Gamma^2$. Let $\phi = \{finite\ sequences\}$.

Consider a double sequence $x = (x_{mn})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\delta_{ij}$ for all $m, n \in \mathbb{N}$,

$$\delta_{mn} = \begin{pmatrix} 0 & 0 & ...0 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & ...1 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \end{pmatrix}$$

with 1 in the $(m,n)^{th}$ position and zero otherwise.

A sequence $x = (x_{mn})$ is called double entire sequence if $|x_{mn}|^{\frac{1}{m+n}} \to 0$ as $m, n \to \infty$. The double entire sequences will be denoted by Γ^2. The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [5] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\},$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.
Here c, c_0 and ℓ_∞ denote the classes of convergent,null and bounded sclar valued single sequences respectively. The spaces $c(\Delta), c_0(\Delta), \ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k\geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}, (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\},$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$. The generalized difference double notion has the following representation: $\Delta^m x_{mn} = \Delta^{m-1} x_{mn} - \Delta^{m-1} x_{mn+1} - \Delta^{m-1} x_{m+1n} + \Delta^{m-1} x_{m+1n+1}$, and also this generalized difference double notion has the following binomial representation: $\Delta^m x_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{m} (-1)^{i+j} \binom{m}{i} \binom{m}{j} x_{m+i,n+j}$.

2. Definitions and Preliminaries

A double sequence $x = (x_{mn})$ has limit 0 (denoted by $P - limx = 0$)
(i.e) $|x_{mn}|^{1/m+n} \to 0$ as $m, n \to \infty$. We shall write more briefly as $P - convergent\ to\ 0$.

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called modulus function.

2.1. **Lemma.** [4,6] Let M be an Orlicz function which satisfies Δ_2- condition and let $0 < \delta < 1$. Then for each $t \geq \delta$, we have $M(t) < K\delta^{-1} M(2)$ for some constant $K > 0$.

A sequence $M = (M_{mn})$ of Orlicz function is called a Musielak-Orlicz function [see [18]]. A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup\{|v|\, u - (M_{mn})(u) : u \geq 0\}, m, n = 1, 2, \cdots$$

is called the complementary function of a sequence of Musielak-Orlicz M. For a given sequence of Musielak-Orlicz function M, the Musielak-Orlicz sequence space t_M is defined as follows

$$t_M = \left\{ x \in w^2 : I_M(|x_{mn}|)^{1/m+n} \to 0\, as\, m, n, k \to \infty \right\},$$

where I_M is a convex modular defined by

$$I_M(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M_{mn}(|x_{mn}|)^{1/m+n}.$$

2.2. **Definition.** A double sequence $x = (x_{mn})$ of real numbers is called almost $P-$ convergent to a limit 0 if

$$P - lim_{p,q \to \infty} sup_{r,s \geq 0} \frac{1}{pq} \sum_{m=r}^{r+p-1} \sum_{n=s}^{s+q-1} |x_{mn}|^{1/m+n} \to 0.$$

that is, the average value of (x_{mn}) taken over any rectangle
$\{(m, n) : r \leq m \leq r + p - 1, s \leq n \leq s + q - 1\}$ tends to 0 as both p and q to ∞, and this $P-$ convergence is uniform in r and s. Let denote the set of sequences with this property as $\left[\widehat{\Gamma^2}\right]$.

2.3. **Definition.** Let $\lambda = (\lambda_m)$ and $\mu = (\mu_n)$ be two non-decreasing sequences of positive real numbers such that each tending to ∞ and
$\lambda_{m+1} \leq \lambda_m + 1, \lambda_1 = 1,\ \ \mu_{n+1} \leq \mu_n + 1, \mu_1 = 1.$
Let $I_m = [m - \lambda_m + 1, m]$ and $I_n = [n - \mu_n + 1, n]$.
For any set $K \subseteq \mathbb{N} \times \mathbb{N}$, the number

$\delta_{\lambda,\mu}(K) = lim_{m,n\to\infty} \frac{1}{\lambda_m \mu_n} |\{(i,j) : i \in I_m, j \in I_n, (i,j) \in K\}|$, is called the $(\lambda,\mu)-$ density of the set K provided the limit exists.

2.4. **Definition.** A double sequence $x = (x_{mn})$ of numbers is said to be $(\lambda,\mu)-$ statistical convergent to a number ξ provided that for each $\epsilon > 0$,
$lim_{m,n\to\infty} \frac{1}{\lambda_m \mu_n} |\{(i,j) : i \in I_m, j \in I_n, |x_{mn} - \xi| \geq \epsilon\}| = 0$,
(i.e) the set $K(\epsilon) = \frac{1}{\lambda_m \mu_n} |\{(i,j) : i \in I_m, j \in I_n, |x_{mn} - \xi| \geq \epsilon\}|$ has $(\lambda,\mu)-$ density zero. In this case the number ξ is called the $(\lambda,\mu)-$ statistical limit of the sequence $x = (x_{mn})$ and we write $St_{(\lambda,\mu)} lim_{m,n\to\infty} = \xi$.

2.5. **Definition.** Let M be an Orlicz functions and $P = (p_{mn})$ be any factorable double sequence of strictly positive real numbers, we define the following sequence space:
$\Gamma^2_M [AC_{\lambda_m\mu_n}, P] =$
$\left\{ P - lim_{m,n} \frac{1}{\lambda_m\mu_n} \sum_{m\in I_{r,s}} \sum_{n\in I_{r,s}} \left[M |x_{m+r,n+s}|^{1/m+n} \right]^{p_{mn}} = 0, \right\}$, uniformly in r and s.

We shall denote $\Gamma^2_M [AC_{\lambda_m\mu_n}, P]$ as $\Gamma^2 [AC_{\lambda_m\mu_n}]$ respectively when $p_{mn} = 1$ for all m and n. If x is in $\Gamma^2 [AC_{\lambda_m\mu_n}, P]$, we shall say that x is almost $(\lambda_m\mu_n)$ in Γ^2 strongly $P-$convergent with respect to the Orlicz function M. Also note if $M(x) = x, p_{mn} = 1$ for all m, n and k then $\Gamma^2_M [AC_{\lambda_m\mu_n}, P] = \Gamma^2 [AC_{\lambda_m\mu_n}, P]$, which are defined as follows:
$\Gamma^2 [AC_{\lambda_m\mu_n}, P] =$
$\left\{ P - lim_{m,n} \frac{1}{\lambda_m\mu_n} \sum_{m\in I_{r,s}} \sum_{n\in I_{r,s}} \left[M |x_{m+r,n+s}|^{1/m+n} \right] = 0, \right\}$, uniformly in r and s.
Again note if $p_{mn} = 1$ for all m and n then $\Gamma^2_M [AC_{\lambda_m\mu_n}, P] = \Gamma^2_M [AC_{\lambda_m\mu_n}]$. We define
$\Gamma^2_M [AC_{\lambda_m\mu_n}, P] =$
$\left\{ P - lim_{m,n} \frac{1}{\lambda_m\mu_n} \sum_{m\in I_{r,s}} \sum_{n\in I_{r,s}} \left[M |x_{m+r,n+s}|^{1/m+n} \right]^{p_{mn}} = 0, \right\}$, uniformly in r and s.

2.6. **Definition.** Let M be an Orlicz functions $P = (p_{mn})$ be any factorable double sequence of strictly positive real numbers, we define the following sequence space:
$\Gamma^2_M [P] =$
$\left\{ P - lim_{p,q\to\infty} \frac{1}{pq} \sum_{m=1}^{p} \sum_{n=1}^{q} \left[M |x_{m+r,n+s}|^{1/m+n} \right]^{p_{mn}} = 0 \right\}$, uniformly in r and s.

If we take $M(x) = x, p_{mn} = 1$ for all m and n then we get $\Gamma^2_M [P] = \Gamma^2$.

2.7. **Definition.** The double number sequence x is $\widehat{S_{\lambda_m\mu_n}} - P-$ convergent to 0 then
$P - lim_{m,n} \frac{1}{\lambda_m\mu_n} max_{r,s} \left| \left\{ (m,n) \in I_{r,s} : M |x_{m+r,n+s}|^{1/m+n} \right\} \right| = 0.$
In this case we write $\widehat{S_{\lambda_m\mu_n}} - lim (M |x_{m+r,n+s}|)^{1/m+n} = 0$.

3. The double almost $(\lambda_m\mu_n)$ in Γ^2 in Riesz space

Let $n \in \mathbb{N}$ and X be a real vector space of dimension m, where $n \leq m$. A real valued function $d_p(x_1, \ldots, x_n) = \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p$ on X satisfying the following four conditions:

(i) $\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p = 0$ if and and only if $d_1(x_1,0),\ldots,d_n(x_n,0)$ are linearly dependent,
(ii) $\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p$ is invariant under permutation,
(iii) $\|(\alpha d_1(x_1,0),\ldots,\alpha d_n(x_n,0))\|_p = |\alpha| \, \|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p, \alpha \in \mathbb{R}$
(iv) $d_p\left((x_1,y_1),(x_2,y_2)\cdots(x_n,y_n)\right) = \left(d_X(x_1,x_2,\cdots x_n)^p + d_Y(y_1,y_2,\cdots y_n)^p\right)^{1/p} for 1 \leq p < \infty$; (or)
(v) $d\left((x_1,y_1),(x_2,y_2),\cdots(x_n,y_n)\right) := \sup\left\{d_X(x_1,x_2,\cdots x_n), d_Y(y_1,y_2,\cdots y_n)\right\}$,
for $x_1,x_2,\cdots x_n \in X, y_1,y_2,\cdots y_n \in Y$ is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces.

A trivial example of p product metric of n metric space is the p norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_E = sup\left(|det(d_{mn}\left(x_{mn},0\right))|\right) =$$
$$sup\left(\begin{vmatrix} d_{11}\left(x_{11},0\right) & d_{12}\left(x_{12},0\right) & ... & d_{1n}\left(x_{1n},0\right) \\ d_{21}\left(x_{21},0\right) & d_{22}\left(x_{22},0\right) & ... & d_{2n}\left(x_{1n},0\right) \\ . & & & \\ . & & & \\ . & & & \\ d_{n1}\left(x_{n1},0\right) & d_{n2}\left(x_{n2},0\right) & ... & d_{nn}\left(x_{nn},0\right) \end{vmatrix}\right)$$

where $x_i = (x_{i1},\cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1,2,\cdots n$.
If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the $p-$ metric. Any complete $p-$ metric space is said to be $p-$ Banach metric space.

3.1. **Definition.** Let L be a real vector space and let $\leq$ be a partial order on this space. L is said to be an ordered vector space if it satisfies the following properties :
(i) If $x,y \in L$ and $y \leq x$, then $y+z \leq x+z$ for each $z \in L$.
(ii) If $x,y \in L$ and $y \leq x$, then $\lambda y \leq \lambda x$ for each $\lambda \geq 0$.
If in addition L is a lattice with respect to the partial ordering, then L is said to be Riesz space.
A subset S of a Riesz space X is said to be solid if $y \in S$ and $|x| \leq |y|$ implies $x \in S$.
A linear topology Z on a Riesz space X is said to be locally solid if Z has a base at zero consisting of solid sets.

3.2. **Definition.** Let $\Gamma^{2\tau}_M\left[AC_{\lambda_m\mu_n}, P, \left\|\left(d\left(x_1,0\right), d\left(x_2,0\right),\cdots, d\left(x_{n-1},0\right)\right)\right\|_p\right]$ be a Riesz space. A sequence (x_{mn}) of points in Γ^2 is said to be $S(\tau)-$ convergent to an element 0 of Γ^2 if for each $\tau-$ neighbourhood V of zero,

$$\delta\left(\left\{m,n \in \mathbb{N} : M_{mn}\left(|x_{mn}|^{1/m+n}\right) \notin V\right\}\right) = 0$$

(i.e)., $\left(P-lim_{m,n}\frac{1}{\lambda_m\mu_n}\left\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}}\left[M\left|x_{m+r,n+s}\right|^{1/m+n}\right]^{p_{mn}}\notin V\right\}\right)=0.$
In this case we write
$S(\tau)-\left(P-lim_{m,n}\frac{1}{\lambda_m\mu_n}\left\{\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}}\left[M\left|x_{m+r,n+s}\right|^{1/m+n}\right]^{p_{mn}}\notin V\right\}\right)=0.$

4. Main Results

4.1. **Theorem.** For bounded factorable positive double number sequence (p_{mn}) the double space
$\Gamma^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]$ is linear space.
Proof: The proof can be established using standard technique.

4.2. **Theorem.** For any Musielak-Orlicz functions M, we have
$\Gamma^{2\tau}\left[AC_{\lambda_m\mu_n},P,\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]\subset$
$\Gamma^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]$
Proof: Let $x\in\Gamma^{2\tau}\left[AC_{\lambda_m\mu_n},P,\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]$ so that for each r and s
$\Gamma^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]=$
$\left\{lim_{m,n}\frac{1}{\lambda_m\mu_n}\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}}\left[M\left|x_{m+r,n+s}\right|^{1/m+n}\right]=0\right\}.$
Since M is continuous at zero, for $\epsilon>0$ and choose δ with $0<\delta<1$ such that $M(t)<\epsilon$ for every t with $0\le t\le\delta$. We obtain the following,
$\frac{1}{\lambda_m\mu_n}(\lambda_m\mu_n\epsilon)+\frac{1}{\lambda_m\mu_n}\sum_{m\in I_{r,s}\sum_{n\in I_{r,s}}\ and\,|x_{m+r,n+s}-0|>\delta}M\left[\left|x_{m+r,n+s}\right|^{1/m+n}\right]$
$\frac{1}{\lambda_m\mu_n}(\lambda_m\mu_n\epsilon)+\frac{1}{\lambda_m\mu_n}K\delta^{-1}f(2)\,\lambda_m\mu_n\Gamma^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right].$
Hence m and n goes to infinity, for each r and s we are granted
$x\in\Gamma^{2\tau}_M\left[AC_{\lambda_m\mu_n},P,\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right].$

4.3. **Theorem.** The double almost $(\lambda_m\mu_n)$ of Riesz sequence space. The following conditions are satisfied:
(i) $(x_{mnk})\xrightarrow{P}\Gamma^{2\tau}\left[\widehat{S_{\lambda_m\mu_n}},\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]$
(ii)$\Gamma^{2\tau}\left[AC_{\lambda_m\mu_n},P,\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]$ is a proper subset of
$\Gamma^{2\tau}\left[\widehat{S_{\lambda_m\mu_n}},\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]$
(iii) If $x\in\Lambda^2$ and $(x_{mn})\xrightarrow{P}\Gamma^{2\tau}\left[\widehat{S_{\lambda_m\mu_n}},\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]$ then $(x_{mn})\xrightarrow{P}$
$\Gamma^{2\tau}\left[AC_{\lambda_m\mu_n},\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]$
(iv) $\Gamma^{2\tau}\left[\widehat{S_{\lambda_m\mu_n}},\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]\bigcap\Lambda^2=$
$\Gamma^{2\tau}\left[AC_{\lambda_m\mu_n},\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]\bigcap\Lambda^2.$
Proof: **(i)** Since for all r and s

$\left|\left\{(m,n)\in I_{r,s}:|x_{m+r,n+s}|^{1/m+n}\right\}\right|=0\leq$
$\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}}|x_{m+r,n+s}|^{1/m+n}$, for all r and s
$P-lim_{m,n}\frac{1}{\lambda_m\mu_n}\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}}|x_{m+r,n+s}|^{1/m+n}=0$
This implies that for all r and s

$$P-lim_{m,n}\frac{1}{\lambda_m\mu_n}\left|\left\{(m,n)\in I_{r,s}:|x_{m+r,n+s}|^{1/m+n}=0\right\}\right|=0.$$

(ii)let $x=(x_{mn})$ be defined as follows:

$$(x_{mn})=\begin{pmatrix} 1 & 2 & 3 & ...\left[\sqrt[4]{\lambda_m\mu_n}\right]^{m+n} & 0 & \dots \\ 1 & 2 & 3 & ...\left[\sqrt[4]{\lambda_m\mu_n}\right]^{m+n} & 0 & \dots \\ . \\ . \\ . \\ 1 & 2 & 3 & ...\left[\sqrt[4]{\lambda_m\mu_n}\right]^{m+n} & 0 & \dots \\ . \\ . \\ . \\ 0 & 0 & 0 & ...0 & 0 & \dots \\ . \\ . \\ . \end{pmatrix};$$

that is $P-lim_{m,n}\frac{1}{\lambda_m\mu_n}\left|\left\{(m,n)\in I_{r,s}:|x_{m+r,n+s}|^{1/m+n}=0\right\}\right|=$
$P-lim_{m,n}\frac{1}{\lambda_m\mu_n}\left(\left[\sqrt[4]{\lambda_m\mu_n}\right]^{m+n}\right)^{1/m+n}=0.$
Therefore $(x_{mnk})\xrightarrow{P}\Gamma^{2\tau}\left[\widehat{S_{\lambda_m\mu_n}},\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]$. Also
$P-lim_{m,n}\frac{1}{\lambda_m\mu_n}\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}}|x_{m+r,n+s}|^{1/m+n}=$
$P-\frac{1}{2}\left(lim_{m,n}\frac{1}{\lambda_m\mu_n}\left(\left[\sqrt[4]{\lambda_m\mu_n}\right]^{m+n}\left[\sqrt[4]{\lambda_m\mu_n}\right]^{m+n}\left[\sqrt[4]{\lambda_m\mu_n}\right]^{m+n}\right)^{1/m+n}+1\right)=\frac{1}{2}.$
Therefore $(x_{mnk})\overset{P}{\nrightarrow}\Gamma^{2\tau}\left[AC_{\lambda_m\mu_n},\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]$.
(iii) Suppose $x\in\Lambda^2$ then for all r and s, $|x_{m+r,n+s}|^{1/m+n}\leq M$ for all m,n. Also for given $\epsilon>0$ and r and s large for all m and n we obtain the following:
$\frac{1}{\lambda_m\mu_n}\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}}|x_{m+r,n+s}|)^{1/m+n}=$
$\frac{1}{\lambda_m\mu_n}\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}}|x_{m+r,n+s}|\geq 0\,|x_{m+r,n+s}|^{1/m+n}+$
$\frac{1}{\lambda_m\mu_n}\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}}|x_{m+r,n+s}|\leq 0\,|x_{m+r,n+s}|^{1/m+n}$
$\leq\frac{M}{\lambda_m\mu_n}\left|\left\{(m,n)\in I_{r,s}:|x_{m+r,n+s}|^{1/m+n}\right\}=0\right|+\epsilon.$
Therefore $x\in\Lambda^2$ and $(x_{mn})\xrightarrow{P}\Gamma^{2\tau}\left[\widehat{S_{\lambda_m\mu_n}},\|(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right]$ then

$(x_{mnk}) \xrightarrow{P} \Gamma^{2\tau}\left[AC_{\lambda_m\mu_n}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$.
(iv)$\Gamma^{2\tau}\left[\widehat{S_{\lambda_m\mu_n}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \bigcap \Lambda^2 =$
$\Gamma^{2\tau}\left[AC_{\lambda_m\mu_n}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \bigcap \Lambda^2$. follows from (i),(ii) and (iii).

4.4. **Theorem.** If M be any Musielak-Orlicz functions then
$\Gamma^{2\tau}_M\left[AC_{\lambda_m\mu_n}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \not\subset$
$\Gamma^{2\tau}_M\left[\widehat{S_{\lambda_m\mu_n}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$
Proof: Let $x \in \Gamma^{2\tau}_M\left[AC_{\lambda_m\mu_n}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$, for all r and s. Therefore we have
$\frac{1}{\lambda_m\mu_n}\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}} M\left[|x_{m+r,n+s}|^{1/m+n}\right] \geq$
$\frac{1}{\lambda_m\mu_n}\sum_{m\in I_{r,s}}\sum_{n\in I_{r,s}} |x_{m+r,n+s}|$
$M\left[|x_{m+r,n+s}|^{1/m+n}\right] >$
$\frac{1}{\lambda_m\mu_n}M(0)\left|\left\{(m,n)\in I_{r,s} : |x_{m+r,n+s}|^{1/m+n}\right\} = 0\right|$.
Hence $x \notin \Gamma^{2\tau}_M\left[\widehat{S_{\lambda_m\mu_n}}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$.

4.5. **Theorem.** Let M be a Mousielak Orlicz function. Then
$\Gamma^{2\tau}_M\left[AC_{\lambda_m\mu_n}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \supseteq$
$\Gamma^{2\tau}\left[AC_{\lambda_m\mu_n}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ if and only if $sup_{mn}M_{mn}(x) < \infty$.
Proof: Let $x \in \Gamma^{2\tau}\left[AC_{\lambda_m\mu_n}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$. Suppose that $\lambda_m\mu_n = sup_{mn}M$ and $\lambda_m\mu_n = sup_{mn}\lambda_m\mu_n$. Since $M \leq \lambda_m\mu_n$ for all m, n. We have
$\Gamma^{2\tau}\left[AC_{\lambda_m\mu_n}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \leq \lambda_m\mu_n^{H_2}|KAC_{\lambda_m\mu_n}(\epsilon)|+|\lambda_m\mu_n(\epsilon)|^{H_2}$.
It follows from $\epsilon \to 0$ that $x \in \Gamma^{2\tau}_M\left[AC_{\lambda_m\mu_n}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$.
Conversely, suppose that $sup_{mn}M_{mn} = \infty$. Let $A = I$, unit matrix, define the sequence x by putting $x_{mn} = u_{rs}$ if $m = 1, 2, \cdots r$ and $n = 1, 2, \cdots s$ and $x_{mn} = 0$ otherwise. Then we have $x \in \Gamma^{2\tau}\left[AC_{\lambda_m\mu_n}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ but $x \notin \Gamma^{2\tau}_M\left[AC_{\lambda_m\mu_n}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$.

4.6. **Theorem.** Let M be a Musielak Orlicz function. Then
$\Gamma^{2\tau}_M\left[AC_{\lambda_m\mu_n}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \subseteq$
$\Gamma^{2\tau}\left[AC_{\lambda_m\mu_n}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ if and only if $lim_{mn\to\infty}M > 0$.
Proof: Let $\epsilon > 0$ and $x = (x_{mn}) \in \Gamma^{2\tau}_M\left[AC_{\lambda_m\mu_n}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$.
If $lim_{mn\to\infty}M > 0$ then there exists a number $d > 0$ such that $M(\epsilon) > d$ and $m, n \in \mathbb{N}$.
Let
$\Gamma^{2\tau}_M\left[AC_{\lambda_m\mu_n}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right] \geq \lambda_m\mu_n d^{H_1}KAC_{\lambda_m\mu_n}(\epsilon)$. It follows

that $\Gamma^{2\tau}\left[AC_{\lambda_m\mu_n}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$.

Conversely, suppose that $lim_{mn\to\infty}M > 0$ does not hold, then there is a number $t > 0$ such that $lim_{mn\to\infty}M(t) = 0$. We can select a almost $(\lambda_m\mu_n)$ sequence $\lambda_m\mu_n = (m_r n_s)$ such that $M(t) < 2^{-rs}$ for any $m > m_r, n > n_s$. Let $A = I$, unit matrix, define the almost $\lambda_m\mu_n$ sequence x by putting

$x_{mn} = t$ if $m_{r-1}, n_{s-1} < m, n < \frac{m_r n_s + m_{r-1}n_{s-1}}{2}$ and $x_{mn} = 0$ if $\frac{m_r n_s + m_{r-1}n_{s-1}}{2} \leq m, n \leq m_r n_s$. We have $x = (x_{mn}) \in \Gamma^{2\tau}_M\left[AC_{\lambda_m\mu_n}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$ but $x \notin \Gamma^{2\tau}\left[AC_{\lambda_m\mu_n}, \|(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right]$.

Competing Interests

The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

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