THE DOUBLE ALMOST \((\lambda_{m\mu n})\) CONVERGENCE IN \(\Gamma^2\)–RIESZ SPACE DEFINED BY A MUSIELAK-ORLICZ FUNCTION

DEEPMALA\(^1\), N. SUBRAMANIAN\(^2\) AND LAKSHMI NARAYAN MISHRA\(^3\),∗

\(^1\)SQC and OR Unit, Indian Statistical Institute, 203 B. T. Road, Kolkata, 700 108, West Bengal, India
\(^2\)Department of Mathematics, SASTRA University, Thanjavur-613 401, India
\(^3\)Department of Mathematics, National Institute of Technology, Silchar 788 010, District Cachar, Assam, India
∗Corresponding author

Abstract. In this paper we introduce a new concept for almost \((\lambda_{m\mu n})\) convergence in \(\Gamma^2\)–Riesz spaces strong \(P\)– convergent to zero with respect to an Musielak Orlicz function and examine some properties of the resulting sequence spaces. We also introduce and study statistical convergence of almost \((\lambda_{m\mu n})\) convergence in \(\Gamma^2\)–Riesz space and also some inclusion theorems are discussed.

2010 Mathematics Subject Classification. 40A05,40C05,40D05.
Key words and phrases. analytic sequence, Museialk-Orlicz function, double sequences, entire sequence, Lambda, Riesz space.

1. Introduction

Throughout \(w, \chi\) and \(\Lambda\) denote the classes of all gai and analytic scalar valued single sequences, respectively. We write \(w^2\) for the set of all complex double sequences \((x_{mn})\), where \(m, n \in \mathbb{N}\), the set of positive integers. Then, \(w^2\) is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [1]. Later on it was investigated by Hardy [3]; Moricz [7]; Moricz and Rhoades [8]; Basarir and Solankan [2]; Tripathy et al. [12-16]; Turkmenoglu [17]; Raj [9-11] and many others.

Let \((x_{mn})\) be a double sequence of real or complex numbers. Then the series \(\sum_{m,n=1}^{\infty} x_{mn}\) is called a double series. The double series \(\sum_{m,n=1}^{\infty} x_{mn}\) give one space is said to be convergent if and only if the double sequence \((S_{mn})\) is convergent, where

\[ S_{mn} = \sum_{i,j=1}^{m,n} x_{ij}(m, n = 1, 2, 3, \ldots) . \]

A sequence \(x = (x_{mn})\) is said to be double analytic if

\[ \sup_{m,n} |x_{mn}|^{\frac{1}{m+n}} < \infty. \]
The vector space of all double analytic sequences are usually denoted by $\Lambda^2$. A sequence $x = (x_{mn})$ is called double entire sequence if

$$|x_{mn}|^{1/m+n} \to 0 \text{ as } m, n \to \infty.$$ 

The vector space of all double entire sequences are usually denoted by $\Gamma^2$. Let the set of sequences with this property be denoted by $\Lambda^2$ and $\Gamma^2$ is a metric space with the metric

$$(1.1) \quad d(x, y) = \sup_{m,n} \{|x_{mn} - y_{mn}|^{1/m+n} : m, n : 1, 2, 3, \ldots\},$$

for all $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in $\Gamma^2$. Let $\phi = \{\text{finite sequences}\}$.

Consider a double sequence $x = (x_{mn})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by

$$x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \delta_{ij}$$

for all $m, n \in \mathbb{N}$,

$$\delta_{mn} = \begin{pmatrix}
0 & 0 & \ldots & 0 & 0 & \ldots \\
0 & 0 & \ldots & 0 & 0 & \ldots \\
\vdots & & & \ddots & & \\
0 & 0 & \ldots & 1 & 0 & \ldots \\
0 & 0 & \ldots & 0 & 0 & \ldots
\end{pmatrix}$$

with 1 in the $(m, n)^{th}$ position and zero otherwise.

A sequence $x = (x_{mn})$ is called double entire sequence if $|x_{mn}|^{1/m+n} \to 0$ as $m, n \to \infty$. The double entire sequences will be denoted by $\Gamma^2$. The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [5] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\},$$

for $Z = c, c_0$ and $\ell_\infty$, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Here $c, c_0$ and $\ell_\infty$ denote the classes of convergent, null and bounded scalar valued single sequences respectively. The spaces $c(\Delta), c_0(\Delta), \ell_\infty(\Delta)$ and $bv_p$ are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k \geq 1} |\Delta x_k| \quad \text{and} \quad \|x\|_{bv_p} = (\sum_{k=1}^{\infty} |x_k|^p)^{1/p}, \quad (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\},$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$. The generalized difference double notion has the following
A double sequence $x = (x_{mn})$ has limit 0 (denoted by $P - \lim x = 0$) (i.e.) $|x_{mn}|^{1/m+n} \to 0$ as $m, n \to \infty$. We shall write more briefly as $P - \text{convergent to } 0$.

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function $M$ is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called modulus function.

2.1. **Lemma.** [4,6] Let $M$ be an Orlicz function which satisfies $\Delta_2$–condition and let 0 $\leq \delta < 1$. Then for each $t \geq \delta$, we have $M(t) < K\delta^{-1}M(2)$ for some constant $K > 0$.

A sequence $M = (M_{mn})$ of Orlicz function is called a Musielak-Orlicz function [see [18]]. A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = \sup \{|v| u - (M_{mn})(u) : u \geq 0\}, m, n = 1, 2, \ldots$$

is called the complementary function of a sequence of Musielak-Orlicz $M$. For a given sequence of Musielak-Orlicz function $M$, the Musielak-Orlicz sequence space $t_M$ is defined as follows

$$t_M = \left\{ x \in R^2 : I_M(|x_{mn}|)^{1/m+n} \to 0 \text{ as } m, n, k \to \infty \right\},$$

where $I_M$ is a convex modular defined by

$$I_M(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M_{mn}(|x_{mn}|)^{1/m+n}.$$  

2.2. **Definition.** A double sequence $x = (x_{mn})$ of real numbers is called almost $P-$convergent to a limit 0 if

$$P - \lim_{p,q \to \infty} \sup_{r,s \geq 0} \frac{1}{pq} \sum_{m=r}^{r+p-1} \sum_{n=s}^{s+q-1} |x_{mn}|^{1/m+n} \to 0.$$ 

that is, the average value of $(x_{mn})$ taken over any rectangle

$$\{(m, n) : r \leq m \leq r+p-1, s \leq n \leq s+q-1\}$$

tends to 0 as both $p$ and $q$ to $\infty$, and this $P-$convergence is uniform in $r$ and $s$. Let denote the set of sequences with this property as $\hat{R}^2$.

2.3. **Definition.** Let $\lambda = (\lambda_m)$ and $\mu = (\mu_n)$ be two non-decreasing sequences of positive real numbers such that each tending to $\infty$ and

$\lambda_{m+1} \leq \lambda_m + 1, \lambda_1 = 1, \mu_{n+1} \leq \mu_n + 1, \mu_1 = 1.$

Let $I_m = [m - \lambda_m + 1, m]$ and $I_n = [n - \mu_n + 1, n]$.

For any set $K \subseteq \mathbb{N} \times \mathbb{N}$, the number
\[ \delta_{\lambda,\mu}(K) = \lim_{m,n \to \infty} \frac{1}{\lambda_{m,\mu}} \left| \{(i,j) : i \in I_m, j \in I_n, (i,j) \in K\} \right|, \]

is called the \((\lambda, \mu) - \) density of the set \(K\) provided the limit exists.

### 2.4. Definition

A double sequence \(x = (x_{mn})\) of numbers is said to be \((\lambda, \mu) - \) statistical convergent to a number \(\xi\) provided that for each \(\epsilon > 0,\)

\[ \lim_{m,n \to \infty} \frac{1}{\lambda_{m,\mu}} \left| \{(i,j) : i \in I_m, j \in I_n, |x_{mn} - \xi| \geq \epsilon\} \right| = 0, \]

(i.e) the set \(K(\epsilon) = \frac{1}{\lambda_{m,\mu}} \left| \{(i,j) : i \in I_m, j \in I_n, |x_{mn} - \xi| \geq \epsilon\} \right|\) has \((\lambda, \mu) - \) density zero. In this case the number \(\xi\) is called the \((\lambda, \mu) - \) statistical limit of the sequence \(x = (x_{mn})\) and we write \(St(\lambda,\mu) lim_{m,n \to \infty} x = \xi.\)

### 2.5. Definition

Let \(M\) be an Orlicz functions and \(P = (p_{mn})\) be any factorable double sequence of strictly positive real numbers, we define the following sequence space:

\[ \Gamma^2_M [AC_{\lambda_{m,\mu}}, P] = \left\{ P - \lim_{m,n \to \infty} \frac{1}{\lambda_{m,\mu}} \sum_{m \in I_r,s} \sum_{n \in I_r,s} M \left| x_{m+r,n+s} \right|^{1/m+n} p_{mn} = 0, \right\}, \text{uniformly in } r \text{ and } s. \]

We shall denote \(\Gamma^2_M [AC_{\lambda_{m,\mu}}, P] \) as \(\Gamma^2 [AC_{\lambda_{m,\mu}}]\) respectively when \(p_{mn} = 1\) for all \(m\) and \(n\). If \(x\) is in \(\Gamma^2 [AC_{\lambda_{m,\mu}}, P]\), we shall say that \(x\) is almost \((\lambda_{m,\mu})\) in \(\Gamma^2\) strongly \(P-\)convergent with respect to the Orlicz function \(M\). Also note if \(M(x) = x, p_{mn} = 1\) for all \(m, n\) and \(k\) then \(\Gamma^2_M [AC_{\lambda_{m,\mu}}, P] = \Gamma^2 [AC_{\lambda_{m,\mu}}, P]\), which are defined as follows:

\[ \Gamma^2_M [AC_{\lambda_{m,\mu}}, P] = \left\{ P - \lim_{m,n \to \infty} \frac{1}{\lambda_{m,\mu}} \sum_{m \in I_r,s} \sum_{n \in I_r,s} M \left| x_{m+r,n+s} \right|^{1/m+n} p_{mn} = 0, \right\}, \text{uniformly in } r \text{ and } s. \]

Again note if \(p_{mn} = 1\) for all \(m, n\) then \(\Gamma^2_M [AC_{\lambda_{m,\mu}}, P] = \Gamma^2_M [AC_{\lambda_{m,\mu}}]\). We define \(\Gamma^2_M [AC_{\lambda_{m,\mu}}, P] = \left\{ P - \lim_{m,n \to \infty} \frac{1}{\lambda_{m,\mu}} \sum_{m \in I_r,s} \sum_{n \in I_r,s} M \left| x_{m+r,n+s} \right|^{1/m+n} p_{mn} = 0, \right\}, \text{uniformly in } r \text{ and } s. \)

### 2.6. Definition

Let \(M\) be an Orlicz functions \(P = (p_{mn})\) be any factorable double sequence of strictly positive real numbers, we define the following sequence space:

\[ \Gamma^2_M [P] = \left\{ P - \lim_{p,q \to \infty} \frac{1}{pq} \sum_{m=1}^{p} \sum_{n=1}^{q} M \left| x_{m+r,n+s} \right|^{1/m+n} p_{mn} = 0, \right\}, \text{uniformly in } r \text{ and } s. \]

If we take \(M(x) = x, p_{mn} = 1\) for all \(m, n\) then we get \(\Gamma^2_M [P] = \Gamma^2.\)

### 2.7. Definition

The double number sequence \(x\) is \(S_{\lambda_{m,\mu}} - \) convergent to \(0\) then

\[ P - \lim_{m,n \to \infty} \frac{1}{\lambda_{m,\mu}} \max_{r,s} \left| \left\{ (m,n) \in I_r,s : M \left| x_{m+r,n+s} \right|^{1/m+n} \right\} p_{mn} = 0, \right\}. \]

In this case we write \(S_{\lambda_{m,\mu}} - \lim (M \left| x_{m+r,n+s} \right|^{1/m+n}) = 0.\)

### 3. The double almost \((\lambda_{m,\mu})\) in \(\Gamma^2\) in Riesz space

Let \(n \in \mathbb{N}\) and \(X\) be a real vector space of dimension \(m\), where \(n \leq m\). A real valued function \(d_p(x_1, \ldots, x_n) = \| (d_1(x_1,0), \ldots, d_n(x_n,0)) \|_p\) on \(X\) satisfying the following four conditions:

- \(d_p(x_1, \ldots, x_n) = \sum_{i=1}^{n} d_i(x_i,0)\)
- \(d_p(ax_1, ax_2, \ldots, ax_n) = |a| d_p(x_1, x_2, \ldots, x_n)\)
- \(d_p(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \leq d_p(x_1, x_2, \ldots, x_n) + d_p(y_1, y_2, \ldots, y_n)\)
- \(d_p(0,0,\ldots,0) = 0\)
(i) \( \| (d_1(x_1, 0), \ldots, d_n(x_n, 0)) \|_p = 0 \) if and only if \( d_1(x_1, 0), \ldots, d_n(x_n, 0) \) are linearly dependent,

(ii) \( \| (d_1(x_1, 0), \ldots, d_n(x_n, 0)) \|_p \) is invariant under permutation,

(iii) \( \| (\alpha d_1(x_1, 0), \ldots, \alpha d_n(x_n, 0)) \|_p = |\alpha| \| (d_1(x_1, 0), \ldots, d_n(x_n, 0)) \|_p, \alpha \in \mathbb{R} \)

(iv) \( d_p ((x_1, y_1), (x_2, y_2) \ldots (x_n, y_n)) = (d_X(x_1, x_2, \ldots x_n)^p + d_Y(y_1, y_2, \ldots y_n)^p)^{1/p} \) for \( 1 \leq p < \infty \); (or)

(v) \( d ((x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)) := \sup \{ d_X(x_1, x_2, \ldots x_n), d_Y(y_1, y_2, \ldots y_n) \} \),

for \( x_1, x_2, \ldots x_n \in X, y_1, y_2, \ldots y_n \in Y \) is called the \( p \) metric of the Cartesian product of \( n \) metric spaces is the \( p \) norm of the \( n \)-vector of the norms of the \( n \) subspaces.

A trivial example of \( p \) metric of \( n \) metric space is the \( p \) norm space is \( X = \mathbb{R} \) equipped with the following Euclidean metric in the product space is the \( p \) norm:

\[
\| (d_1(x_1, 0), \ldots, d_n(x_n, 0)) \|_E = \sup \left( |\text{det}(d_{mn}(x_{mn}, 0))| \right) = \\
\sup \begin{pmatrix}
d_{11} (x_{11}, 0) & d_{12} (x_{12}, 0) & \cdots & d_{1n} (x_{1n}, 0) \\
d_{21} (x_{21}, 0) & d_{22} (x_{22}, 0) & \cdots & d_{2n} (x_{2n}, 0) \\
\vdots & \vdots & \ddots & \vdots \\
d_{n1} (x_{n1}, 0) & d_{n2} (x_{n2}, 0) & \cdots & d_{nn} (x_{nn}, 0)
\end{pmatrix}
\]

where \( x_i = (x_{i1}, \ldots x_{in}) \in \mathbb{R}^n \) for each \( i = 1, 2, \ldots n \).

If every Cauchy sequence in \( X \) converges to some \( L \in X \), then \( X \) is said to be complete with respect to the \( p \)- metric. Any complete \( p \)-metric space is said to be a \( p \)-Banach metric space.

3.1. Definition. Let \( L \) be a real vector space and let \( \leq \) be a partial order on this space. \( L \) is said to be an ordered vector space if it satisfies the following properties:

(i) If \( x, y \in L \) and \( y \leq x \), then \( y + z \leq x + z \) for each \( z \in L \).

(ii) If \( x, y \in L \) and \( y \leq x \), then \( \lambda y \leq \lambda x \) for each \( \lambda \geq 0 \).

If in addition \( L \) is a lattice with respect to the partial ordering, then \( L \) is said to be Riesz space.

A subset \( S \) of a Riesz space \( X \) is said to be solid if \( y \in S \) and \( |x| \leq |y| \) implies \( x \in S \).

A linear topology \( Z \) on a Riesz space \( X \) is said to be locally solid if \( Z \) has a base at zero consisting of solid sets.

3.2. Definition. Let \( \Gamma^2_M \left[ AC_{\lambda_{m,n}}, P, \| (d(x_1, 0), d(x_2, 0), \ldots, d(x_{n-1}, 0)) \|_p \right] \) be a Riesz space. A sequence \( (x_{mn}) \) of points in \( \Gamma^2 \) is said to be \( S(\tau) \) convergent to an element \( 0 \) of \( \Gamma^2 \) if for each \( \tau \)-neighbourhood \( V \) of zero,

\[
\delta \left( \{ m, n \in \mathbb{N} : M_{mn} (|x_{mn}|^{1/m+n}) \notin V \} \right) = 0
\]
(i.e.), \( P - \lim_{m,n} \frac{1}{\lambda_{m\mu_n}} \left\{ \sum_{m \in I_r,s} \sum_{n \in I_r,s} M \left| x_{m+r,n+s} \right|^{1/m+n} \right\}^{p_{mn}} \notin V \} \) = 0.

In this case we write
\[ S(\tau) - \left( P - \lim_{m,n} \frac{1}{\lambda_{m\mu_n}} \left\{ \sum_{m \in I_r,s} \sum_{n \in I_r,s} M \left| x_{m+r,n+s} \right|^{1/m+n} \right\}^{p_{mn}} \notin V \} \) = 0.

4. Main Results

4.1. Theorem. For bounded factorable positive double number sequence \((p_{mn})\) the double space
\[ \Gamma_{M}^{2r}[AC_{\lambda_{m\mu_n}}, P, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \] is linear space.

Proof: The proof can be established using standard technique.

4.2. Theorem. For any Musielak-Orlicz functions \(M\), we have
\[ \Gamma_{M}^{2r}[AC_{\lambda_{m\mu_n}}, P, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \subseteq \Gamma_{M}^{2r}[AC_{\lambda_{m\mu_n}}, P, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \]

Proof: Let \( x \in \Gamma_{M}^{2r}[AC_{\lambda_{m\mu_n}}, P, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \) so that for each \( r \) and \( s \)
\[ \Gamma_{M}^{2r}[AC_{\lambda_{m\mu_n}}, P, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] = \left\{ \lim_{m,n} \frac{1}{\lambda_{m\mu_n}} \sum_{m \in I_r,s} \sum_{n \in I_r,s} M \left| x_{m+r,n+s} \right|^{1/m+n} \right\} = 0 \].

Since \( M \) is continuous at zero, for \( \epsilon > 0 \) and choose \( \delta \) with \( 0 < \delta < 1 \) such that \( M(t) < \epsilon \) for every \( t \) with \( 0 \leq t \leq \delta \). We obtain the following,
\[ \frac{1}{\lambda_{m\mu_n}} (\lambda_{m\mu_n} + \frac{1}{\lambda_{m\mu_n}} \sum_{m \in I_r,s} \sum_{n \in I_r,s} M |x_{m+r,n+s}|^{1/m+n} ) \]
\[ \frac{1}{\lambda_{m\mu_n}} (\lambda_{m\mu_n} + \frac{1}{\lambda_{m\mu_n}} K\delta^{-1} M[\lambda_{m\mu_n} P, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \]

Hence \( m \) and \( n \) goes to infinity, for each \( r \) and \( s \) we are granted
\[ x \in \Gamma_{M}^{2r}[AC_{\lambda_{m\mu_n}}, P, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \].

4.3. Theorem. The double almost \((\lambda_{m\mu_n})\) of Riesz sequence space. The following conditions are satisfied:
(i) \((x_{mnk}) \xrightarrow{P} \Gamma_{M}^{2r}[\widehat{S}_{\lambda_{m\mu_n}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \)

(ii) \[ \Gamma_{M}^{2r}[AC_{\lambda_{m\mu_n}}, P, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \] is a proper subset of
\[ \Gamma_{M}^{2r}[\widehat{S}_{\lambda_{m\mu_n}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \]

(iii) If \( x \in \Lambda^2 \) and \((x_{mn}) \xrightarrow{P} \Gamma_{M}^{2r}[\widehat{S}_{\lambda_{m\mu_n}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \) then \((x_{mn}) \xrightarrow{P} \Gamma_{M}^{2r}[AC_{\lambda_{m\mu_n}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \)

(iv) \[ \Gamma_{M}^{2r}[\widehat{S}_{\lambda_{m\mu_n}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \] \( \cap \Lambda^2 = \Gamma_{M}^{2r}[AC_{\lambda_{m\mu_n}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \) \( \cap \Lambda^2 \).

Proof: (i) Since for all \( r \) and \( s \)
\( \left\{ (m, n) \in I_{r,s} : |x_{m+r,n+s}|^{1/(m+n)} \right\} = 0 \leq \sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} |x_{m+r,n+s}|^{1/(m+n)} \), for all \( r \) and \( s \)

\( P - \lim_{m,n \to \lambda \mu n} \sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} |x_{m+r,n+s}|^{1/(m+n)} = 0 \)

This implies that for all \( r \) and \( s \)

\( P - \lim_{m,n \to \lambda \mu n} \frac{1}{\lambda \mu n} \left( \left\{ (m, n) \in I_{r,s} : |x_{m+r,n+s}|^{1/(m+n)} = 0 \right\} \right) = 0. \)

(ii) let \( x = (x_{mn}) \) be defined as follows:

\[
\begin{pmatrix}
1 & 2 & 3 & \cdots & \left[ \sqrt{\frac{\lambda m \mu n}{m+n}} \right]^{m+n} & 0 & \cdots \\
1 & 2 & 3 & \cdots & \left[ \sqrt{\frac{\lambda m \mu n}{m+n}} \right]^{m+n} & 0 & \cdots \\
\vdots & & & \ddots & \vdots & \vdots & \ddots \\
\vdots & & & & \vdots & \vdots & \ddots \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\
\vdots & & & & \vdots & \vdots & \ddots \\
\end{pmatrix}
\]

that is

\[ P - \lim_{m,n \to \lambda \mu n} \left( \left\{ (m, n) \in I_{r,s} : |x_{m+r,n+s}|^{1/(m+n)} = 0 \right\} \right) = 0. \]

\[ P - \lim_{m,n \to \lambda \mu n} \frac{1}{\lambda \mu n} \left( \left\{ (m, n) \in I_{r,s} : |x_{m+r,n+s}|^{1/(m+n)} = 0 \right\} \right) = 0. \]

Therefore \( (x_{mn}) \xrightarrow{P} \Gamma^{2r} \left[ \lambda \mu n, \|d(x_1,0),d(x_2,0),\ldots,d(x_n,0)\|_p \right]. \) Also

\[ P - \lim_{m,n \to \lambda \mu n} \frac{1}{\lambda \mu n} \sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} |x_{m+r,n+s}|^{1/(m+n)} = \]

\[ P - \frac{1}{2} \left( \lim_{m,n \to \lambda \mu n} \frac{1}{\lambda \mu n} \left( \left[ \sqrt{\frac{\lambda m \mu n}{m+n}} \right]^{m+n} \left[ \sqrt{\frac{\lambda m \mu n}{m+n}} \right]^{m+n} \left[ \sqrt{\frac{\lambda m \mu n}{m+n}} \right]^{m+n} + 1 \right) \right) = \frac{1}{2}. \]

Therefore \( (x_{mn}) \xrightarrow{P} \Gamma^{2r} \left[ AC_{\lambda \mu n}, \|d(x_1,0),d(x_2,0),\ldots,d(x_n,0)\|_p \right]. \)

(iii) Suppose \( x \in \Lambda^2 \) then for all \( r \) and \( s \), \( |x_{m+r,n+s}|^{1/(m+n)} \leq M \) for all \( m, n \). Also for given \( \epsilon > 0 \) and \( r \) and \( s \) large for all \( m \) and \( n \) we obtain the following:

\[ \frac{1}{\lambda \mu n} \sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} |x_{m+r,n+s}|^{1/(m+n)} = \]

\[ \frac{1}{\lambda \mu n} \sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} |x_{m+r,n+s}|^{1/(m+n)} \geq 0 |x_{m+r,n+s}|^{1/(m+n)} + \]

\[ \frac{1}{\lambda \mu n} \sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} |x_{m+r,n+s}|^{1/(m+n)} \leq 0 |x_{m+r,n+s}|^{1/(m+n)} \]

\[ \leq M \sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} |x_{m+r,n+s}|^{1/(m+n)} \right\} = 0 \right| + \epsilon. \]

Therefore \( x \in \Lambda^2 \) and \( (x_{mn}) \xrightarrow{P} \Gamma^{2r} \left[ S_{\lambda \mu n}, \|d(x_1,0),d(x_2,0),\ldots,d(x_n,0)\|_p \right] \) then
(x_{mnk}) \rightarrow_{\Gamma_{2r}} \Gamma_{2r} [AC_{\lambda_{m},\mu_{n}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p].

(iv) \Gamma_{2r} [S_{\lambda_{m},\mu_{n}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \cap \Lambda^2 = \Gamma_{2r} [AC_{\lambda_{m},\mu_{n}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \cap \Lambda^2. \text{ follows from (i), (ii) and (iii).}

4.4. Theorem. If M be any Musielak-Orlicz functions then
\begin{align*}
\Gamma_{2r} [AC_{\lambda_{m},\mu_{n}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] &\not\subseteq M \\
\Gamma_{2r} [S_{\lambda_{m},\mu_{n}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p]
\end{align*}

Proof: Let \( x \in \Gamma_{2r} [AC_{\lambda_{m},\mu_{n}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \), for all r and s. Therefore we have
\begin{align*}
\frac{1}{\lambda_{m,n}} \sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} M \left[ \left|x_{m+r,n+s}\right|^{1/m+n}\right] &\geq \frac{1}{\lambda_{m,n}} \sum_{m \in I_{r,s}} \sum_{n \in I_{r,s}} M \left| x_{m+r,n+s} \right|^{1/m+n} \\
\frac{1}{\lambda_{m,n}} M(0) \left\{ (m,n) \in I_{r,s} : \left|x_{m+r,n+s}\right|^{1/m+n} = 0 \right\}
\end{align*}

Hence \( x \notin \Gamma_{2r} [S_{\lambda_{m},\mu_{n}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \).

4.5. Theorem. Let M be a Mousielak Orlicz function. Then
\begin{align*}
\Gamma_{2r} [AC_{\lambda_{m},\mu_{n}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] &\supseteq M \\
\Gamma_{2r} [AC_{\lambda_{m},\mu_{n}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p]
\end{align*}

if and only if \( \sup_{mn}M_{mn}(x) < \infty \).

Proof: Let \( x \in \Gamma_{2r} [AC_{\lambda_{m},\mu_{n}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \). Suppose that \( \lambda_{m,\mu_{n}} = \sup_{mn}M_{mn} \) and \( \lambda_{m,\mu_{n}} = \sup_{mn}\lambda_{m,\mu_{n}} \). Since \( M \leq \lambda_{m,\mu_{n}} \) for all m, n. We have
\begin{align*}
\Gamma_{2r} [AC_{\lambda_{m},\mu_{n}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] &\leq \lambda_{m,\mu_{n}} |AC_{\lambda_{m},\mu_{n}}(\varepsilon)| + |\lambda_{m,\mu_{n}}(\varepsilon)|^{H_2} \\
\end{align*}

It follows from \( \varepsilon \to 0 \) that \( x \in \Gamma_{2r} [AC_{\lambda_{m},\mu_{n}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \).

Conversely, suppose that \( \sup_{mn}M_{mn} = \infty \). Let \( A = I \), unit matrix, define the sequence \( x \) by putting \( x_{mn} = u_{rs} \) if \( m = 1, 2, \ldots r \) and \( n = 1, 2, \ldots s \) and \( x_{mn} = 0 \) otherwise. Then we have \( x \in \Gamma_{2r} [AC_{\lambda_{m},\mu_{n}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \) but \( x \notin \Gamma_{2r} [AC_{\lambda_{m},\mu_{n}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \).

4.6. Theorem. Let M be a Musielak Orlicz function. Then
\begin{align*}
\Gamma_{2r} [AC_{\lambda_{m},\mu_{n}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] &\subseteq M \\
\Gamma_{2r} [AC_{\lambda_{m},\mu_{n}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p]
\end{align*}

if and only if \( \lim_{mn \to \infty}M > 0 \).

Proof: Let \( \varepsilon > 0 \) and \( x = (x_{mn}) \in \Gamma_{2r} [AC_{\lambda_{m},\mu_{n}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] \).

If \( \lim_{mn \to \infty}M > 0 \) then there exists a number \( d > 0 \) such that \( M(\varepsilon) > d \) and \( m, n \in \mathbb{N} \).

Let
\begin{align*}
\Gamma_{2r} [AC_{\lambda_{m},\mu_{n}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0))\|_p] &\geq \lambda_{m,\mu_{n}}d^{H_1}KAC_{\lambda_{m},\mu_{n}}(\varepsilon). \text{ It follows}
\end{align*}
that $\Gamma^{2r} \left[ AC_{\lambda m \mu n}, \| (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0)) \|_p \right]$. 

Conversely, suppose that $\lim_{mn \to \infty} M > 0$ does not hold, then there is a number $t > 0$ such that $\lim_{mn \to \infty} M(t) = 0$. We can select a almost $(\lambda m \mu n)$ sequence $\lambda m \mu n = (m_r, n_s)$ such that $M(t) < 2^{-rs}$ for any $m > m_r, n > n_s$. Let $A = I$, unit matrix, define the almost $\lambda m \mu n$ sequence $x$ by putting 

$x_{mn} = 0 \text{ if } m_r-1, n_s-1 < m, n \leq m_r n_s$ and $x_{mn} = 0 \text{ if } m_r n_s + m_r-1 n_s-1 \leq m, n \leq m_r n_s$. We have $x = (x_{mn}) \in \Gamma^{2r}_M \left[ AC_{\lambda m \mu n}, \| (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0)) \|_p \right] \text{ but } x \notin \Gamma^{2r} \left[ AC_{\lambda m \mu n}, \| (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0)) \|_p \right]$.

**Competing Interests**

The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

**Acknowledgement**

The research of the first author Deepmala is supported by the Science and Engineering Research Board, Department of Science and Technology, Government of India under SERB National Post-Doctoral fellowship scheme File Number: PDF/2015/000799.

**References**


