THE $\int \chi^{2\lambda I}$ STATISTICAL CONVERGENCE OF FUZZY NUMBERS OVER $p-$ METRIC SPACE-II

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Abstract. In this paper of part-II, we introduce the concepts of $\chi^{2\lambda I}$ statistical convergence and strongly $\chi^{2\lambda I}$ of fuzzy numbers. It is also shown that $\chi^{2\lambda I}$ statistical convergence and strongly $\chi^{2\lambda I}$ are equivalent for analytic sequences of fuzzy numbers.

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1. Introduction

Throughout $w, \chi$ and $\Lambda$ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

We write $w^2$ for the set of all complex sequences $(x_{mn})$, where $m, n \in \mathbb{N}$, the set of positive integers. Then, $w^2$ is a linear space under the coordinate wise addition and scalar multiplication. Let $(x_{mn})$ be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is said to be convergent if and only if the double sequence $(S_{mn})$ is convergent, where

$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij}(m, n = 1, 2, 3, ...).$$

We denote $w^2$ as the class of all complex double sequences $(x_{mn})$. A sequence $x = (x_{mn})$ is said to be double analytic if

$$\sup_{mn} |x_{mn}|^{1/m+n} < \infty.$$

The vector space of all prime sense double analytic sequences are usually denoted by $\Lambda^2$. A sequence $x = (x_{mn})$ is called double entire sequence if

$$|x_{mn}|^{1/m+n} \to 0 \text{ as } m, n \to \infty.$$
The vector space of all prime sense double entire sequences are usually denoted by $\Gamma^2$. The spaces $\Lambda^2$ and $\Gamma^2$ are metric spaces with the metric

\begin{equation}
 d(x, y) = \sup_{mn} \left\{ |x_{mn} - y_{mn}|^{1/(m+n)} : m, n : 1, 2, 3, \ldots \right\},
\end{equation}

for all $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in $\Gamma^2$.

A sequence $x = (x_{mn})$ is called double gai sequence if \((m + n)! |x_{mn}|^{1/(m+n)} \to 0\) as $m, n \to \infty$. The double gai sequences will be denoted by $\chi^2$. Let $\phi = \{\text{finite sequences}\}$.

Consider a double sequence $x = (x_{ij})$. The $(m, n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \delta_{ij}$ for all $m, n \in \mathbb{N}$,

\[
\delta_{mn} = \begin{pmatrix}
0 & 0 & \ldots & 0 & 0 & \ldots \\
0 & 0 & \ldots & 0 & 0 & \ldots \\
& & \ddots & & \ddots & \\
& & & 0 & 0 & \ldots 1 \\
& & & & 0 & 0 & \ldots 0
\end{pmatrix}
\]

with 1 in the $(m, n)^{th}$ position and zero otherwise.

Let $M$ and $\Phi$ are mutually complementary modulus functions. Then, we have:

(i) For all $u, y \geq 0$,

\begin{equation}
 uy \leq M(u) + \Phi(y), \text{ (Young's inequality)} \quad [\text{See} \ [16]]
\end{equation}

(ii) For all $u \geq 0$,

\begin{equation}
 u\eta(u) = M(u) + \Phi(\eta(u)).
\end{equation}

(iii) For all $u \geq 0$, and $0 < \lambda < 1$,

\begin{equation}
 M(\lambda u) \leq \lambda M(u)
\end{equation}

Lindenstrauss and Tzafriri used the idea of Orlicz function to construct Orlicz sequence space

$$
\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\},
$$

The space $\ell_M$ with the norm

$$
\| x \| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) \leq 1 \right\}.
$$
becomes a Banach space which is called an Orlicz sequence space. For \( M(t) = t^p \) \((1 \leq p < \infty)\), the spaces \( \ell_M \) coincide with the classical sequence space \( \ell_p \).

A sequence \( f = (f_{mn}) \) of Orlicz function is called a Musielak-Orlicz function. A sequence \( g = (g_{mn}) \) defined by

\[
g_{mn}(v) = \sup \{|v - f_{mn}(u) : u \geq 0\}, m, n = 1, 2, \ldots
\]

is called the complementary function of a Musielak-Orlicz function \( f \). For a given Musielak-Orlicz function \( f \), the Musielak-Orlicz sequence space \( t_f \) is defined as follows [see 20]

\[
t_f = \left\{ x \in w^2 : M_f (|x_{mn}|^{1/m+n}) \rightarrow 0 \text{ as } m, n \rightarrow \infty \right\},
\]

where \( M_f \) is a convex modular defined by

\[
M_f(x) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f_{mn} (|x_{mn}|^{1/m+n}), x = (x_{mn}) \in t_f.
\]

We consider \( t_f \) equipped with the Luxemburg metric

\[
d(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn} \left( \frac{|x_{mn}|^{1/m+n}}{m+n} \right)
\]

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz as follows

\[
Z(\Delta) = \{ x = (x_k) \in w : (\Delta x_k) \in Z \}
\]

for \( Z = c, c_0 \) and \( \ell_\infty \), where \( \Delta x_k = x_k - x_{k+1} \) for all \( k \in \mathbb{N} \).

Here \( c, c_0 \) and \( \ell_\infty \) denote the classes of convergent, null and bounded scalar valued single sequences respectively. The difference sequence space \( bv_p \) of the classical space \( \ell_p \) is introduced and studied in the case \( 1 \leq p \leq \infty \) by Başar and Altay and in the case \( 0 < p < 1 \) by Altay and Başar. The spaces \( c(\Delta), c_0(\Delta), \ell_\infty(\Delta) \) and \( bv_p \) are Banach spaces normed by

\[
\|x\| = |x_1| + \sup_{k \geq 1} |\Delta x_k| \quad \text{and} \quad \|x\|_{bv_p} = \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}, (1 \leq p < \infty).
\]

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

\[
Z(\Delta) = \{ x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z \}
\]

where \( Z = A^2, \chi^2 \) and \( \Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1} \) for all \( m, n \in \mathbb{N} \).

2. Definition and Preliminaries

Let \( n \in \mathbb{N} \) and \( X \) be a real vector space of dimension \( w \), where \( n \leq m \). A real valued function \( d_p(x_1, \ldots, x_n) = \|d_1(x_1, 0), \ldots, d_n(x_n, 0)\|_p \) on \( X \) satisfying the following four conditions:

(i) \( \|d_1(x_1, 0), \ldots, d_n(x_n, 0)\|_p = 0 \) if and only if \( d_1(x_1, 0), \ldots, d_n(x_n, 0) \) are linearly
dependent,
(ii) \( \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p \) is invariant under permutation,
(iii) \( \|(\alpha d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = |\alpha| \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p, \alpha \in \mathbb{R} \)
(iv) \( d_p ((x_1, y_1), (x_2, y_2) \cdots (x_n, y_n)) = (d_X(x_1, x_2, \cdots x_n)^p + d_Y(y_1, y_2, \cdots y_n)^p)^{1/p} \) for \( 1 \leq p < \infty \); (or)
(v) \( d ((x_1, y_1), (x_2, y_2), \cdots (x_n, y_n)) := \sup \{d_X(x_1, x_2, \cdots x_n), d_Y(y_1, y_2, \cdots y_n)\} \),
for \( x_1, x_2, \cdots x_n \in X, y_1, y_2, \cdots y_n \in Y \) is called the \( p \) product metric of the Cartesian product of \( n \) metric spaces is the \( p \) norm of the \( n \)-vector of the norms of the \( n \) subspaces.

A trivial example of \( p \) product metric of \( n \) metric space is the \( \mathbb{R} \) equipped with the following Euclidean metric in the product space is the \( p \) norm:

\[
\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_E = \sup (|det(d_{mn}(x_{mn}, 0))| = \\
\left| \begin{array}{cccc}
d_{11}(x_{11}, 0) & d_{12}(x_{12}, 0) & \cdots & d_{1n}(x_{1n}, 0) \\
d_{21}(x_{21}, 0) & d_{22}(x_{22}, 0) & \cdots & d_{2n}(x_{1n}, 0) \\
\vdots & \vdots & \ddots & \vdots \\
d_{n1}(x_{n1}, 0) & d_{n2}(x_{n2}, 0) & \cdots & d_{nn}(x_{nn}, 0)
\end{array} \right| \\
\]

where \( x_i = (x_{i1}, \cdots x_{in}) \in \mathbb{R}^n \) for each \( i = 1, 2, \cdots n \).

If every Cauchy sequence in \( X \) converges to some \( L \in X \), then \( X \) is said to be complete with respect to the \( p \)-metric. Any complete \( p \)-metric space is said to be \( p \)-Banach metric space.

2.1. Definition. Let \( X \) be a linear metric space. A function \( \rho : X \to \mathbb{R} \) is called paranorm, if

(1) \( \rho (x) \geq 0, \text{ for all } x \in X; \)
(2) \( \rho (-x) = \rho (x), \text{ for all } x \in X; \)
(3) \( \rho (x + y) \leq \rho (x) + \rho (y), \text{ for all } x, y \in X; \)
(4) If \( (\sigma_{mn}) \) is a sequence of scalars with \( \sigma_{mn} \to \sigma \text{ as } m, n \to \infty \) and \( (x_{mn}) \) is a sequence of vectors with \( \rho (x_{mn} - x) \to 0 \text{ as } m, n \to \infty \), then \( \rho (\sigma_{mn}x_{mn} - \sigma x) \to 0 \text{ as } m, n \to \infty \).

A paranorm \( w \) for which \( \rho (x) = 0 \) implies \( x = 0 \) is called total paranorm and the pair \( (X, w) \) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [28], Theorem 10.4.2, p.183).

The notion double sequence of \( \chi \) and double \( \Gamma \) was introduced by Subramanian et al. [21-24] and the notion of ideal convergence was studied by Esi et al. [1], Kostyrko et al.[17] and others as a generalization of statistical convergence which was further studied in topological spaces by Kumar et al.[18,19] and also more applications of ideals.
can be deals with various authors by B.Hazarika [2-14] and B.C.Tripathy and B. Hazarika [15,25-27].

2.2. Definition. A family $I \subset 2^Y$ of subsets of a non empty set $Y$ is said to be an ideal in $Y$ if
1. $\phi \in I$
2. $A, B \in I$ imply $A \cup B \in I$
3. $A \in I, B \subset A$ imply $B \in I$.

while an admissible ideal $I$ of $Y$ further satisfies \{x\} $\in I$ for each $x \in Y$. Given $I \subset 2^{N \times N}$ be a non trivial ideal in $N \times N$. A sequence $(x_{mn})_{m,n \in N \times N}$ in $X$ is said to be $I$-convergent to $0 \in X$, if for each $\epsilon > 0$ the set $A(\epsilon) = \{(m, n) \in N \times N : \|d_1(x_1, 0), \ldots, d_n(x_n, 0)\| \geq \epsilon\}$ belongs to $I$.

2.3. Definition. A non-empty family of sets $F \subset 2^X$ is a filter on $X$ if and only if
1. $\phi \in F$
2. for each $A, B \in F$, we have imply $A \cap B \in F$
3. each $A \in F$ and each $A \subset B$, we have $B \in F$.

2.4. Definition. An ideal $I$ is called non-trivial ideal if $I \neq \phi$ and $X \notin I$. Clearly $I \subset 2^X$ is a non-trivial ideal if and only if $F = F(I) = \{X - A : A \in I\}$ is a filter on $X$.

2.5. Definition. A non-trivial ideal $I \subset 2^X$ is called (i) admissible if and only if \{x\} $\in I$ \forall x \in X \subset I$. (ii) maximal if there cannot exists any non-trivial ideal $J \neq I$ containing $I$ as a subset.

If we take $I = I_f = \{A \subseteq N \times N : A is a finite subset \}$. Then $I_f$ is a non-trivial admissible ideal of $N$ and the corresponding convergence coincides with the usual convergence. If we take $I = I_\delta = \{A \subseteq N \times N : \delta(A) = 0\}$ where $\delta(A)$ denote the asyptotic density of the set $A$. Then $I_\delta$ is a non-trivial admissible ideal of $N \times N$ and the corresponding convergence coincides with the statistical convergence.

Let $D$ denote the set of all closed and bounded intervals $X = [x_1, x_2]$ on the real line $\mathbb{R} \times N$. For $X, Y \in D$, we define $X \leq Y$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$, $d(X, Y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$, where $X = [x_1, x_2]$ and $Y = [y_1, y_2]$.

Then it can be easily seen that $d$ defines a metric on $D$ and $(D, d)$ is a complete metric space. Also the relation ”$\leq$” is a partial order on $D$. A fuzzy number $X$ is a fuzzy subset of the real line $\mathbb{R} \times \mathbb{R}$ i.e. a mapping $X : R \to J (= [0, 1])$ associating each real number $t$ with its grade of membership $X(t)$. 


2.6. **Definition.** A fuzzy number $X$ is said to be (i) convex if $X(t) \geq X(s) \land X(r) = \min \{ X(s), X(r) \}$, where $s < t < r$. (ii) normal if there exists $t_0 \in \mathbb{R} \times \mathbb{R}$ such that $X(t_0) = 1$. (iii) upper semi-continuous if for each $\epsilon > 0, X^{-1}(t) = 0$, for all $a \in [0, 1]$ is open in the usual topology of $\mathbb{R} \times \mathbb{R}$.

Let $R(J)$ denote the set of all fuzzy numbers which are upper semi continuous and have compact support, i.e. if $X \in R(J)$ the for any $\alpha \in [0, 1], [X]^{\alpha}$ is compact, where $[X]^{\alpha} = \{ t \in \mathbb{R} \times \mathbb{R} : X(t) \geq \alpha, \text{if } \alpha \in [0, 1] \}, [X]^0 = \text{closure of } (\{ t \in \mathbb{R} \times \mathbb{R} : X(t) > \alpha, i.e \alpha = 0 \})$.

The set $\mathbb{R}$ of real numbers can be embedded $R(J)$ if we define $\bar{r} \in R(J) \times R(J)$ by

$$\bar{r}(t) = \begin{cases} 1, & \text{if } t = r : \\ 0, & \text{if } t \neq r \end{cases}$$

The absolute value, $|X|$ of $X \in R(J)$ is defined by

$$|X|(t) = \begin{cases} \max \{ X(t), X(-t) \}, & \text{if } t \geq 0; \\ 0, & \text{if } t < 0 \end{cases}$$

Define a mapping $\bar{d} : R(J) \times R(J) \rightarrow \mathbb{R}^+ \cup \{ 0 \}$ by

$$\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d([X]^{\alpha}, [Y]^{\alpha}).$$

It is known that $(R(J), \bar{d})$ is a complete metric space.

2.7. **Definition.** A metric on $R(J)$ is said to be translation invariant if $\bar{d}(X + Z, Y + Z) = \bar{d}(X, Y)$, for $X, Y, Z \in R(J)$.

2.8. **Definition.** A sequence $X = (X_{mn})$ of fuzzy numbers is said to be (i) convergent to a fuzzy number $X_0$ if for every $\epsilon > 0$, there exists a positive integer $n_0$ such that $\bar{d}(X_{mn}, X_0) < \epsilon$ for all $n \geq n_0$. (ii) bounded if the set $\{ X_{mn} : m, n \in \mathbb{N} \}$ of fuzzy numbers is bounded.

2.9. **Definition.** A sequence $X = (X_{mn})$ of fuzzy numbers is said to be (i) $I$-convergent to a fuzzy number $X_0$ if for each $\epsilon > 0$ such that

$$A = \{ m, n \in \mathbb{N} : \bar{d}(X_{mn}, X_0) \geq \epsilon \} \in I.$$  \hspace{1cm}

The fuzzy number $X_0$ is called $I$-limit of the sequence $(X_{mn})$ of fuzzy numbers and we write $\lim I X_{mn} = X_0$. (ii) $I$-bounded if there exists $M > 0$ such that

$$\{ m, n \in \mathbb{N} : \bar{d}(X_{mn}, \bar{0}) > M \} \in I.$$  \hspace{1cm}

2.10. **Definition.** A sequence space $E_F$ of fuzzy numbers is said to be (i) solid (or normal) if $(Y_{mn}) \in E_F$ whenever $(X_{mn}) \in E_F$ and $\bar{d}(Y_{mn}, \bar{0}) \leq \bar{d}(X_{mn}, \bar{0})$ for all $m, n \in \mathbb{N}$. (ii) symmetric if $(X_{mn}) \in E_F$ implies $(X_{\pi(mn)}) \in E_F$ where $\pi$ is a permutation of $\mathbb{N} \times \mathbb{N}$.

Let $K = \{ k_1 < k_2 < \ldots \} \subseteq \mathbb{N}$ and $E$ be a sequence space. A $K$-step space of $E$ is a sequence space.
\[ \lambda^E_{mn} = \{ (X_{m,p}n_p) \in w^2 : (m,pn_p) \in E \} . \]

A canonical preimage of a sequence \( \{ (x_{m,p}n_p) \} \in \lambda^E_K \) is a sequence \( \{ y_{mn} \} \in w^2 \) defined as

\[
y_{mn} = \begin{cases} 
  x_{mn}, & \text{if } m, n \in E \\
  0, & \text{otherwise}.
\end{cases}
\]

A canonical preimage of a step space \( \lambda^E_K \) is a set of canonical preimages of all elements in \( \lambda^E_K \), i.e. \( y \) is in canonical preimage of \( \lambda^E_K \) if and only if \( y \) is canonical preimage of some \( x \in \lambda^E_K \).

2.11. **Definition.** A sequence space \( E \) is said to be monotone if \( E \) contains the canonical pre-images of all its step spaces.

The following well-known inequality will be used throughout the article. Let \( p = (p_{mn}) \) be any sequence of positive real numbers with \( 0 \leq p_{mn} \leq sup_{mn}p_{mn} = G, D = max \{ 1, 2G - 1 \} \) then

\[
|a_{mn} + b_{mn}|^{p_{mn}} \leq D (|a_{mn}|^{p_{mn}} + |b_{mn}|^{p_{mn}}) \text{ for all } m, n \in \mathbb{N} \text{ and } a_{mn}, b_{mn} \in \mathbb{C}.
\]

Also \( |a_{mn}|^{p_{mn}} \leq max \{ 1, |a|^G \} \) for all \( a \in \mathbb{C} \).

First we procure some known results; those will help in establishing the results of this article.

2.12. **Lemma.** A sequence space \( E \) is normal implies \( E \) is monotone. (For the crisp set case, one may refer to Kamthan and Gupta [16], page 53).

2.13. **Lemma.** (Kostyrko et al., [17], Lemma 5.1). If \( I \subset 2^\mathbb{N} \) is a maximal ideal, then for each \( A \subset \mathbb{N} \) we have either \( A \in I \) or \( \mathbb{N} - A \in I \).

2.14. **Definition.** A sequence \( X = (X_{mn}) \) of fuzzy numbers is a function \( X \) from the set \( \mathbb{N} \times \mathbb{N} \) of natural numbers into \( L(\mathbb{R}) \times L(\mathbb{R}) \). The fuzzy number \( X_{mn} \) denotes the value of the function \( m, n \in \mathbb{N} \).

We denote \( W^{2F} \) denotes the set of all sequences \( X = (X_{mn}) \) of fuzzy numbers.

2.15. **Definition.** A sequence \( X = (X_{mn}) \) of fuzzy numbers is said to be analytic if the set \( \{ X_{mn} : m, n \in \mathbb{N} \} \) of fuzzy numbers is analytic.

2.16. **Definition.** The sequence \( X = (X_{mn}) \) of fuzzy numbers is said to be almost convergent to a fuzzy number \( \bar{0} \) if \( \lim_{m,n \to \infty} d(t_{pm,qn}(X), \bar{0}) = 0 \) uniformly in \( m, n \), where

\[
t_{pm,qn}(X) = \frac{1}{(m+1)(n+1)} \sum_{i=0}^{p} \sum_{j=0}^{q} ((i + m) + (j + n))!X_{i+m,j+n}^{1/(i+m)+(j+n)}.
\]

This means that for every \( \epsilon > 0 \), there exists a \( p_0q_0 \in \mathbb{N} \) such that \( d(t_{pm,qn}(X), \bar{0}) < \epsilon \) whenever \( p, q \geq p_0q_0 \) and for all \( m, n \).
2.17. **Definition.** A sequence \( X = (X_{mn}) \) of fuzzy numbers is said to be statistically convergent to a fuzzy number \( \bar{0} \) if for every \( \epsilon > 0 \),

\[
\lim_{r,s} \left| \left\{ m \leq r, n \leq s : d \left( \left( (m + n)!X_{mn} \right)^{1/(m+n)}, \bar{0} \right) \geq \epsilon \right\} \right| = 0.
\]

The set of all statistically convergent sequences of fuzzy numbers is denoted by \( S^{2F} \).

We note that if a sequence \( X = (X_{mn}) \) of fuzzy numbers converges to a fuzzy number \( \bar{0} \), then it is statistically converges to \( \bar{0} \). But the converse statement is not necessarily valid.

Let \( \mu = (\lambda_{rs}) \) be a non-decreasing sequence of positive real numbers tending to infinity and \( \lambda_{11} = 1 \) and \( \lambda_{r+1,s+1} \leq \lambda_{rs} + 1 \), for all \( r, s \in \mathbb{N} \).

The generalized de la Vallee-Poussin means is defined by

\[
t_{rs}(x) = \frac{1}{\lambda_{rs}} \sum_{p \in I_r} \sum_{q \in I_s} ((m + n)!|x_{mn}|)^{1/(m+n)}
\]

where \( I_{rs} = [r, s - \lambda_{rs} + 1, rs] \). A sequence \( x = (x_{mn}) \) of complex numbers is said to be \((V, \lambda) -\) summable to a number if \( t_{rs}(x) \to L \) as \( r, s \to \infty \).

3. **Some new integrated statistical convergence sequence spaces of fuzzy numbers**

The main aim of this article to introduce the following sequence spaces and examine topological and algebraic properties of the resulting sequence spaces. Let \( p = (p_{mn}) \) be a sequence of positive real numbers for all \( m, n \in \mathbb{N} \). \( f = (f_{mn}) \) be a Musielak-Orlicz function, \( (X, \|(d(x_1,0), d(x_2,0), \ldots, d(x_n-1,0))\|_p) \) be a \( p \)-metric space, and \( (\lambda_{rs}^{-1}) \) be a sequence of non-zero scalars and \( \mu_{mn}(X) = d(t_{rs}, \bar{0}) \) be a sequence of fuzzy numbers, we define the following sequence spaces as follows:

\[
\left[ \left. \frac{2q}{\lambda_{f\mu}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_n-1,0))\|_p \right] = \lim_{r,s \to \infty} \left| \left\{ m, n \in I_{rs} : \left[ f_{mn} \left( \|\mu_{mn}(x), (d(x_1,0), d(x_2,0), \ldots, d(x_n-1,0))\|_p \right) \right]^{q_{mn}} \geq \epsilon \right\} \right| = 0, \text{ uniformly in } r, s.
\]

In this case, we write \( X_{mn} \to \bar{0} \left( S^F_\lambda \right) \). The set of all statistically convergent sequences is denoted by \( S^F_\lambda \).

Let \( X = (X_{mn}) \) be a sequence of fuzzy numbers and \( q = (q_{mn}) \) be a sequence of strictly positive real numbers. Then the sequence \( X = (X_{mn}) \) is said to be strongly \( \lambda \)-convergent if there is a fuzzy number \( \bar{0} \) such that

\[
\left[ \left. \frac{2q}{\lambda_{f\mu}}, \|(d(x_1,0), d(x_2,0), \ldots, d(x_n-1,0))\|_p \right] = \lim_{r,s \to \infty} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} f_{mn} \left( \|\mu_{mn}(x), (d(x_1,0), d(x_2,0), \ldots, d(x_n-1,0))\|_p \right)^{q_{mn}} = 0, \text{ uniformly in } r, s.
\]
In this case, we write \( X_{mn} \to \overline{0}(\bar{w}_\lambda^F, q) \). The set of all strongly \( \lambda \)-convergent sequences is denoted by \( (\bar{w}_\lambda^F, q) \).

Let \( X = (X_{mn}) \) be a sequence of fuzzy numbers. Then the sequence \( X = (X_{mn}) \) of fuzzy numbers is said to be double analytic if the set \( \{r_{rs} : r, s \in \mathbb{N} \} \) of fuzzy numbers is double analytic and it is denoted by \( \Lambda^2 \). In this section we give some inclusion relations between strongly \( \lambda \)-convergence and \( \lambda \)-statistically convergence and show that they are equivalent for almost bounded sequences of fuzzy numbers. We also study the inclusion \( \bar{S}_\lambda^2 \subseteq \bar{S}_\lambda^2 \) under certain restrictions on the sequence \( \Lambda^2 = (\lambda_{rs}) \).

### 3.1. Theorem
If \( \chi^2 (X) \in \bar{S}_\lambda^2 \) and \( c \in \mathbb{R} \), then

(a) \( \bar{S}_\lambda^2 - \lim \chi^2 (X) = c \bar{S}_\lambda^2 - \lim \chi^2 (X) \)

(b) \( \bar{S}_\lambda^2 - \lim \chi^2 (X + Y) = \bar{S}_\lambda^2 - \lim \chi^2 (X) + \bar{S}_\lambda^2 - \lim \chi^2 (Y) \)

**Proof (a):** Let \( \chi^2 (X) \in \bar{S}_\lambda^2 \) so that \( \bar{S}_\lambda^2 - \lim \chi^2 (X) = 0, c \in \mathbb{R} \) and \( \epsilon > 0 \). Then the inequality

\[
\left\{ m, n \in I_{rs} : \left[ f_{mn}\left( \|\mu_{mn} (x + y), (d(x,0), d(x,1), \cdots, d(x_{n-1},0))\|_p \right) \right]^{q_{mn}} \geq \epsilon \right\} \leq
\]

\[
\left\{ m, n \in I_{rs} : \left[ f_{mn}\left( \|\mu_{mn} (x), (d(x,0), d(x,1), \cdots, d(x_{n-1},0))\|_p \right) \right]^{q_{mn}} \geq \frac{\epsilon}{2} \right\}. 
\]

**Proof (b):** Suppose that \( \chi^2 (X), \chi^2 (Y) \in \bar{S}_\lambda^2 \) so that \( \bar{S}_\lambda^2 - \lim \chi^2 (X) = 0 \) and \( \bar{S}_\lambda^2 - \lim \chi^2 (Y) = 0 \). By Minkowski’s inequality, we get

\[
\left[ f_{mn}\left( \|\mu_{mn} (x + y), (d(x,0), d(x,1), \cdots, d(x_{n-1},0))\|_p \right) \right]^{q_{mn}} \leq
\]

\[
\left[ f_{mn}\left( \|\mu_{mn} (x), (d(x,0), d(x,1), \cdots, d(x_{n-1},0))\|_p \right) \right]^{q_{mn}} +
\]

\[
\left[ f_{mn}\left( \|\mu_{mn} (y), (d(x,0), d(x,1), \cdots, d(x_{n-1},0))\|_p \right) \right]^{q_{mn}} .
\]

Therefore given \( \epsilon > 0 \), for all \( r, s \in \mathbb{N} \), we have

\[
\left\{ m, n \in I_{rs} : \left[ f_{mn}\left( \|\mu_{mn} (x + y), (d(x,0), d(x,1), \cdots, d(x_{n-1},0))\|_p \right) \right]^{q_{mn}} \geq \epsilon \right\} \leq
\]

\[
\left\{ m, n \in I_{rs} : \left[ f_{mn}\left( \|\mu_{mn} (x), (d(x,0), d(x,1), \cdots, d(x_{n-1},0))\|_p \right) \right]^{q_{mn}} \geq \frac{\epsilon}{2} \right\} +
\]

\[
\left\{ m, n \in I_{rs} : \left[ f_{mn}\left( \|\mu_{mn} (y), (d(x,0), d(x,1), \cdots, d(x_{n-1},0))\|_p \right) \right]^{q_{mn}} \geq \frac{\epsilon}{2} \right\} .
\]

The completes the proof.

The following theorem shows that \( \lambda \)-statistical convergence and strongly \( \lambda \)-convergence are equivalent for double analytic sequences of fuzzy numbers.

### 3.2. Theorem
Let the sequence \( \mu = (\mu_{mn}) \) be double analytic and \( \chi^2 (X) \) be a sequence of fuzzy numbers. Then (a) \( \chi^2 (X) \to \overline{0}(\bar{w}_\lambda^2, \mu) \) implies \( \chi^2 (X) \to \overline{0}(\bar{S}_\lambda^2, \mu) \).

(b) \( \Lambda^2 (X) \to \overline{0}(\bar{S}_\lambda^2, \mu) \) imply \( \Lambda^2 (X) \to \overline{0}(\bar{w}_\lambda^2, \mu) \).

(c) \( \bar{S}_\lambda^2 \cap \Lambda^2 = (\bar{w}_\lambda^2, \mu) \cap \Lambda^2 \).

**Proof (a):** Let \( \epsilon > 0 \) and \( \chi^2 (X) \to \overline{0}(\bar{w}_\lambda^2, \mu) \) for all \( r, s \in \mathbb{N} \), we have

\[
\sum_{m \in I_{rs}} \sum_{n \in I_{rs}} f_{mn}\left( \|\mu_{mn} (x), (d(x,0), d(x,1), \cdots, d(x_{n-1},0))\|_p \right) \right]^{q_{mn}} \geq
\]
Theorem 3.3.

\[
\sum_{m \in I_r} \sum_{n \in I_s} \left[ f_{mn} \left( \left\| \mu_{mn} (x), (d(x_1, 0), d(x_2, 0), \ldots, d(x_{n-1}, 0)) \right\|_p \right) \right]^{q_{mn}} \geq \epsilon
\]

\[
\sum_{m \in I_r} \sum_{n \in I_s} \left[ f_{mn} \left( \left\| \mu_{mn} (x), (d(x_1, 0), d(x_2, 0), \ldots, d(x_{n-1}, 0)) \right\|_p \right) \right]^{q_{mn}} \geq \epsilon
\]

\[
\left\{ (m, n) \in I_r : \left[ f_{mn} \left( \left\| \mu_{mn} (x), (d(x_1, 0), d(x_2, 0), \ldots, d(x_{n-1}, 0)) \right\|_p \right) \right]^{q_{mn}} \geq \epsilon \right\} \cdot \min \left( \epsilon^h, \epsilon^H \right).
\]

Hence \( \chi^2 (X) \in \tilde{S}^2_F \).

**Proof (b):** Suppose that \( \chi^2 (X) \in \tilde{S}^2_F \cap \Lambda^2_F \). Since \( \chi^2 (X) \in \Lambda^2_F \), we write

\[
\left[ f_{mn} \left( \left\| \mu_{mn} (x), (d(x_1, 0), d(x_2, 0), \ldots, d(x_{n-1}, 0)) \right\|_p \right) \right]^{q_{mn}} \leq T, \text{ for all } r, s \in \mathbb{N}, \text{ let}
\]

\[
G_{rs} = \left\{ (m, n) \in I_r : \left[ f_{mn} \left( \left\| \mu_{mn} (x), (d(x_1, 0), d(x_2, 0), \ldots, d(x_{n-1}, 0)) \right\|_p \right) \right]^{q_{mn}} \geq \epsilon \right\} \text{ and}
\]

\[
H_{rs} = \left\{ (m, n) \in I_r : \left[ f_{mn} \left( \left\| \mu_{mn} (x), (d(x_1, 0), d(x_2, 0), \ldots, d(x_{n-1}, 0)) \right\|_p \right) \right]^{q_{mn}} < \epsilon \right\}.
\]

Then we have

\[
\sum_{m \in I_r} \sum_{n \in I_s} f_{mn} \left( \left\| \mu_{mn} (x), (d(x_1, 0), d(x_2, 0), \ldots, d(x_{n-1}, 0)) \right\|_p \right) \geq 0
\]

\[
\sum_{m \in G_{rs}} \sum_{n \in G_{rs}} f_{mn} \left( \left\| \mu_{mn} (x), (d(x_1, 0), d(x_2, 0), \ldots, d(x_{n-1}, 0)) \right\|_p \right) \geq 0
\]

\[
\sum_{m \in H_{rs}} \sum_{n \in H_{rs}} f_{mn} \left( \left\| \mu_{mn} (x), (d(x_1, 0), d(x_2, 0), \ldots, d(x_{n-1}, 0)) \right\|_p \right) \leq max (T^h, T^H) G_{rs} + max (\epsilon^h, \epsilon^H).
\]

Taking the limit as \( \epsilon \to 0 \) and \( r, s \to \infty \), it follows that \( \chi^2 (X) \in \tilde{S}^2_F \).

(c) Follows from (a) and (b).

### 3.3. Theorem

If \( \liminf_{rs} (\lambda_{rs}) > 0 \), then \( \tilde{S}^2_F \subset \tilde{S}^2 \).

**Proof:** Let \( \chi^2 (X) \in \tilde{S}^2_F \). For given \( \epsilon > 0 \), we get

\[
\left\{ m \leq r, n \leq s : \left[ f_{mn} \left( \left\| \mu_{mn} (x), (d(x_1, 0), d(x_2, 0), \ldots, d(x_{n-1}, 0)) \right\|_p \right) \right]^{q_{mn}} \geq \epsilon \right\} \supset G_{rs} \text{ where}
\]

\( G_{rs} \) is in the Theorem of 3.2 (b). Thus,

\[
\left\{ m \leq r, n \leq s : \left[ f_{mn} \left( \left\| \mu_{mn} (x), (d(x_1, 0), d(x_2, 0), \ldots, d(x_{n-1}, 0)) \right\|_p \right) \right]^{q_{mn}} \geq \epsilon \right\} \supset G_{rs} = \lambda^{rs} \text{ Taking limit as } r, s \to \infty \text{ and using } \liminf_{rs} (\lambda^{rs}) > 0, \text{ we get } \chi^2 (X) \in \tilde{S}^2 \).

### 3.4. Theorem

Let \( 0 < u_{mn} \leq v_{mn} \) and \( (u_{mn}, v_{mn}^{-1}) \) be double analytic. Then \( (\tilde{w}^2_F, v) \subset (\tilde{w}^2, u) \).

**Proof:** Let \( \chi^2 (X) \in \tilde{w}^2_F \). Let

\[
w_{mn} = f_{mn} \left( \left\| \mu_{mn} (x), (d(x_1, 0), d(x_2, 0), \ldots, d(x_{n-1}, 0)) \right\|_p \right) \geq 0 \text{ for all } r, s \in \mathbb{N} \text{ and } \lambda_{mn} = u_{mn}v_{mn}^{-1} \text{ for all } m, n \in \mathbb{N} \text{ Then } 0 < \lambda_{mn} \leq 1 \text{ for all } m, n \in \mathbb{N} \text{. Let } b \text{ be a constant such that } 0 < b \leq \lambda_{mn} \leq 1 \text{ for all } m, n \in \mathbb{N} \text{.}
\]

Define the sequences \( (k_{mn}) \) and \( (\ell_{mn}) \) as follows:

For \( w_{mn} \geq 1 \), let \( k_{mn} = (w_{mn}) \) and \( \ell_{mn} = 0 \) and for \( w_{mn} < 1 \), let \( k_{mn} = 0 \) and \( \ell_{mn} = w_{mn} \). Then it is clear that for all \( m, n \in \mathbb{N} \), we have \( w_{mn} = k_{mn} + \ell_{mn} \) and \( w_{mn}^{\lambda_{mn}} = k_{mn}^{\lambda_{mn}} + \ell_{mn}^{\lambda_{mn}} \). Now it follows that \( k_{mn}^{\lambda_{mn}} \leq k_{mn} \leq w_{mn} \) and \( \ell_{mn}^{\lambda_{mn}} \leq \ell_{mn}^{\lambda_{mn}} \). Therefore
\[ \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left[ f_{mn} \left( \| \mu_{mn}(w_{mn}^{\lambda_{mn}}), (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0)) \|_p \right) \right]^{q_{mn}} = \]
\[ \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left[ f_{mn} \left( \| \mu_{mn}(k_{mn} + \ell_{mn})^{\lambda_{mn}}, (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0)) \|_p \right) \right]^{q_{mn}} = \]
\[ \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left[ f_{mn} \left( \| \mu_{mn}(w_{mn}), (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0)) \|_p \right) \right]^{q_{mn}} + \]
\[ \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left[ f_{mn} \left( \| \mu_{mn}(\ell_{mn})^{\lambda_{mn}}, (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0)) \|_p \right) \right]^{q_{mn}}. \]

Now for each \( r, s, \)
\[ \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left[ f_{mn} \left( \| \mu_{mn}(\ell_{mn})^{\lambda}, (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0)) \|_p \right) \right]^{q_{mn}} = \]
\[ \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left[ f_{mn} \left( \| \mu_{mn}(\ell_{mn})^{\lambda}, (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0)) \|_p \right) \right]^{q_{mn}} \leq \]
\[ \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left[ f_{mn} \left( \| \mu_{mn}(\ell_{mn})^{\lambda}, (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0)) \|_p \right) \right]^{q_{mn}} \lambda. \]

### 3.5. Theorem.
\( \hat{\Lambda}^2 F = \hat{w}_{\lambda, \Lambda^2}^{2F}, \) where \( \hat{w}_{\lambda, \Lambda^2}^{2F} = \)
\[ \left\{ \sup_{rs} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} f_{mn} \left( \| \mu_{mn}(x), (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0)) \|_p \right) \right\}^{q_{mn}} < \infty \]

**Proof:** Let \( X = (X_{mn}) \in \hat{w}_{\lambda, \Lambda^2}^{2F}. \) Then there exists a constant \( T_1 > 0 \) such that
\[ \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} f_{mn} \left( \| \mu_{mn}(x), (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0)) \|_p \right) \right\}^{q_{mn}} \leq \]
\[ \sup_{rs} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} f_{mn} \left( \| \mu_{mn}(x), (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0)) \|_p \right) \right\}^{q_{mn}} \leq T_1 \text{ for all } r, s \in \mathbb{N}. \]
Therefore we have \( X = (X_{mn}) \in \hat{\Lambda}^2 F. \) Conversely, let \( X = (X_{mn}) \in \hat{\Lambda}^2 F. \) Then there exists a constant \( T_2 > 0 \) such that
\[ \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} f_{mn} \left( \| \mu_{mn}(x), (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0)) \|_p \right) \right\}^{q_{mn}} \leq T_2 \text{ for all } m, n \text{ and } r, s. \]
\[ \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} f_{mn} \left( \| \mu_{mn}(x), (d(x_1,0), d(x_2,0), \ldots, d(x_{n-1},0)) \|_p \right) \right\}^{q_{mn}} \leq T_2 \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} 1 \leq T_2, \text{ for all } m, n \text{ and } r, s. \]
Thus \( X = (X_{mn}) \in \hat{w}_{\lambda, \Lambda^2}^{2F}. \)

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