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THE $\int \chi^{2\lambda I}$ STATISTICAL CONVERGENCE OF FUZZY NUMBERS OVER $p-$ METRIC SPACE-II

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Abstract. In this paper of part-II, we introduce the concepts of $\chi^{2\lambda I}$ statistical convergence and strongly $\chi^{2\lambda I}$ of fuzzy numbers. It is also shown that $\chi^{2\lambda I}$ statistical convergence and strongly $\chi^{2\lambda I}$ are equivalent for analytic sequences of fuzzy numbers.

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1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication. Let (x_{mn}) be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$is said to be convergent if and only if the double sequence $(S_{mn})$is convergent, where

$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n = 1, 2, 3, ...) .$$

We denote w^2 as the class of all complex double sequences (x_{mn}). A sequence $x = (x_{mn})$is said to be double analytic if

$$sup_{mn} |x_{mn}|^{1/m+n} < \infty.$$

The vector space of all prime sense double analytic sequences are usually denoted by Λ^2. A sequence $x = (x_{mn})$ is called double entire sequence if

$$|x_{mn}|^{1/m+n} \to 0 \text{ as } m, n \to \infty.$$

The vector space of all prime sense double entire sequences are usually denoted by Γ^2. The spaces Λ^2 and Γ^2 are metric spaces with the metric

$$d(x,y) = sup_{mn}\left\{|x_{mn} - y_{mn}|^{1/m+n} : m,n : 1,2,3,...\right\}, \tag{1.1}$$

forall $x = \{x_{mn}\}$ and $y = \{y_{mn}\}\, in\, \Gamma^2$.

A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)!\,|x_{mn}|)^{1/m+n} \to 0$ as $m,n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{finite\ sequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\delta_{ij}$ for all $m,n \in \mathbb{N}$,

$$\delta_{mn} = \begin{pmatrix} 0 & 0 & ...0 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & ...1 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ & & & & \end{pmatrix}$$

with 1 in the $(m,n)^{th}$ position and zero other wise.

Let M and Φ are mutually complementary modulus functions. Then, we have:

(i) For all $u, y \geq 0$,

$$uy \leq M(u) + \Phi(y), (Young's\, inequality)[See[16]] \tag{1.2}$$

(ii) For all $u \geq 0$,

$$u\eta(u) = M(u) + \Phi(\eta(u)). \tag{1.3}$$

(iii) For all $u \geq 0$, and $0 < \lambda < 1$,

$$M(\lambda u) \leq \lambda M(u) \tag{1.4}$$

Lindenstrauss and Tzafriri used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{x \in w : \textstyle\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty,\ for\, some\, \rho > 0\right\},$$

The space ℓ_M with the norm

$$\|x\| = inf\left\{\rho > 0 : \textstyle\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p (1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

A sequence $f = (f_{mn})$ of Orlicz function is called a Musielak -Orlicz function . A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup\{|v| u - (f_{mn})(u) : u \geq 0\}, m, n = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function f. For a given Musielak-Orlicz function f, the Musielak-Orlicz sequence space t_f is defined as follows [see 20]

$$t_f = \left\{ x \in w^2 : M_f(|x_{mn}|)^{1/m+n} \to 0 \, as \, m, n \to \infty \right\},$$

where M_f is a convex modular defined by

$$M_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn}(|x_{mn}|)^{1/m+n}, x = (x_{mn}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn}\left(\frac{|x_{mn}|^{1/m+n}}{mn}\right)$$

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Here c, c_0 and ℓ_∞ denote the classes of convergent,null and bounded sclar valued single sequences respectively. The difference sequence space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by Başar and Altay and in the case $0 < p < 1$ by Altay and Başar. The spaces $c(\Delta), c_0(\Delta), \ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k \geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}, (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$.

2. Definition and Preliminaries

Let $n \in \mathbb{N}$ and X be a real vector space of dimension w, where $n \leq m$. A real valued function $d_p(x_1, \ldots, x_n) = \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p$ on X satisfying the following four conditions:

(i) $\|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = 0$ if and and only if $d_1(x_1, 0), \ldots, d_n(x_n, 0)$ are linearly

dependent,
(ii) $\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p$ is invariant under permutation,
(iii) $\|(\alpha d_1(x_1,0),\ldots,d_n(x_n,0))\|_p = |\alpha| \, \|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p, \alpha \in \mathbb{R}$
(iv) $d_p\left((x_1,y_1),(x_2,y_2)\cdots(x_n,y_n)\right) = \left(d_X(x_1,x_2,\cdots x_n)^p + d_Y(y_1,y_2,\cdots y_n)^p\right)^{1/p} for 1 \le p < \infty$; (or)
(v) $d\left((x_1,y_1),(x_2,y_2),\cdots(x_n,y_n)\right) := \sup\left\{d_X(x_1,x_2,\cdots x_n), d_Y(y_1,y_2,\cdots y_n)\right\},$
for $x_1,x_2,\cdots x_n \in X, y_1,y_2,\cdots y_n \in Y$ is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces.

A trivial example of p product metric of n metric space is the p norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_E = sup\left(|det(d_{mn}\left(x_{mn},0\right))|\right) =$$
$$sup\begin{pmatrix}\begin{vmatrix} d_{11}\left(x_{11},0\right) & d_{12}\left(x_{12},0\right) & \ldots & d_{1n}\left(x_{1n},0\right) \\ d_{21}\left(x_{21},0\right) & d_{22}\left(x_{22},0\right) & \ldots & d_{2n}\left(x_{1n},0\right) \\ . & & & \\ . & & & \\ . & & & \\ d_{n1}\left(x_{n1},0\right) & d_{n2}\left(x_{n2},0\right) & \ldots & d_{nn}\left(x_{nn},0\right) \end{vmatrix}\end{pmatrix}$$

where $x_i = (x_{i1},\cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1,2,\cdots n.$
If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the $p-$ metric. Any complete $p-$ metric space is said to be $p-$ Banach metric space.

2.1. **Definition.** Let X be a linear metric space. A function $\rho : X \to \mathbb{R}$ is called paranorm, if
(1) $\rho(x) \ge 0$, for all $x \in X$;
(2) $\rho(-x) = \rho(x)$, for all $x \in X$;
(3) $\rho(x+y) \le \rho(x) + \rho(y)$, for all $x, y \in X$;
(4) If (σ_{mn}) is a sequence of scalars with $\sigma_{mn} \to \sigma$ as $m,n \to \infty$ and (x_{mn}) is a sequence of vectors with $\rho(x_{mn} - x) \to 0$ as $m,n \to \infty$, then $\rho(\sigma_{mn}x_{mn} - \sigma x) \to 0$ as $m,n \to \infty$.
A paranorm w for which $\rho(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X,w) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [28], Theorem 10.4.2, p.183).

The notion double sequence of χ and double Γ was introduced by Subramanian et al. [21-24] and the notion of ideal convergence was studied by Esi et al. [1], Kostyrko et al.[17] and others as a generalization of statistical convergence which was further studied in toplogical spaces by Kumar et al.[18,19] and also more applications of ideals can be deals with various authors by B.Hazarika [2-14] and B.C.Tripathy and B. Hazarika [15,25-27].

2.2. **Definition.** A family $I \subset 2^Y$ of subsets of a non empty set Y is said to be an ideal in Y if
(1) $\phi \in I$
(2) $A, B \in I$ imply $A \bigcup B \in I$
(3) $A \in I, B \subset A$ imply $B \in I$.

while an admissible ideal I of Y further satisfies $\{x\} \in I$ for each $x \in Y$. Given $I \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a non trivial ideal in $\mathbb{N} \times \mathbb{N}$. A sequence $(x_{mn})_{m,n\in\mathbb{N}\times\mathbb{N}}$ in X is said to be $I-$ convergent to $0 \in X$, if for each $\epsilon > 0$ the set
$A(\epsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|(d_1(x_1,0),\ldots,d_n(x_n,0)) - 0\|_p \geq \epsilon\}$ belongs to I.

2.3. **Definition.** A non-empty family of sets $F \subset 2^X$ is a filter on X if and only if
(1) $\phi \in F$
(2) for each $A, B \in F$, we have imply $A \bigcap B \in F$
(3) each $A \in F$ and each $A \subset B$, we have $B \in F$.

2.4. **Definition.** An ideal I is called non-trivial ideal if $I \neq \phi$ and $X \notin I$. Clearly $I \subset 2^X$ is a non-trivial ideal if and only if $F = F(I) = \{X - A : A \in I\}$ is a filter on X.

2.5. **Definition.** A non-trivial ideal $I \subset 2^X$ is called (i) admissible if and only if $\{\{x\} : x \in X\} \subset I$. (ii) maximal if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset.

If we take $I = I_f = \{A \subseteq \mathbb{N}\times\mathbb{N} : A\, is\, a\, finite\, subset\}$. Then I_f is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincides with the usual convergence. If we take $I = I_\delta = \{A \subseteq \mathbb{N}\times\mathbb{N} : \delta(A) = 0\}$ where $\delta(A)$ denote the asyptotic density of the set A. Then I_δ is a non-trivial admissible ideal of $\mathbb{N}\times\mathbb{N}$ and the corresponding convergence coincides with the statistical convergence.

Let D denote the set of all closed and bounded intervals $X = [x_1, x_2]$ on the real line $\mathbb{R}\times\mathbb{N}$. For $X, Y \in D$, we define $X \leq Y$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$, $d(X,Y) = max\{|x_1 - y_1|, |x_2 - y_2|\}$, where $X = [x_1, x_2]$ and $Y = [y_1, y_2]$.

Then it can be easily seen that d defines a metric on D and (D,d) is a complete metric space. Also the relation " $\leq$ " is a partial order on D. A fuzzy number X is a fuzzy subset of the real line $\mathbb{R}\times\mathbb{R}$ i.e. a mapping $X : R \to J\,(=[0,1])$ associating each real number t with its grade of membership $X(t)$.

2.6. **Definition.** A fuzzy number X is said to be (i) convex if $X(t) \geq X(s) \wedge X(r) = min\{X(s), X(r)\}$, where $s < t < r$. (ii) normal if there exists $t_0 \in \mathbb{R} \times \mathbb{R}$ such that $X(t_0) = 1$. (iii) upper semi-continuous if for each $\epsilon > 0, X^{-1}([0, a+\epsilon])$ for all $a \in [0,1]$ is open in the usual topology of $\mathbb{R} \times \mathbb{R}$.

Let $R(J)$ denote the set of all fuzzy numbers which are upper semi continuous and have compact support, i.e. if $X \in \mathbb{R}(J) \times \mathbb{R}(J)$ the for any $\alpha \in [0,1], [X]^\alpha$ is compact, where $[X]^\alpha = \{t \in \mathbb{R} \times \mathbb{R} : X(t) \geq \alpha, \ if \ \alpha \in [0,1]\}$, $[X]^0$ =closure of $(\{t \in \mathbb{R} \times \mathbb{R} : X(t) > \alpha, if \alpha = 0\})$.

The set $\mathbb{R}$ of real numbers can be embedded $\mathbb{R}(J)$ if we define $\bar{r} \in \mathbb{R}(J) \times \mathbb{R}(J)$ by

$$\bar{r}(t) = \begin{cases} 1, & \text{if } t = r : \\ 0, & \text{if } t \neq r \end{cases}$$

The absolute value, $|X|$ of $X \in \mathbb{R}(J)$ is defined by

$$|X|(t) = \begin{cases} max\{X(t), X(-t)\}, & \text{if } t \geq 0; \\ 0, & \text{if } t < 0 \end{cases}$$

Define a mapping $\bar{d} : \mathbb{R}(J) \times \mathbb{R}(J) \to \mathbb{R}^+ \cup \{0\}$by

$$\bar{d}(X,Y) = sup_{0 \leq \alpha \leq 1} d([X]^\alpha, [Y]^\alpha).$$

It is known that $(\mathbb{R}(J), \bar{d})$ is a complete metric space.

2.7. **Definition.** A metric on $\mathbb{R}(J)$ is said to be translation invariant if $\bar{d}(X+Z, Y+Z) = \bar{d}(X,Y)$, for $X, Y, Z \in \mathbb{R}(J)$.

2.8. **Definition.** A sequence $X = (X_{mn})$ of fuzzy numbers is said to be (i) convergent to a fuzzy number X_0 if for every $\epsilon > 0$, there exists a positive integer n_0 such that $\bar{d}(X_{mn}, X_0) < \epsilon$ for all $n \geq n_0$. (ii) bounded if the set $\{X_{mn} : m, n \in \mathbb{N}\}$ of fuzzy numbers is bounded.

2.9. **Definition.** A sequence $X = (X_{mn})$ of fuzzy numbers is said to be (i) I-convergent to a fuzzy number X_0 if for each $\epsilon > 0$ such that

$$A = \{m, n \in \mathbb{N} : \bar{d}(X_{mn}, X_0) \geq \epsilon\} \in I.$$

The fuzzy number X_0 is called I-limit of the sequence (X_{mn}) of fuzzy numbers and we write $I - lim X_{mn} = X_0$. (ii) I-bounded if there exists $M > 0$ such that

$$\{m, n \in \mathbb{N} : \bar{d}(X_{mn}, \bar{0}) > M\} \in I.$$

2.10. **Definition.** A sequence space E_F of fuzzy numbers is said to be (i) solid (or normal) if $(Y_{mn}) \in E_F$ whenever $(X_{mn}) \in E_F$ and $\bar{d}(Y_{mn}, \bar{0}) \leq \bar{d}(X_{mn}, \bar{0})$ for all $m, n \in \mathbb{N}$. (ii) symmetric if $(X_{mn}) \in E_F$ implies $(X_{\pi(mn)}) \in E_F$ where π is a permutation of $\mathbb{N} \times \mathbb{N}$.

Let $K = \{k_1 < k_2 < ...\} \subseteq \mathbb{N}$ and E be a sequence space. A K-step space of E is a sequence space

$$\lambda_{mn}^{E} = \left\{ \left(X_{m_p n_p} \right) \in w^2 : \left(m_p n_p \right) \in E \right\}.$$

A canonical preimage of a sequence $\left\{ \left(x_{m_p n_p} \right) \right\} \in \lambda_K^E$ is a sequence $\{y_{mn}\} \in w^2$ defined as

$$y_{mn} = \begin{cases} x_{mn}, & \text{if } m, n \in E \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of canonical preimages of all elements in λ_K^E, i.e. y is in canonical preimage of λ_K^E if and only if y is canonical preimage of some $x \in \lambda_K^E$.

2.11. **Definition.** A sequence space E is said to be monotone if E contains the canonical pre-images of all its step spaces.

The following well-known inequality will be used throughout the article. Let $p = (p_{mn})$ be any sequence of positive real numbers with $0 \leq p_{mn} \leq sup_{mn} p_{mn} = G, D = max\{1, 2G-1\}$ then

$$|a_{mn} + b_{mn}|^{p_{mn}} \leq D \left(|a_{mn}|^{p_{mn}} + |b_{mn}|^{p_{mn}} \right) \text{ for all } m, n \in \mathbb{N} \text{ and } a_{mn}, b_{mn} \in \mathbb{C}.$$

Also $|a_{mn}|^{p_{mn}} \leq max \left\{ 1, |a|^G \right\}$ for all $a \in \mathbb{C}$.

First we procure some known results; those will help in establishing the results of this article.

2.12. **Lemma.** A sequence space E is normal implies E is monotone. (For the crisp set case, one may refer to Kamthan and Gupta [16], page 53).

2.13. **Lemma.** (Kostyrko et al., [17], Lemma 5.1). If $I \subset 2^{\mathbb{N}}$ is a maximal ideal, then for each $A \subset \mathbb{N}$ we have either $A \in I$ or $\mathbb{N} - A \in I$.

2.14. **Definition.** A sequence $X = (X_{mn})$ of fuzzy numbers is a function X from the set $\mathbb{N} \times \mathbb{N}$ of natural numbers into $L(\mathbb{R}) \times L(\mathbb{R})$. The fuzzy number X_{mn} denotes the value of the function $m, n \in \mathbb{N}$.
We denote W^{2F} denotes the set of all sequences $X = (X_{mn})$ of fuzzy numbers.

2.15. **Definition.** A sequence $X = (X_{mn})$ of fuzzy numbers is said to be analytic if the set $\{X_{mn} : m, n \in \mathbb{N}\}$ of fuzzy numbers is analytic.

2.16. **Definition.** The sequence $X = (X_{mn})$ of fuzzy numbers is said to be almost convergent to a fuzzy number $\bar{0}$ if $lim_{m,n \to \infty} d\left(t_{pm,qn}(X), \bar{0}\right) = 0$ uniformly in m, n, where $t_{pm,qn}(X) = \frac{1}{(m+1)(n+1)} \sum_{i=0}^{p} \sum_{j=0}^{q} \left(((i+m)+(j+n))! X_{i+m,j+n} \right)^{1/((i+m)+(j+n))}$.

This means that for every $\epsilon > 0$, there exists a $p_0 q_0 \in \mathbb{N}$ such that $d\left(t_{pm,qn}(X), \bar{0}\right) < \epsilon$. whenever $p, q \geq p_0 q_0$ and for all m, n.

2.17. **Definition.** A sequence $X = (X_{mn})$ of fuzzy numbers is said to be statistically convergent to a fuzzy number $\bar{0}$ if for every $\epsilon > 0$,

$$lim_{\frac{1}{rs}} \left| \left\{ m \leq r, n \leq s : d \left(((m+n)! X_{mn})^{1/m+n}, \bar{0} \right) \geq \epsilon \right\} \right| = 0.$$

The set of all statistically convergent sequences of fuzzy numbers is denoted by S^{2F}. We note that if a sequence $X = (X_{mn})$ of fuzzy numbers converges to a fuzzy number $\bar{0}$, then it is statistically converges to $\bar{0}$. But the converse statement is not necessarily valid.

Let $\mu = (\lambda_{rs})$ be a non-decreasing sequence of positive real numbers tending to infinity and $\lambda_{11} = 1$ and $\lambda_{r+1,s+1} \leq \lambda_{rs} + 1$, for all $r, s \in \mathbb{N}$.

The generalized de la Vallee-Poussin means is defined by

$$t_{rs}(x) = \frac{1}{\lambda_{rs}} \sum_{p \in I_r} \sum_{q \in I_s} ((m+n)! |x_{mn}|)^{1/m+n}$$

where $I_{rs} = [r, s - \lambda_{rs} + 1, rs]$. A sequence $x = (x_{mn})$ of complex numbers is said to be $(V, \lambda)-$ summable to a number if $t_{rs}(x) \to L$ as $r, s \to \infty$.

3. Some new integrated statistical convergence sequence spaces of fuzzy numbers

The main aim of this article to introduce the following sequence spaces and examine topological and algebraic properties of the resulting sequence spaces. Let $p = (p_{mn})$ be a sequence of positive real numbers for all $m, n \in \mathbb{N}$. $f = (f_{mn})$ be a Musielak-Orlicz function, $\left(X, \|(d(x_1, 0), d(x_2, 0), \cdots, d(x_{n-1}, 0))\|_p \right)$ be a $p-$metric space, and $\left(\lambda_{rs}^{-1} \right)$ be a sequence of non-zero scalars and $\mu_{mn}(X) = \bar{d}(t_{rs}, \bar{0})$ be a sequence of fuzzy numbers, we define the following sequence spaces as follows:

$\left[\chi_{f\mu}^{2q}, \|(d(x_1, 0), d(x_2, 0), \cdots, d(x_{n-1}, 0))\|_p \right] =$
$lim_{r,s \to \infty} \left| \left\{ m, n \in I_{rs} : \left[f_{mn} \left(\|\mu_{mn}(x), (d(x_1, 0), d(x_2, 0), \cdots, d(x_{n-1}, 0))\|_p \right) \right]^{q_{mn}} \geq \epsilon \right\} \right| =$
0, uniformly in r, s.

In this case, we write $X_{mn} \to \bar{0} \left(\check{S}_{\lambda}^{F} \right)$. The set of all statistically convergent sequences is denoted by $\check{S}_{\lambda}^{F}$.

Let $X = (X_{mn})$ be a sequence of fuzzy numbers and $q = (q_{mn})$ be a sequence of strictly positive real numbers. Then the sequence $X = (X_{mn})$ is said to be strongly $\lambda-$ convergent if there is a fuzzy number $\bar{0}$ such that

$\left[\chi_{f\mu}^{2q}, \|(d(x_1, 0), d(x_2, 0), \cdots, d(x_{n-1}, 0))\|_p \right] =$
$lim_{r,s \to \infty} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left[f_{mn} \left(\|\mu_{mn}(x), (d(x_1, 0), d(x_2, 0), \cdots, d(x_{n-1}, 0))\|_p \right) \right]^{q_{mn}} = 0$, uniformly in r, s.

In this case, we write $X_{mn} \to \bar{0}\left(\breve{w}_{\lambda}^{F}, q\right)$. The set of all strongly $\lambda-$ convergent sequences is denoted by $\left(\breve{w}_{\lambda}^{F}, q\right)$.

Let $X = (X_{mn})$ be a sequence of fuzzy numbers. Then the sequence $X = (X_{mn})$ of fuzzy numbers is said to be double analytic if the set $\{t_{rs} : r, s \in \mathbb{N}\}$ of fuzzy numbers is double analytic and it is denoted by $\breve{\Lambda}^{2F}$. In this section we give some inclusion relations between strongly $\lambda-$ convergence and $\lambda-$ statistically convergence and show that they are equivalent for almost bounded sequences of fuzzy numbers. We also study the inclusion $\breve{S}^{2F} \subset \breve{S}^{2F}$ under certain restrictions on the sequence $\Lambda^2 = (\lambda_{rs})$.

3.1. **Theorem.** If $\chi^2(X) \in \breve{S}_{\lambda}^{2F}$ and $c \in \mathbb{R}$, then
(a) $\breve{S}_{\lambda}^{2F} - lim c\chi^2(X) = c\breve{S}_{\lambda}^{2F} - lim\chi^2(X)$
(b) $\breve{S}_{\lambda}^{2F} - lim\chi^2(X+Y) = \breve{S}_{\lambda}^{2F} - lim\chi^2(X) + \breve{S}_{\lambda}^{2F} - lim\chi^2(Y)$
Proof: (a) Let $\chi^2(X) \in \breve{S}_{\lambda}^{2F}$ so that $\breve{S}_{\lambda}^{2F} - lim\chi^2(X) = \bar{0}, c \in \mathbb{R}$ and $\epsilon > 0$. Then the inequality
$\left|\left\{m, n \in I_{rs} : \left[f_{mn}\left(\|\mu_{mn}(cx), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)\right]^{q_{mn}} \geq \epsilon\right\}\right| \leq$
$\left|\left\{m, n \in I_{rs} : \left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)\right]^{q_{mn}} \geq \frac{\epsilon}{|c|}\right\}\right|$, for all $r, s \in \mathbb{N}$.

Proof (b): Suppose that $\chi^2(X), \chi^2(Y) \in \breve{S}_{\lambda}^{2F}$ so that $\breve{S}_{\lambda}^{2F} - lim\chi^2(X) = \bar{0}$ and $\breve{S}_{\lambda}^{2F} - lim\chi^2(Y) = \bar{0}$. By Minkowski's inequality, we get
$\left[f_{mn}\left(\|\mu_{mn}(x+y), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)\right]^{q_{mn}} \leq$
$\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)\right]^{q_{mn}} +$
$\left[f_{mn}\left(\|\mu_{mn}(y), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)\right]^{q_{mn}}$. Therefore given $\epsilon > 0$, for all $r, s \in \mathbb{N}$, we have
$\left|\left\{m, n \in I_{rs} : \left[f_{mn}\left(\|\mu_{mn}(x+y), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)\right]^{q_{mn}} \geq \epsilon\right\}\right| \leq$
$\left|\left\{m, n \in I_{rs} : \left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)\right]^{q_{mn}} \geq \frac{\epsilon}{2}\right\}\right| +$
$\left|\left\{m, n \in I_{rs} : \left[f_{mn}\left(\|\mu_{mn}(y), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)\right]^{q_{mn}} \geq \frac{\epsilon}{2}\right\}\right|$.
The completes the proof.
The following theorem shows that $\lambda-$ statistical convergence and strongly $\lambda-$ convergence are equivalent for double analytic sequences of fuzzy numbers.

3.2. **Theorem.** Let the sequence $\mu = (\mu_{mn})$ be double analytic and $\chi^2(X)$ be a sequence of fuzzy numbers. Then (a) $\chi^2(X) \to \bar{0}\left(\breve{w}_{\lambda}^{2F}, \mu\right)$ implies $\chi^2(X) \to \bar{0}\left(\breve{S}_{\lambda}^{2F}, \mu\right)$.
(b)$\Lambda^2(X) \to \bar{0}\left(\breve{S}_{\lambda}^{2F}, \mu\right)$ imply $\Lambda^2(X) \to \bar{0}\left(\breve{w}_{\lambda}^{2F}, \mu\right)$.
(c)$\breve{S}_{\lambda}^{2F} \bigcap \Lambda_{\lambda}^{2F} = \left(\breve{w}_{\lambda}^{2F}, \mu\right) \bigcap \Lambda_{\lambda}^{2F}$.
Proof:(a) Let $\epsilon > 0$ and $\chi^2(X) \to \bar{0}\left(\breve{w}_{\lambda}^{2F}, \mu\right)$ for all $r, s \in \mathbb{N}$, we have
$\sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0))\|_p\right)\right]^{q_{mn}} \geq$

$\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}$
$\left[f_{mn}\left(\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right)\right]^{q_{mn}}\geq\epsilon$
$\left[f_{mn}\left(\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right)\right]^{q_{mn}}\geq$
$\left|\left\{(m,n)\in I_{rs}:\left[f_{mn}\left(\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right)\right]^{q_{mn}}\geq\epsilon\right\}\right|\cdot min\left(\epsilon^h,\epsilon^H\right).$
Hence $\chi^2(X)\in\breve{S}^{2F}_{\lambda}$.

Proof (b): Suppose that $\chi^2(X)\in\breve{S}^{2F}_{2\lambda}\bigcap\Lambda^{2F}$. Since $\chi^2(X)\in\Lambda^{2F}$, we write
$\left[f_{mn}\left(\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right)\right]^{q_{mn}}\leq T$, for all $r,s\in\mathbb{N}$, let
$G_{rs}=\left|\left\{(m,n)\in I_{rs}:\left[f_{mn}\left(\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right)\right]^{q_{mn}}\geq\epsilon\right\}\right|$ and
$H_{rs}=\left|\left\{(m,n)\in I_{rs}:\left[f_{mn}\left(\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right)\right]^{q_{mn}}<\epsilon\right\}\right|$. Then we have
$\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[f_{mn}\left(\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right)\right]^{q_{mn}}=$
$\sum_{m\in G_{rs}}\sum_{n\in G_{rs}}\left[f_{mn}\left(\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right)\right]^{q_{mn}}+$
$\sum_{m\in H_{rs}}\sum_{n\in H_{rs}}\left[f_{mn}\left(\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right)\right]^{q_{mn}}\leq$
$max\left(T^h,T^H\right)G_{rs}+max\left(\epsilon^h,\epsilon^H\right)$. Taking the limit as $\epsilon\to0$ and $r,s\to\infty$, it follows that $\chi^2(X)\in\left(\breve{w}^F_{\lambda},q\right)$.
(c) Follows from (a) and (b).

3.3. **Theorem.** If $liminf_{rs}\left(\frac{\lambda_{rs}}{rs}\right)>0$, then $\breve{S}^{2F}\subset\breve{S}^{2F}_{\lambda}$.
Proof: Let $\chi^2(X)\in\breve{S}^{2F}$. For given $\epsilon>0$, we get
$\left|\left\{m\leq r,n\leq s:\left[f_{mn}\left(\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right)\right]^{q_{mn}}\geq\epsilon\right\}\right|\supset G_{rs}$ where G_{rs} is in the Theorem of 3.2 (b). Thus,
$\left|\left\{m\leq r,n\leq s:\left[f_{mn}\left(\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right)\right]^{q_{mn}}\geq\epsilon\right\}\right|\geq G_{rs}=$ $\frac{\lambda_{rs}}{rs}$. Taking limit as $r,s\to\infty$ and using $liminf_{rs}\left(\frac{\lambda_{rs}}{rs}\right)>0$, we get $\chi^2(X)\in\breve{S}^{2F}_{\lambda}$.

3.4. **Theorem.** Let $0<u_{mn}\leq v_{mn}$ and $(u_{mn}v^{-1}_{mn})$ be double analytic. Then $\left(\breve{w}^{2F}_{\lambda},v\right)\subset\left(\breve{w}^{2F}_{\lambda},u\right)$
Proof: Let $\chi^2(X)\in\left(\breve{w}^{2F}_{\lambda},v\right)$. Let
$w_{mn}=\left[f_{mn}\left(\|\mu_{mn}(x),(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0))\|_p\right)\right]^{q_{mn}}$ for all $r,s\in\mathbb{N}$ and $\lambda_{mn}=u_{mn}v^{-1}_{mn}$ for all $m,n\in\mathbb{N}$. Then $0<\lambda_{mn}\leq1$ for all $m,n\in\mathbb{N}$. Let b be a constant such that $0<b\leq\lambda_{mn}\leq1$ for all $m,n\in\mathbb{N}$.

Define the sequences (k_{mn}) and (ℓ_{mn}) as follows:

For $w_{mn}\geq1$, let $(k_{mn})=(w_{mn})$ and $\ell_{mn}=0$ and for $w_{mn}<1$, let $k_{mn}=0$ and $\ell_{mn}=w_{mn}$. Then it is clear that for all $m,n\in\mathbb{N}$, we have $w_{mn}=k_{mn}+\ell_{mn}$ and $w^{\lambda_{mn}}_{mn}=k^{\lambda_{mn}}_{mn}+\ell^{\lambda_{mn}}_{mn}$. Now it follows that $k^{\lambda_{mn}}_{mn}\leq k_{mn}\leq w_{mn}$ and $\ell^{\lambda_{mn}}_{mn}\leq\ell^{\lambda}_{mn}$. Therefore

$\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[f_{mn}\left(\left\|\mu_{mn}\left(w_{mn}^{\lambda_{mn}}\right),\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\right]^{q_{mn}}=$
$\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[f_{mn}\left(\left\|\mu_{mn}\left(k_{mn}+\ell_{mn}\right)^{\lambda_{mn}},\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\right]^{q_{mn}}=$
$\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[f_{mn}\left(\left\|\mu_{mn}\left(w_{mn}\right),\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\right]^{q_{mn}}+$
$\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[f_{mn}\left(\left\|\mu_{mn}\left(\ell_{mn}\right)^{\lambda_{mn}},\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\right]^{q_{mn}}$. Now for each r,s,
$\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[f_{mn}\left(\left\|\mu_{mn}\left(\ell_{mn}\right)^{\lambda},\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\right]^{q_{mn}}=$
$\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[f_{mn}\left(\left\|\mu_{mn}\left(\left(\ell_{mn}\right)^{\lambda}\left(\frac{1}{\lambda_{rs}}\right)^{1-\lambda}\right),\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\right]^{q_{mn}}\leq$
$\left[\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[f_{mn}\left\|\left(\left(\mu_{mn}\left(\left(\ell_{mn}\right)^{\lambda}\right)^{\lambda}\right)^{1/\lambda},\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\right]^{q_{mn}}\right]^{\lambda}.$

3.5. **Theorem.** $\breve{\Lambda}^{2F}=\breve{w}^{2F}_{\lambda,\Lambda^2}$, where $\breve{w}^{2F}_{\lambda,\Lambda^2}=$
$\left\{sup_{rs}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[f_{mn}\left(\left\|\mu_{mn}\left(x\right),\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\right]^{q_{mn}}<\infty\right\}$

Proof: Let $X=(X_{mn})\in\breve{w}^{2F}_{\lambda,\Lambda^2}$. Then there exists a constant $T_1>0$ such that
$\left[f_{mn}\left(\left\|\mu_{mn}\left(x\right),\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\right]^{q_{mn}}\leq$
$sup_{rs}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[f_{mn}\left(\left\|\mu_{mn}\left(x\right),\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\right]^{q_{mn}}\leq T_1$ for all $r,s\in\mathbb{N}$. Therefore we have $X=(X_{mn})\in\breve{\Lambda}^{2F}$. Conversely, let $X=(X_{mn})\in\breve{\Lambda}^{2F}$. Then there exists a constant $T_2>0$ such that
$\left[f_{mn}\left(\left\|\mu_{mn}\left(x\right),\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\right]^{q_{mn}}\leq T_2$ for all m,n and r,s. So,
$\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left[f_{mn}\left(\left\|\mu_{mn}\left(x\right),\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\right]^{q_{mn}}\leq T_2\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}1$
$\leq T_2$, for all m,n and r,s. Thus $X=(X_{mn})\in\breve{w}^{2F}_{\lambda,\Lambda^2}$.

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References

[1] A. Esi and B. Hazarika , On ideal convergent sequence spaces of fuzzy real numbers associated with multiplier sequences defined by sequence of Orlicz functions, *Ann. Fuzzy Math. Inform*, **7(1)**(2014), 289-301.

[2] B. Hazarika, On fuzzy real valued $I-$ convergent double sequence spaces, *The journal of Nonlinear Sciences and Its Applications* **(in press)**.

[3] B. Hazarika, On fuzzy real valued $I-$ convergent double sequence spaces defined by Musielak-Orlicz function, *J. Intell. Fuzzy Systems*, **25(1)** (2013), 9-15, DOI: 10.3233/IFS-2012-0609.

[4] B. Hazarika, Lacunary difference ideal convergent sequence spaces of fuzzy numbers, *J. Intell. Fuzzy Systems*, **25(1)** (2013), 157-166, DOI: 10.3233/IFS-2012-0622.

[5] B.Hazarika, On $\sigma-$ uniform density and ideal convergent sequences of fuzzy real numbers, *J. Intell. Fuzzy Systems*, DOI: 10.3233/IFS-130769.

[6] B.Hazarika, Fuzzy real valued lacunary $I-$ convergent sequences, *Applied Math. Letters*, **25(3)** (2012), 466-470.

[7] B.Hazarika, Lacunary $I-$ convergent sequence of fuzzy real numbers, *The Pacific J. Sci. Techno.*, **10(2)** (2009), 203-206.

[8] B.Hazarika, On generalized difference ideal convergence in random 2-normed spaces, *Filomat*, **26(no.6)** (2012), 1273-1282.

[9] B.Hazarika, Some classes of ideal convergent difference sequence spaces of fuzzy numbers defined by Orlicz function, *Fasciculi Mathematici*, **52** (2014), 46-63.

[10] B.Hazarika, $I-$ convergence and summability in topological group, *J. Informa. Math. Sci.*, **4(3)** (2012), 269-283.

[11] B. Hazarika, Classes of generalized difference ideal convergent sequence of fuzzy numbers, *Ann. Fuzzy Math. Inform*, (in press).

[12] B.Hazarika, On ideal convergent sequences in fuzzy normed linear spaces, *Afrika Matematika*, DOI: 10.1007/s13370-013-0168-0.

[13] B. Hazarika and E. Savas, Some $I-$ convergent lambda-summable difference sequence spaces of fuzzy real numbers defined by a sequence of Orlicz functions, *Math. Comp. Modell.*, **54(11-12)** (2011), 2986-2998.

[14] B. Hazarika,K.Tamang and B.K.Singh, Zweier ideal convergent sequence spaces defined by Orlicz function, *The J. Math. and Computer Sci.*, (Accepted).

[15] B. Hazarika and V. Kumar, Fuzzy real valued $I-$ convergent double sequences in fuzzy normed spaces, *J. Intell. Fuzzy Systems*, (accepted).

[16] P.K. Kamthan and M. Gupta, Sequence Spaces and Series, Lecture Notes, Pure and Applied Mathematics, 65 Marcel Dekker, In c., New York , (1981).

[17] P. Kostyrko, T. Salat and W.Wilczynski, $I-$ convergence, *Real Anal. Exchange*, **26(2)** (2000-2001), 669-686, MR 2002e:54002.

[18] V. Kumar and K. Kumar, On the ideal convergence of sequences of fuzzy numbers, *Inform. Sci.*, **178(24)** (2008), 4670-4678.

[19] V. Kumar , On I and I^*- convergence of double sequences, *Mathematical Communications*, **12** (2007), 171-181.

[20] J. Lindenstrauss and L. Tzafriri, On Orlicz sequence spaces, Israel J. Math., 10 (1971), 379-390.

[21] J. Musielak, Orlicz spaces and modular spaces, *Lecture Notes in Mathematics, Springer,Berlin, Germany*, **1034**, (1983).

[22] N. Subramanian and U.K. Misra, The generalized semi normed difference of double gai Sequence spaces defined by a modulus function, *Stud. Univ. Babes, Bolyai Math.* **156**(2011), 63-73.

[23] N. Subramanian, The semi normed space defined by modulus function, *Southeast Asian Bulletin of Mathematics*, **32** (2008), 1161-1166.

[24] N. Subramanian and U.K. Misra, Characterization of gai sequences via double Orlicz space, *Southeast Asian Bulletin of Mathematics*, **35**(2011), 687-697.

[25] N. Subramanian,P. Angalagan and P.Thirunavukarasu,The ideal convergence of strongly of Γ^2 in $p-$ metric spaces defined by modulus, *Southeast Asian Bulletin of Mathematics*, **37** (2013), 919-930.

[26] B.C.Tripathy and B.Hazarika, $I-$ convergent sequence spaces associated with multiplier sequences, *Math. Ineq. Appl.*, **11(3)** (2008), 543-548.

[27] B.C. Tripathy and B. Hazarika, Paranorm $I-$ convergent sequence spaces, *Math. Slovaca*, **59(4)** (2009), 485-494.

[28] B.C. Tripathy and B. Hazarika, Some $I-$ convergent sequence spaces defined by Orlicz functions, *Acta Math. Appl. Sinica*, **27(1)** (2011), 149-154.

[29] A. Wilansky, Summability through functional analysis, North-Holland mathematical studies, North-Holland publishing, Amsterdam, 85(1984).