Asia Pacific Journal of Mathematics, Vol. 3, No. 2 (2016), 173-185 ISSN 2357-2205

SOME RESULTS ON THE EDGE HUB-INTEGRITY OF GRAPHS

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Abstract. The edge hub-integrity of a graph is given by the minimum of $|S|+m(G-S)$, where S is an edge hub set and $m(G-S)$ is the maximum order of the components of $G-S$. This paper discusses edge hub-integrity of splitting graph and duplication of an edge by vertex and duplication of a vertex by an edge of some graphs.
2010 Mathematics Subject Classification. 05C40, 05C99, 05C76.
Key words and phrases. edge integrity; edge hub-integrity; splitting graph; duplication graph.

1. Introduction

In 1987, Barefoot, Entringer and Swart [2], defined the edge-integrity of a graph G with edge set $E(G)$ by $I'(G) = \min\{|S|+m(G-S) : S \subseteq E(G)\}$. Any set $S \subseteq E(G)$ of edges which realizes this value is called an $I'-set$. Sultan et al. [6], have introduced the concept of hub-integrity of a graph as a new measure of vulnerability. The hub-integrity of a graph G denoted by $HI(G)$ is defined by, $HI(G) = \min\{|S|+m(G-S)\}$, where S is a hub set and $m(G-S)$ is the order of a maximum component of $G-S$. Further Sultan Mahde and Veena Mathad [7] have studied hub-integrity of some operations of graphs. In [8] they discussed hub-integrity of a graph obtained by duplication of an edge by vertex and duplication of a vertex by an edge and splitting graph of some graphs. Sultan Mahde and Veena Mathad [9] defined the edge hub-integrity of a graph G. Let $e = (u, v)$ and $f = (u', v')$, a path between the two edges e and f is a path between one end vertex from e and another end vertex from f such that $d(e, f) = min\{d(u, u'), (u, v'), (v, u'), (v, v')\}$. Internal edges of a path between two edges e and f are all the edges of the path except e and f. Suppose that $S \subseteq E(G)$. An S-path between edges e and f is a path in which all its edges except e and f are in S. (This includes if the path contains two adjacent edges or single edge, such cases the S-path is trivial.)A subset $S \subseteq E(G)$ is called an edge hub set of G if every pair of edges $e, f \in E-S$ are connected by a path where all internal edges are from S. The minimum cardinality of an edge hub set is called edge hub number of G, and is denoted by $h_e(G)$. If G is a disconnected graph then any edge hub set must contain

all of the edges in all but one of the components, as well as an edge hub set in the remaining component. The edge hub-integrity of a graph G denoted by $EHI(G)$ is defined as $EHI(G) = \min\{|S| + m(G-S), S \ is \ an \ edge \ hub \ set \ of \ G\}$, where $m(G-S)$ is the order of a maximum component of $G-S$. Any set $S \subseteq E(G)$ with property that $|S| + m(G-S) = EHI(G)$ is called an $EHI-$ set of G.
In the present work edge hub-integrity of splitting graphs and a graph obtained by duplication of an edge by vertex and duplication of a vertex by an edge in some graphs is investigated. In this paper, a graph is considered as finite, undirected, with single lines and no loops with p vertices and q edges. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. The cardinalities of $V(G)$ and $E(G)$ are called respectively the order and size of G. Let us denote by $G-e$ the graph obtained from G by removing the edge $e \in E(G)$. The symbols $\Delta(G)$, $\delta(G)$, $\alpha(G)$, $\beta(G)$ and $\chi(G)$ denote the maximum degree, the minimum degree, the vertex cover number, the independence number and chromatic number of G, respectively. Also we denote the minimum number of edges in edge cover of G (i.e., edge cover number) by $\alpha_1(G)$ and the maximum number of edges in independent set of edges of G (i.e., edge independence number) by $\beta_1(G)$. For the terminology and notation not defined in this paper, the reader is referred to [3, 4]

Definition 1.1. [5] *For a graph G the splitting graph $S'(G)$ of a graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$, where $N(v)$ and $N(v')$ are the neighborhood sets of v and v', respectively.*

Definition 1.2. [10] *Duplication of a vertex v_i by a new edge $e = (v_i', v_i'')$ in graph G produces a new graph G' such that $N(v_i') = \{v_i'', v_i\}$ and $N(v_i'') = \{v_i', v_i\}$.*

Definition 1.3. [10] *Duplication of an edge $e = (u, v)$ by a new vertex w in graph G produces a new graph G' such that $N(w) = \{u, v\}$.*

The following are some fundamental results which will be required for many of our arguments in this paper.

Proposition 1.1. [5] *For any graph G with p points,*

(1) $\alpha(S'(G)) = p = \beta(S'(G))$*,*
(2) $\alpha_1(S'(G)) = 2\alpha_1(G)$ *and* $\beta_1(S'(G)) = 2\beta_1(G)$*.*

Proposition 1.2. [5] $\chi(S'(G)) = \chi(G)$.

Theorem 1.1. [1] *The edge-integrity of*

(1) *the path P_p is $\lceil 2\sqrt{p} \rceil - 1$,*
(2) *the cycle C_p is $\lceil 2\sqrt{p} \rceil$.*

Lemma 1.1. [9] *For any graph G, $EHI(G) \geq \Delta(G) + 1$.*

Proposition 1.3. [9] *For any graph G, $EHI(G) \geq \gamma(G)$.*

Corollary 1.1. [9] *For any graph G, $EHI(G) \geq \chi(G)$.*

Corollary 1.2. [9] *For any graph G, $EHI(G) \geq \alpha_1(G)$.*

Lemma 1.2. [9] *For any graph G, $EHI(G) \geq \beta_1(G)$.*

2. The edge hub-integrity of splitting graph

Lemma 2.1. *For any proper subgraph H of G, $EHI(S'(H)) < EHI(S'(G))$.*

Proposition 2.1. *For any graph G, $p+1 \leq EHI(S'(G)) \leq 3p-1$. The lower bound is sharp for $G = P_2$, and the upper bound is sharp for $G = K_p$.*

Theorem 2.1. *For any graph G, $EHI(S'(G-e)) \geq EHI(S'(G)) - 1$.*

Proof. Let S' be an EHI-set of $S'(G)-e$, so $EHI(S'(G)-e) = |S'|+m((S'(G)-e)-S')$. Consider $S'' = S' \cup \{e\}$, then $|S''| = |S'| + 1$. Hence, S'' is an EHI-set of $S'(G)$ and $m(S'(G) - S'') = m((S'(G) - e) - S')$. Thus,

$$\begin{aligned} EHI(S'(G)) &\leq |S''| + m(S'(G) - S'') \\ &= |S'| + 1 + m(S'(G) - e) - S') \\ &= EHI(S'(G) - e) + 1. \end{aligned} \tag{1}$$

□

Theorem 2.2. $EHI(S'(G)) = EHI(L(S'(G))$ *if and only if* $G \cong P_2$.

By Corollary 1.1 and Proposition 1.2, the proof of the following result is straight forward.

Proposition 2.2. *For any graph G, $EHI(G) \geq \chi(S'(G))$.*

Lemma 2.2. (1) *For any graph G, $EHI(G) \geq \frac{\alpha_1(S(G))}{2}$.*
(2) *For any graph G, $EHI(G) \geq \frac{\beta_1(S(G))}{2} + 1$.*

Proof. (1)Proof follows from Corollary 1.2 and Proposition 1.1.
(2) Proof follows from Lemma 1.2 and Proposition 1.1. □

Observation 2.1. *For any graph G, $EHI(S'(G)) \geq EHI(G)+1$. The bound is sharp for $G = P_2$.*

Lemma 2.3. *For any graph $G, EHI(S'(G)) \geq \delta(S'(G)) + 2$.*

Proof. Let S be an EHI-set of $S'(G)$ satisfies $EHI(S'(G)) = |S| + m(S'(G) - S)$, since $m(S'(G) - S) \geq \delta(S'(G) - S) + 1 \geq \delta(S'(G) - S) - |S| + 2$, we conclude that $EHI(S'(G)) = |S| + m(S'(G) - S) \geq \delta(S'(G)) + 2$. □

Corollary 2.1. $EHI(S'(G)) \geq \Delta(G) + 2.$

Proof. Lemma 1.1 and Observation 2.1, lead to the result. □

Corollary 2.2. *For any graph* G*,* $EHI(S'(G)) \geq \gamma(G) + 1.$

Proof. Proof follows from Proposition 1.3 and Observation 2.1. □

Theorem 2.3. *For* $p \geq 2$*,*

$$EHI(S'(P_p)) = \begin{cases} 2p-1, \text{ if } p = 2,3,4,5\,; \\ p + \lceil \frac{p}{i} \rceil + 2i - 3, \; i(2i-1) \leq p \leq i((2i-1)) + 4, i \in Z^+/\{1\}. \end{cases}$$

Proof. Let $\{u_1, u_2, ..., u_p\}$ be the vertices set of path P_p and $\{v_1, v_2, v_3, ..., v_p\}$ be the new vertices corresponding to $\{u_1, u_2, ..., u_p\}$ which are added to obtain $S'(P_p)$. We have the following cases:

Case 1: For $p = 2$, consider $S = \{(u_1, u_2)\}$, an edge hub set of $S'(P_2)$, then $m(S'(P_2) - S) = 2$. This implies that $EHI(S'(P_2)) \leq |S| + m(S'(P_2) - S) = 3$. If S_1 is any edge hub set other than S with $m(S'(P_2) - S_1) = 1$, then $|S_1| = 3$, so $EHI(S'(P_2)) = 3$. Clearly there does not exist any edge hub set S_1 of $S'(P_2)$ such that $|S_1| + m(S'(P_2) - S_1) > |S| + m(S'(P_2) - S)$. Hence, $EHI(S'(P_2) = 3$.

Case 2: For $p = 3$, consider $S = \{(u_1, u_2), (u_2, u_3)\}$, an edge hub set of $S'(P_3)$, then $S'(P_3) - S = 2P_3$, $m(S'(P_3) - S) = 3$, and $|S| = 2$. Thus

$$EHI(S'(P_3)) \leq |S| + m(S'(P_3) - S) = 5. \tag{2}$$

If $m(S'(P_3) - S) = 2$, then $|S| \geq 4$, Therefore,

$$EHI(S'(P_3)) \geq |S| + m(S'(P_3) - S) = 6. \tag{3}$$

Consider $m(S'(P_3) - S) = 1$, then $|S| = 6$. Therefore, $EHI(S'(P_3) = |S| + m(S'(P_3) - S) = 7$. Consider $m(S'(P_3) - S) \geq 3$, then trivially, $EHI(S'(P_3)) \geq 2p - 1$. So $EHI(S'(P_3)) = 5$.

Case 3: For $p = 4$, consider $S = \{(u_1, u_2), (u_2, u_3), (u_3, u_4)\}$, an edge hub set of $S'(P_4)$, $|S| = 3$, then $m(S'(P_4) - S) = 4$, thus

$$EHI(S'(P_4)) \leq |S| + m(S'(P_4) - S) = 7 = 2p - 1. \tag{4}$$

Clearly there does not exist any edge hub set S_1 of $S'(P_4)$ such that $|S_1| + m(S'(P_4) - S_1) < |S| + m(S'(P_4) - S)$. Hence, $EHI(S'(P_4) = 7$.

Case 4: For $p = 5$, consider $S = \{(u_1, u_2), (u_2, u_3), (u_3, u_4), (u_4, u_5)\}$, an edge hub set of $S'(P_5)$, $|S| = 4$, then $m(S'(P_5) - S) = 5$, this implies that

$$EHI(S'(P_5)) \leq |S| + m(S'(P_5) - S) = 9. \tag{5}$$

Clearly there does not exist any edge hub set S_1 of $S'(P_5)$ such that $|S_1| + m(S'(P_5) - S_1) < |S| + m(S'(P_5) - S)$. Hence, $EHI(S'(P_5) = 9$.

Case 5: For $p \geq 6$, in case $p = 6$. Consider $S_1 = \{(u_1, u_2), (u_2, u_3), (u_3, u_4), (u_4, u_5), (u_5, u_6)\}$, an edge hub set of $S'(P_6)$, $|S_1| = 5$, then $S'(P_6) - S_1 = 2P_6$ and $m(S'(P_6) - S_1) = 6$. We choose one edge from each component P_6, so $|S| = 2$, and $m(P_6 - S) = 3$, then $I'(2C_6) = 5$. So $EHI(S'(P_6)) = |S_1| + m(S'(P_6) - S) = 11$. But this value is not minimum, therefore, we can do follows: let $S_2 = \{e_k = \{(u_k, v_{k+1}), (v_j, u_{j+1}), 1 \leq k, j \leq 5 : e_k \in I'-\ set\ of\ 2P_6\}$. Take $E_1 = \{e_k : e_k \in I'-\ set\ of\ 2P_6\}$, thus $|S_2| = |E_1|$. Consider $S = S_1 \cup S_2$. Then S is an edge hub set of P_6, and $|S| = |S_1| + |S_2| = |S_1| + |E_1|$, and $S'(P_6) - S = 2P_6 - E_1$. So $m(S'(P_6) - S) = m(2P_6 - E_1) = 5$. Therefor,

$$
\begin{aligned}
|S| + m(S'(P_6) - S) &= |S_1| + |E_1| + m(2P_6 - E_1) \\
&= |S_1| + I'(2P_6) \\
&= p - 1 + \lceil \frac{p}{2} \rceil + 2 = 10.
\end{aligned} \tag{6}
$$

In general if $p \geq 6$, for $i \in Z^+/\{1\}$, $i(2i-1) \leq p \leq i((2i-1)+4)$. Then

$$
I'(2P_p) = \lceil \frac{p}{i} \rceil + 2(i-1). \tag{7}
$$

For more details, if $i = 2$, then $p = 6, 8, 10, 12, 14$, $|S| = \lceil \frac{p}{2} \rceil$, and $m(S'(P_P) - S) = 2$. Thus, consider $S_1 = \{(u_1, u_2), (u_2, u_3), (u_3, u_4), ..., (u_{p-1}, u_p)\}$, an edge hub set of $S'(P_p)$, $|S_1| = p - 1$, and $S'(P_p) - S_1 = 2P_p$. Let $S_2 = \{(u_k, v_{k+1}), (v_j, u_{j+1}), 1 \leq i, k \leq p - 1 \in I' - set\ of\ 2P_p\}$. Take $E_1 = \{e_k, e_j : e_k, e_j \in I' - set\ of\ 2P_p\}$. Thus $|S_2| = |E_1|$. Consider $S = S_1 \cup S_2$. Then S is an edge hub set of $S'(P_p)$, thus $|S| = |S_1| + |S_2| = |S_1| + |E_1|, S'(P_p) - S = 2P_p - E_1$ and $m(S'(P_p) - S) = m(2P_p - E_1)$. By Equation (7),

$$
\begin{aligned}
|S| + m(S'(P_p) - S) &= |S_1| + |E_1| + m(2P_p - E_1) \\
&= |S_1| + I'(2P_p) \\
&= p - 1 + \lceil \frac{p}{i} \rceil + 2(i-1).
\end{aligned} \tag{8}
$$

The minimality of $|S|+m(S'(P_p)-S)$ is discussed. Consider S_3 is any edge hub set of G such that $S_3 = S_1$, thus $|S_3| = p-1$. Then $S'(P_p) - S_3 = 2P_p$ and $m(S'(P_p) - S_3) = p$, this implies that

$$
|S_3| + m(S'(P_p) - S_3) = 2p - 1 > |S| + m(S'(P_p) - S). \tag{9}
$$

Let S_5 be another an edge hub set of $S'(P_p)$ such that $S_5 = S_4 \cup S_1$, where $S_4 \subset S_2$ with $|S_4| < \lceil \frac{p}{i} \rceil$. Therefore, $m(S'(P_p) - S_5) = p - |S_4|$. Hence,

$$
\begin{aligned}
|S_5| + m(S'(P_p) - S_5) &= |S_4| + |S_1| + p - |S_4| \\
&= |S_1| + p \\
&> |S| + m(S'(P_p) - S).
\end{aligned} \tag{10}
$$

Consider $S_7 = S_6 \cup S_1$, where $S_6 = \{(u_1, v_2), (v_2, u_3), (u_3, v_4), (v_4, u_5), ..., (u_{p-1}, v_p), (v_{p-1}, u_p)\}$, an edge hub set of $S'(P_p)$, $|S_6| = p - 1$. Then $m(S'(P_p) - S_7) = |S_1| + p - |S_6|$, hence,

$$
|S_7| + m(S'(P_p) - S_7) = |S_6| + |S_1| + |S_1| + p - |S_6| = 2|S_1| + p > |S| + m(S'(P_p) - S). \tag{11}
$$

Thus, from the above discussion and (9), (10) and (11), it follows that $|S|+m(S'(P_p)-S)$ is minimum. Then, from (8) and the minimality of $|S|+m(S'(P_p)-S)$,

$$
\begin{aligned}
EHI(S'(P_p) &= min\{|X| + m(S'(P_p) - X) : X \text{ is an edge hub set}\} \\
&= |S| + m(S'(P_p) - S) \qquad (12)\\
&= p + \lceil \frac{p}{i} \rceil + 2i - 3.
\end{aligned}
$$

□

Theorem 2.4. *For* $p \geq 3$,

$$
EHI(S'(C_p)) = \begin{cases} 2p, \text{ if } p = 4, 6\ ; \\ p - 1 + \lceil 2\sqrt{2p} \rceil, \text{if } p \text{ is odd;} \\ p + \lceil \frac{p}{i} \rceil + 2i, \text{if } p \text{ is even, } 2i^2 \leq p \leq 2((i+1)^2 - 2), i \in Z^+/\{1\}. \end{cases}
$$

Proof. Let $\{u_1, u_2, ..., u_p\}$ be the vertices of cycle C_p and $\{v_1, v_2, ..., v_p\}$ be the new vertices corresponding to $\{u_1, u_2, ..., u_p\}$ which are added to obtain $S'(C_p)$. The following cases are available:

Case 1: For p is odd, consider $S_1 = \{(u_1, u_2), (u_2, u_3), ..., (u_{p-1}, u_p)\}$, an edge hub set of $S'(C_p), thus\ |S_1| = p - 1$ and $G - S_1 = C_{2p} + e$, where e is an edge $v_p v_1$ shown as Figure 1. From the definition of edge integrity of a cycle C_{2p}, $m(C_{2p} - S) \geq 3$, where S is an edge-set. So we can use the same method for find $I'(C_{2p} + e)$, this means that $I'(C_{2p} + e) = I'(C_{2p})$ such that $e = (v_1, v_p)$. Let $S_2 = \{e_i = (v_i, u_i), 1 \leq i \leq p : e_i \in I' - \ set\}$. Take $E_1 = \{e_i : e_i \in I' - \ set\ of\ C_{2p}\}$, so $|S_2| = |E_1|$. Consider $S = S_1 \cup S_2$. Then S is an edge hub set of $S'(C_p)$, and $|S| = |S_1| + |S_2| = |S_1| + |E_1|$ and $S'(C_p) - S = C_{2p} - E_1$, so $m(S'(C_p) - S) = m(C_{2p} - E_1)$. By Theorem 1.1,

$$
\begin{aligned}
|S| + m(S'(C_p) - S) &= |S_1| + |E_1| + m(C_{2p} - E_1) \\
&= |S_1| + I'(C_{2p}) \qquad (13)\\
&= p - 1 + \lceil 2\sqrt{2p} \rceil.
\end{aligned}
$$

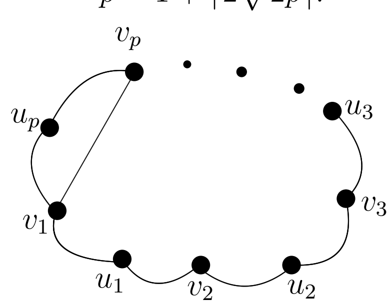

Figure 1: $C_{2p} + e$

Now the following cases are discussed for the minimality of $|S|+m(S'(C_p)-S)$. If S_3 is any edge hub set of $S'(C_p)$ such that $S_3 = S' \cup S_2$, where $S' = \{(v_1, v_2), (v_2, v_3), ..., (v_{p-1}, v_p), (v_p, v_1)\}$, $|S'| = p$. This implies that

$$
|S_3| + m(S'(C_p) - S_3) = p + \lceil 2\sqrt{2p} \rceil > |S| + m(S'(C_p) - S). \qquad (14)
$$

If the set $S_5 = S_4 \cup S_2$, where $S_4 \subset S_1$ with $|S_4| < p-1$ is considered as an edge hub set, then there does not exists a path between some edges in $S'(C_p)$, so the set S is the minimum an edge hub set. Hence from the above discussion and (14), we get that $|S| + m(S'(C_p) - S)$ is minimum. Thus, from equation (13) and the minimality of $|S| + m(S'(C_p) - S)$,

$$\begin{aligned} EHI(S'(C_p)) &= min\{|X| + m(S'(C_p) - S) : X \ is \ an \ edge \ hub \ set)\} \\ &= |S| + m(S'(C_p) - S) \\ &= p - 1 + \lceil 2\sqrt{2p} \rceil. \end{aligned} \tag{15}$$

Case 2: For p is even, We have the following subcases:

Subcase 2.1: For $p = 4$, consider $S = \{(u_1, u_2), (u_2, u_3), (u_3, u_4), (u_4, u_1)\}$, an edge hub set of $S'(C_4), |S| = 4$ and $S'(C_4) - S = 2C_4$. So $m(S'(C_4) - S) = 4$. This implies that $EHI(S'(C_4)) \leq |S| + m(S'(C_4) - S) = 8$. Clearly does not exist any edge hub set S_1 of $S'(C_4)$ such that $|S_1| + m(S'(C_4) - S_1) < |S| + m(S'(C_4) - S)$. Hence $EHI(S'(C_4)) = 8$.

Subcase 2.2: For $p = 6$, consider $S = \{(u_1, u_2), (u_2, u_3), (u_3, u_4), (u_4, u_5), (u_5, u_6), (u_6, u_1)\}$, an edge hub set of $S'(C_6), |S| = 6$ and $m(S'(C_6) - S) = 6$. This implies that $EHI(S'(C_6)) \leq |S| + m(S'(C_6) - S) = 12$. Clearly does not exist any edge hub set S_1 of $S'(C_6)$ such that $|S_1| + m(S'(C_6) - S_1) < |S| + m(S'(C_6) - S)$. Hence $EHI(S'(C_6)) = 12$.

Subcase 2.3: For $p \geq 8$, let $i \in Z^+/\{1\}$.

Consider $S_1 = \{(u_1, u_2), (u_2, u_3), (u_3, u_4), ..., (u_{p-1}, u_p), (u_p, u_1)\}$, an edge hub set of $S'(C_p)$, $|S_1| = p$ and $S'(C_4) - S_1 = 2C_p$, and $m(S'(C_p) - S_1) = p$. Let $S_2 = \{e_k = (v_k, u_j) : 1 \leq k, j \leq p, e_j = (u_n, u_m) : 1 \leq n, m \leq p\}$. Take $E_1 = \{e_k, e_j : e_k, e_j \in I' - set \ of \ 2C_p\}$. If $8 \leq p \leq 16$, in this case, we choose two edges from each component, so $|S| = 4$, and $m(2C_p - S) = \lceil \frac{p}{2} \rceil$, where i denotes to the number of edges removed from $2C_p$ and $i = \lfloor \sqrt{\frac{p}{2}} \rfloor$. For $p = 8, |S_1| = 8$ and $S'(C_8) - S_1 = 2C_8$, So $m(S'(C_8) - S_1) = 8, I'(2C_8) = \frac{p}{2} + 2$. In this case, we delete two edges of each component C_8 and we get four components of order 4. Hence $I'(2C_8) = |S_2| + m(2C_8 - S_2) = 8$. In general, for $2i^2 \leq p \leq 2((i+1)^2 - 2)$, such that $i \in Z^+/\{1\}$,

$$I'(2C_p) = \lceil \frac{p}{i} \rceil + 2i. \tag{16}$$

Consider $S = S_1 \cup S_2$, then S is also an edge hub set of $S'(C_p)$, $|S| = |S_1| + |S_2| = |S_1| + |E_1|$ and $S'(C_p) - S = 2C_p - E_1$, So $m(S'(C_p) - S) = m(2C_p - E_1)$. By Equation (16),

$$\begin{aligned} |S| + m(S'(C_p) - S) &= |S_1| + |E_1| + m(C_{2p} - E_1) \\ &= |S_1| + I'(2C_p) \\ &= p + \lceil \frac{p}{i} \rceil + 2i. \end{aligned} \tag{17}$$

To show $|S| + m(S'(C_p) - S)$ is minimum, consider $S_3 = S_1$, $|S_3| = p$ and $m(S'(C_p) - S_3) = p$, this implies that

$$|S_3| + m(S'(C_p) - S_3) = 2p > |S| + m(S'(C_p) - S). \tag{18}$$

Consider $S_5 = S_4 \cup S_1$, where $S_4 = \{(v_1, u_2), (u_2, v_3), ..., (u_{p-2}, v_{p-1}), (v_{p-1}, u_p), (u_p, v_1)\}$, an edge hub set of $S'(C_p)$ such that $|S_4| = p$ and $S'(C_p) - S_5 = C_p$, then $m(S'(C_p) - S_5) = p$. Hence,

$$|S_5| + m(S'(C_p) - S_5) = |S_4| + |S_1| + p = 3p > |S| + m(S'(C_p) - S). \tag{19}$$

Thus, from the above argumentation and (18) and (19), we obtain that $|S| + m(S'(C_p) - S)$ is minimum. Then, from equation (17) and minimality of $|S| + m(S'(C_p) - S)$,

$$\begin{aligned} EHI(S'(C_p) &= min\{|X| + m(S'(C_p) - X) : X \ is \ an \ edge \ hub \ set\} \\ &= |S| + m(S'(C_p) - S) \\ &= p + \lceil \frac{p}{i} \rceil + 2i. \end{aligned} \tag{20}$$

□

3. THE EDGE HUB-INTEGRITY OF DUPLICATION OF GRAPH ELEMENTS

Lemma 3.1. *Let $(P_p)_v$ be a graph obtained from P_p by duplication of each edge by vertex. Then $h_e(P_p)_v = p - 1, p \geq 3$.*

Theorem 3.1. *Let $(P_p)_v$ be a graph obtained from P_p by duplication of each edge by vertex. Then $EHI((P_p)_v = p + \lceil 2\sqrt{2p-1} \rceil - 2$.*

Proof. Let $(P_p)_v$ be a graph obtained by duplication of each edge (v_i, v_{i+1}) of path $P_p, for\ (1 \leq i \leq p-1)$ by vertex w_i as in Figure 2,

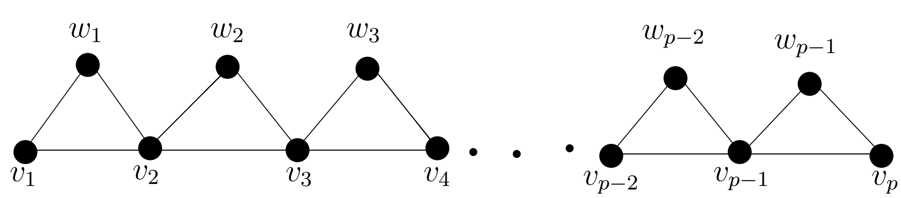

Figure 2: $(P_p)_v$

Hence, $|V((P_p)_v)| = 2p-1\ and\ |E((P_p)_v)| = 3p-3$. Consider $S_1 = \{(v_1, v_2), (v_2, v_3), ..., (v_{p-1}, v_p)\}$, an edge hub set of $(P_p)_v$, and $|S_1| = p - 1$, then $m((P_p)_v - S_1) = 2p - 1$.
Let $S_2 = \{e_k = (v_k, w_k) \cup (w_{p-1}, v_p) : 1 \leq k \leq p-1,\ and\ e_k \in I' -\ set\ of\ P_{2p-1}\}$. Take $E_1 = \{e_k : e_k \in I' -\ set\ of\ P_{2p-1}\}$, then $|S_2| = |E_1|$. Consider $S = S_1 \cup S_2$, S is also an edge hub set of $(P_p)_v$, $|S| = |S_1| + |S_2| = |S_1| + |E_1|$ and $(P_p)_v - S = P_{2p-1} - E_1$, thus

$m((P_p)_v - S) = m(P_{2p-1} - E_1)$. By Theorem 1.1,

$$(21)\quad \begin{aligned} |S| + m((P_p)_v - S) &= |S_1| + |E_1| + m(P_{2p-1} - E_1) \\ &= |S_1| + I'(P_{2p-1}) = p - 1 + \lceil 2\sqrt{2p-1} \rceil - 1 \\ &= p + \lceil 2\sqrt{2p-1} \rceil - 2. \end{aligned}$$

Now, the following cases are discussed :

Case 1. If S_3 is any edge hub set of $(P_p)_v$ which is not containing S_1 or S_2 as a proper subset and $|S_3| = k < 3p - 3$. Thus,

$$(22)\quad |S_3| + m((P_p)_v - S_3) > |S| + m((P_p)_v - S).$$

Case 2. Consider $S_4 = \{(v_1, w_1), (v_1, v_2), (v_2, w_2), (v_3, w_3), ..., (v_{p-1}, w_{p-1}), (v_{p-1}, v_p)\}$, an edge hub set of $(P_p)_v$, $|S_4| = 2p - 2$, then $m((P_p)_v - S_4) = 2$. This implies that

$$(23)\quad |S_4| + m((P_p)_v - S_4) = 2p > |S| + m((P_p)_v - S).$$

Hence, from the above discussion and (22) and (23), it lead to that $|S| + m((P_p)_v - S)$ is minimum. Hence, from equation (21) and the minimality of $|S| + m((P_p)_v - S)$,

$$(24)\quad \begin{aligned} EHI((P_p)_v) &= min\{|X| + m((P_p)_v - S) : X \ is \ an \ edge \ hub \ set)\} \\ &= |S| + m((P_p)_v - S) \\ &= p + \lceil 2\sqrt{2p-1} \rceil - 2. \end{aligned}$$

□

Lemma 3.2. *Let $(P_p)_e$ be a graph obtained from P_p by duplication of each vertex by an edge. Then $h_e((P_p)_e) = 2p - 1$.*

Theorem 3.2. *Let $(P_p)_e$ be a graph obtained from P_p by duplication of each vertex by an edge. Then $EHI((P_p)_e) = 2p + 2$.*

Proof. Let $(P_p)_e$ be a graph obtained by duplication of each vertex v_i of path P_p by an edge $(u_{2i-1}, u_{2i})(1 \le i \le p)$ as shown in Figure 3. Then the number of vertices of $(P_p)_e$ is $3p$ and $|E((P_p)_e)| = 4p - 1$.

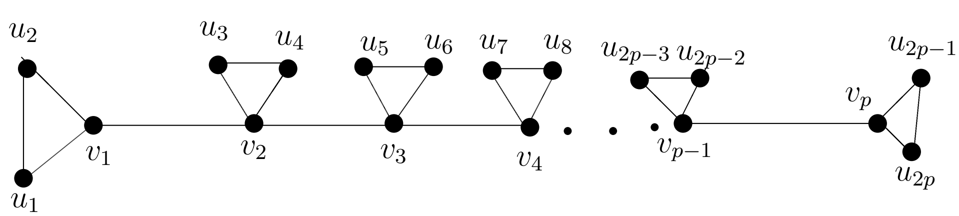

Figure 3: $(P_p)_e$

Consider $S = \{(u_1, v_1), (v_1, v_2), (v_2, u_3), (v_2, v_3), ..., (v_{p-1}, v_p), (v_p, v_{2p-1})\}$, an edge hub set of $(P_p)_e$, then $|S| = 2p - 1$, and $m((P_p)_e - S) = 3$. This implies that, $EHI((P_p)_e) \le |S| + m((P_p)_e - S) = 2p + 2$. For showing the number $|S| + m((P_p)_e - S)$ is minimum, the minimality of both $|S|$ and $m((P_p)_e - S)$ is taken into consideration. The minimality of $|S|$ is guaranteed as S is h_e-set from Lemma 3.2. It remains to show that if S_1 is any edge hub set other than S, then

$$|S_1| + m((P_p)_e - S_1) \geq 2p + 2. \tag{25}$$

In case if S_1 is any edge hub set other than S and $m((P_p)_e - S_1) = 2$, then $|S_1| \geq 2p+2$ and

$$|S_1| + m((P_p)_e - S_1) = 2p + 4 > |S| + m((P_p)_e - S), \tag{26}$$

now, if $m((P_p)_e - S_1) = 1$, then $|S_1| \geq 4p - 1 > 2p + 2$, so

$$|S_1| + m((P_p)_e - S_1) > 2p + 2, \tag{27}$$

finally, if $m((P_p)_e - S_1) \geq 3$, then trivially

$$|S_1| + m((P_p)_e - S_1) \geq 2p + 2. \tag{28}$$

Hence for any edge hub set S_1, $|S_1| + m((P_p)_e - S_1) \geq 2p + 2$. From (25), (26), (27) and (28), $EHI((P_p)_e) = 2p + 2$. □

Lemma 3.3. *Let $(C_p)_v$ be a graph obtained from C_p by duplication of each edge by vertex. Then $h_e((C_p)_v) = p - 1, p \geq 3$.*

Theorem 3.3. *Let $(C_p)_v$ be a graph obtained from C_p by duplication of each edge by vertex. Then $EHI((C_p)_v) = p - 1 + \lceil 2\sqrt{2p} \rceil, p \geq 3$.*

Proof. Let $(C_p)_v$ be a graph obtained by duplication of each edge (v_i, v_{i+1}) of a cycle $C_p, for\ (1 \leq i \leq p-1)$ by vertex u_i and edge (v_p, v_1) by vertex u_p. Then $|V((C_p)_v)| = 2p$ and $|E((C_p)_v)| = 3p$. Consider $S = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), ..., (v_{p-1}, v_p)\}$, an edge hub set of $(C_p)_v, then\ |S| = p-1$ and $(C_p)_v - S = C_{2p} + e$, where e is an edge (v_p, v_1) shown in Figure 1 in Theorem 2.4, and the proof is similar to that the proof of Theorem 2.4. Hence, $EHI((C_p)_v) = p - 1 + \lceil 2\sqrt{2p} \rceil$. □

Lemma 3.4. *Let $(C_p)_e$ be a graph obtained from C_p by duplication of each vertex by an edge. Then $h_e((C_p)_e) = 2p - 1$.*

Theorem 3.4. *Let $(C_p)_e$ be a graph obtained from C_p by duplication of each vertex by an edge. Then $EHI((C_p)_e) = 2p + 3$.*

Proof. Let $(C_p)_e$ be a graph obtained by duplication of vertices v_i of a cycle C_p by an edge $(u_{2i-1}, u_{2i}), 1 \leq i \leq p$.

Consider $S = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), ..., (v_p, v_1)\} \cup \{(v_1, u_2), (v_2, u_4), (v_3, u_6), ..., (v_p, u_{2p})\}$, an edge hub set of $(C_p)_e$, $|S| = 2p$, then $m((C_p)_e - S) = 3$. This implies that

$$EHI((C_p)_e) \leq |S| + m((C_p)_e - S) = 2p + 3. \tag{29}$$

To show that the number $|S| + m((C_p)_e - S)$ is minimum. Consider S_1 is any edge hub set other than S and $m((C_p)_e - S_1) = 2$, then $|S_1| \geq 3p \geq 2p + 3$, thus

$$|S_1| + m((C_p)_e - S_1) > 2p + 3. \tag{30}$$

If S_2 is any edge hub set other than S and S_1 with $m((C_p)_e - S_2) = 1$, then $|S_2| \geq 4p > 2p + 3$, thus

$$|S_2| + m((C_p)_e - S_2) > 2p + 3. \tag{31}$$

Finally, if $m((C_p)_e - S_4) \geq 3$, then trivially

$$|S_4| + m((C_p)_e - S_4) \geq 2p + 3. \tag{32}$$

Thus for any edge hub set S_1, $|S_1| + m((C_p)_e - S_1) \geq 2p + 3$. From (30), (31), and (32), $EHI((C_p)_e) = 2p + 3$. □

Lemma 3.5. *Let $(K_{1,p-1})_e$ be a graph obtained from star graph $K_{1,p-1}$ by duplication of each vertex by an edge. Then $h_e((K_{1,p-1})_e) = 2p - 1$.*

Theorem 3.5. *Let $(K_{1,p-1})_e$ be a graph obtained from star graph $K_{1,p-1}$ by duplication of each vertex by an edge. Then $EHI((K_{1,p-1})_e) = 2p + 2$.*

Proof. Let $(K_{1,p-1})_e$ be a graph obtained by duplication of vertices $\{v_0, v_1, ..., v_{p-1}\}$ of star graph $K_{1,p-1}$ by an edge $(u_{2i}, u_{2i+1}), 0 \leq i \leq p - 1$.
Consider $S = \{(v_0, v_1), (v_0, v_2), (v_0, v_3), (v_0, v_{p-1}), (v_1, u_2), (v_2, u_4), ..., (v_{p-1}, u_{2p})\}$, an edge hub set of $(K_{1,p-1})_e$ such that $|S| = 2p - 1$, then $m((K_{1,p-1})_e - S) = 3$, which implies that

$$EHI((K_{1,p-1})_e) \leq |S| + m((K_{1,p-1})_e - S) = 2p + 2. \tag{33}$$

For showing the number $|S| + m((K_{1,p-1})_e - S)$ is minimum, the minimality of both $|S|$ and $m((K_{1,p-1})_e - S)$ is taken into consideration. The minimality of $|S|$ is ensured as S is h_e-set from Lemma 3.5, it remains to show that if S_1 is any edge hub set other than S and $m((K_{1,p-1})_e - S_1) = 2$. Then $|S_1| \geq 3p - 1$, hence

$$|S_1| + m((K_{1,p-1})_e - S_1) > 2p + 2. \tag{34}$$

If $m((K_{1,p-1})_e - S_1) = 1$, then $|S_1| \geq 4p - 1$, hence

$$|S_1| + m((K_{1,p-1})_e - S_1) > 2p + 2. \tag{35}$$

Finally, if $m((K_{1,p-1})_e - S_1) \geq 3$, then trivially

$$|S_1| + m((K_{1,p-1})_e - S_1) \geq 2p + 2. \tag{36}$$

Hence for any edge hub set S_1, $|S_1| + m((K_{1,p-1})_e - S_1) \geq 2p + 2$. From (33), (34), (35) and (36), $EHI((K_{1,p-1})_e) = 2p + 2$. □

Lemma 3.6. *Let* $(K_{1,p-1})_v$ *be a graph obtained by duplication of each edge of star* $K_{1,p-1}$ *by vertex. Then* $h_e((K_{1,p-1})_v) = p-1$.

Theorem 3.6. *Let* $(K_{1,p-1})_v$ *be a graph obtained by duplication of each edge of star* $K_{1,p-1}$ *by vertex. Then* $EHI((K_{1,p-1})_v) = 2p$.

Proof. Let $(K_{1,p-1})_v$ be a graph obtained by duplication of each edge (v, v_i) of star $K_{1,p-1}$ by vertex $u_i, 1 \leq i \leq p-1$. The number of vertices of $(K_{1,p-1})_v$ is $2p-1$ and the number of edges is $3p-3$. Consider $S = \{(v,v_1),(v,v_2),...,(v,v_{p-1})\} \cup \{(v,u_1),(v,u_2),...,(v,u_{p-1})\}$, an edge hub set of $(K_{1,p-1})_v$ such that $|S| = 2p-2$, then $m((K_{1,p-1})_v - S) = 2$ which implies that

(37) $$EHI((K_{1,p-1})_v) \leq |S| + m((K_{1,p-1})_v - S) = 2p.$$

To show the number $|S| + m((K_{1,p-1})_v - S)$ is minimum, it is assumed that S_1 is any edge hub set other than S with $m((K_{1,p-1})_v - S_1) = 1$, then $|S_1| \geq 2p+1 > 2p$, which implies that $|S_1| + m((K_{1,p-1})_v - S_1) \geq 2p+2$. If $m((K_{1,p-1})_v - S_1) \geq 2$, thus trivially $|S_1| + m((K_{1,p-1})_v - S_1) \geq 2p$. Hence, for any edge hub set S_1,

(38) $$|S_1| + m((K_{1,p-1})_v - S_1) \geq 2p.$$

From (37) and (38), $EHI((K_{1,p-1})_v) = 2p$.

□

4. Conclusion

The results presented in this paper complement the results from [9]. Here we have investigated edge hub-integrity of splitting graph and duplication of an edge by vertex and duplication of vertex by an edge of of path, cycle and star graphs. To investigate similar results for different graph families obtained by various graph operations is an open area of research.

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