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SOME PROPERTIES OF CERTAIN SUBCLASSES OF HARMONIC UNIVALENT FUNCTIONS

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ABSTRACT. For harmonic univalent functions of the form $f(z) = z+\sum_{n=2}^{\infty} a_n z^n+\sum_{n=1}^{\infty} \bar{b}_n \bar{z}^n$, normalized with $f(0) = f'_z(0) - 1 = 0$, in the open unit disk U,the author investigated the class $H(\zeta_1, \zeta_2; \gamma)$ of $f(z)$, obtain a characterization and its closure under convolution for some complex number ζ_1, ζ_2 and some real number γ. We also give a condition under which the integral operator of the form $H(z) = \int_0^z \prod_{i=1}^{k} \left(\frac{f_i(s)}{s}\right)^{1/\alpha} ds$ is starlike.

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1. Introduction and Preliminaries

Let A denote the class of functions of the form;

(1)
$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \; (a_n \in C)$$

which are analytic in the open unit disk $U := \{z \in C : |z| < 1\}$

$$S = \{f \in A : f \text{ is univalent in } U\}$$

$$S^* = \{f \in S : \text{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in U\},$$

The class of all complex-valued, harmonic functions are represented in a simply connected domain D in the form:

$$f(z) = h(z) + \overline{g(z)}$$

where h and g are analytic in the domain.

Let H denote the family of functions f denoted by $f = h+\overline{g}$ that are harmonic, complex-valued, orientation preserving and univalent in the unit disk $U = \{z : |z| < 1\}$, with the normalization:

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=1}^{\infty} b_n z^n$$

We call h the analytic and g the co-analytic of f. Note that H reduces to the class S of normalised analytic univalent function whenever the co-analytic part of f is identically zero.

Xiao - Fei Li and An - Ping [10] denoted $L_1^*(\zeta_1, \zeta_2, \gamma)$ to be the subclass of A defined as follows;

(2) $$L_1^*(\zeta_1, \zeta_2, \gamma) = \left\{ f \in A : \left| \frac{f'(z) - 1}{\zeta_1 f'(z) + \zeta_2} \right| \leq \gamma \right\}$$

for some complex ζ_1, ζ_2 and for some real γ

Let T denote the subclass of A consisting of functions of the form:

(3) $$f(z) = z - \sum_{n=2}^{\infty} a_n z^n (a_n \geq 0)$$

Further, he denoted $L^*(\zeta_1, \zeta_2, \gamma)$ to be the subclass of $L_1^*(\zeta_1, \zeta_2, \gamma)$ defined by

$$L^*(\zeta_1, \zeta_2, \gamma) = L_1^*(\zeta_1, \zeta_2, \gamma) \cap T$$

for some real number $\zeta_1 (0 \leq \zeta_1 \leq 1)$ and $\zeta_2 (0 < \zeta_2 \leq 1)$ and for some real number $\gamma (0 < \gamma \leq 1)$. The class $L^*(\zeta_1, \zeta_2, \gamma)$ was studied by Kim and Lee [7] (see also [1,3]).
The class $G^*(\gamma)$ was introduced by Kim and Lee [7] and He noted that:
i) $L^*(0, 1, \gamma) = G^*(\gamma)$, where $G^*(\gamma)$ is the class of functions $f(z) \in T$ which satisfy $|f'(z) - 1| \leq \gamma$.
ii) $L^*(1, 1, \gamma) = D^*(\gamma)$, where $D^*(\gamma)$ is the class of functions $f(z) \in T$ which satisfy $\left| \frac{f'(z)-1}{f'(z)+1} \right| \leq \gamma$.
Jahangiri [5,6] denoted T to be the univalent subclass of A consisting of functions $f(z)$ satfying

(4) $$\left| \frac{z^2 f'(z)}{f(z)^2} - 1 \right| < 1, (z \in U)$$

Lemma 1 [4] Let M and N be analytic in U with$M(0) = N(0) = 0$. If $N(z)$ maps onto a many sheeted region which is starlike with respect to the origin and *Re*$\{\frac{M'(z)}{N'(z)}\} > 0$ in U, then *Re*$\{\frac{M(z)}{N(z)}\} > 0$ in U.
Lemma 2 [9] Let $f_i \in T_n, \mu$ $(i = 1, 2, ..., k; k \in N^*)$ be defined by

(5) $$f_i(z) = z + \sum_{n=2}^{\infty} a_n^i z^n$$

then, the integral operator

$$F_{\alpha,\beta}(z) = \{\beta \int_0^z t^{\beta-1} \prod_{i=1}^{k} (\frac{f_i(t)}{t})^{\frac{1}{\alpha}} dt\}^{\frac{1}{\beta}} \tag{6}$$

is univalent for all $i = 1, 2, ..., k; \alpha, \beta \in C$ $R\{\beta\} \geq \gamma$ and $\gamma = \sum_{i=1}^{k} \frac{1+(1+\mu_i)M}{|\alpha|}$ $(M \geq 1, 0 < \mu_i < 1, k \in N^*)$. If $|f_i(z)| \leq M(z \in U) i = 1, 2, ..., k$.
We now define A to be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n}\overline{z^n}\ (a_n, b_n) \in C \tag{7}$$

which are analytic in the open unit disk $U := \{z \in c : |z| < 1\}$.

We also denote $H(\zeta_1, \zeta_2, \gamma)$ to be the subclass of A defined as follows:

$$H(\zeta_1, \zeta_2, \gamma) = \{f \in A : \left|\frac{f'(z) - 1}{\zeta_1 f'(z) + \zeta_2}\right| \leq \gamma\} \tag{8}$$

for some complex ζ_1, ζ_2 and for some real γ.
We also define:

$$f_i(z) = z + \sum_{n=2}^{\infty} a_n^i z^n + \sum_{n=1}^{\infty} \overline{b_n^i}\overline{z}^n \tag{9}$$

for all i=1,2,...,k.
For the complex valued functions, $f_1(z)$ and $f_2(z)$ defined respectively by:

$$f_1(z) = z + \sum_{n=2}^{\infty} a_n^1 z^n + \sum_{n=1}^{\infty} \overline{b_n^1 z^n}$$

$$f_2(z) = z + \sum_{n=2}^{\infty} a_n^2 z^n + \sum_{n=1}^{\infty} \overline{b_n^2 z^n}$$

The Hadamard product of $f_1(z)$ and $f_2(z)$ is given by

$$f_1(z) * f_2(z) = (f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_n^1 a_n^2 z^n + \sum_{n=1}^{\infty} \overline{b_n^1 b_n^2 z^n} \tag{10}$$

We shall now present our main results.

2. Main Results

Theorem 2.1 Let f be in A. Then f is in the class $H(\zeta_1, \zeta_2, \gamma)$ if and only if

$$|b_1| + \sum_{n=2}^{\infty} n(1+\gamma|\zeta_1|)(|a_n| + |b_n|) \leq \gamma(|\zeta_1|(1+|b_1|) + |\zeta_2|) \tag{11}$$

for some complex $\zeta_1,\ \zeta_2$ and $\gamma (0 < \gamma \leq 1)$

Proof

From (7) and (8) we have

$$\begin{aligned}
\left|\frac{f'(z)-1}{\zeta_1 f'(z)+\zeta_2}\right| &= \left|\frac{\bar{b}_1 + \sum_{n=2}^{\infty} n(a_n z^{n-1} + \bar{b}_n \overline{z}^{n-1})}{\zeta_1(1+\bar{b}_1) + \zeta_2 + \zeta_1 \sum_{n=2}^{\infty} n(a_n z^{n-1} + \bar{b}_n \overline{z}^{n-1})}\right| \\
&\leq \frac{|\bar{b}_1| + \sum_{n=2}^{\infty} n(|a_n||z^{n-1}| + |\bar{b}_n||\overline{z}^{n-1}|)}{|\zeta_1||(1+\bar{b}_1)| + |\zeta_2| - |\zeta_1| \sum_{n=2}^{\infty} n(|a_n||z^{n-1}| + |\overline{b_n}||\overline{z}^{n-1}|)} \\
&\leq \frac{\gamma(|\zeta_1|(1+|b_1|) + |\zeta_2|)}{|\zeta_1|(1+|b_1|) + |\zeta_2| - |\zeta_1| \sum_{n=2}^{\infty} n(|a_n| + |b_n|)}
\end{aligned}$$

which shows that $f(z) \in H(\zeta_1, \zeta_2, \gamma)$.

Conversely, suppose $f(z) \in H(\zeta_1, \zeta_2, \gamma)$,

we have

$$\left|\frac{f'(z)-1}{\zeta_1 f'(z)+\zeta_2}\right| < 1$$

simplifying we obtained

$$|b_1| + \sum_{n=2}^{\infty} n(1+\gamma|\zeta_1|)(|a_n||b_n|) \leq \gamma(|\zeta_1|(1+|b_1|) + |\zeta_2|)$$

This ends the proof of the Theorem 1.

Corollary 2.1 If $f(z) \in H(\zeta_1, \zeta_2, \gamma)$, then we have

$$|a_n| + |b_n| \leq \frac{\gamma(|\zeta_1|(1+|b_1|) + |\zeta_2|) - |b_1|}{n(1+\gamma|\zeta_1|)}\ (n = 2, 3, \ldots)$$

Theorem 2.2 Let the function $f_1(z)$ belong to the class $H(\beta_1, \beta_2, \gamma)$ and $f_2(z)$ belong to the class $H(\beta_1, \beta_2, \gamma)$. Then $f_1(z) * f_2(z) = (f_1 * f_2)(z)$ belong to the class $H(\beta_1, \beta_2, \lambda)$ if

$$\frac{\gamma^2 \sum_{n=2}^{\infty} n[|\beta_1|(1+b_1) + \beta_1 - |b_1|]}{(\sum_{n=2}^{\infty} n(1+\gamma|\beta_1|))^2 - \gamma^2 \sum_{n=2}^{\infty} n\beta 1[|\beta_1|(1+b_1) + \beta_2 - |b_1|]} \leq \lambda$$

Proof

$f_1(z)$ belong to the class $H(\beta_1, \beta_2, \gamma) \Rightarrow \frac{\sum_{n=2}^{\infty} n(1+\gamma|\beta_1)(|a_n^1| + |b_n^1|)}{\gamma|\beta_1|(1+|b_1^1|) + |\beta_2| - |b_1^1|} < 1$

and

$f_2(z)$ belong to the class $H(\beta_1, \beta_2, \gamma) \Rightarrow \frac{\sum_{n=2}^{\infty} n(1+\gamma|\beta_1)(|a_n^2|+|b_n^2|)}{\gamma|\beta_1|(1+|b_1^1|)+|\beta_2|_-|b_1^1|} < 1$
Thus, we need to find the smallest λ such that

$$\frac{\sum_{n=2}^{\infty} n(1+\lambda|\beta_1)(|a_n^1|+|b_n^1|)(|a_n^2|+|b_n^2|)}{\lambda|\beta_1|(1+|b_1^1|)+|\beta_2|_-|b_1^1|} < 1$$

By Cauchy-Schwartz inequality we have,

$$\frac{\sum_{n=2}^{\infty} n(1+\gamma|\beta_1)\sqrt{(|a_n^1|+|b_n^1|)(|a_n^2|+|b_n^2|)}}{\gamma|\beta_1|(1+|b_1^1|)+|\beta_2|_-|b_1^1|} < 1$$

Therefore, it suffices to show that

$$\frac{\sum_{n=2}^{\infty} n(1+\lambda|\beta_1)(|a_n^1|+|b_n^1|)(|a_n^2|+|b_n^2|)}{\lambda|\beta_1|(1+|b_1^1|)+|\beta_2|_-|b_1^1|} \leq \frac{\sum_{n=2}^{\infty} n(1+\gamma|\beta_1)\sqrt{(|a_n^1|+|b_n^1|)(|a_n^2|+|b_n^2|)}}{\gamma|\beta_1|(1+|b_1^1|)+|\beta_2|_-|b_1^1|}$$

Thus

$$\begin{aligned}
\frac{(|a_n^1|+|b_n^1|)(|a_n^2|+|b_n^2|)}{\sqrt{(|a_n^1|+|b_n^1|)(|a_n^2|+|b_n^2|)}} &\leq \frac{\sum_{n=2}^{\infty} n(1+\gamma|\beta_1)\lambda(|\beta_1|(1+|b_1^1|)+|\beta_2|_-|b_1^1|)}{\sum_{n=2}^{\infty} n(1+\lambda|\beta_1)\gamma(|\beta_1|(1+|b_1^1|)+|\beta_2|_-|b_1^1|)} \\
\sqrt{(|a_n^1|+|b_n^1|)(|a_n^2|+|b_n^2|)} &= \frac{\lambda\sum_{n=2}^{\infty} n(1+\gamma|\beta_1)}{\gamma\sum_{n=2}^{\infty} n(1+\lambda|\beta_1)}
\end{aligned}$$

But

$$\sqrt{(|a_n^1|+|b_n^1|)(|a_n^2|+|b_n^2|)} \leq \frac{\gamma(|\beta_1|(1+|b_1^1|)+|\beta_2|_-|b_1^1|)}{\sum_{n=2}^{\infty} n(1+\gamma|\beta_1)}$$

$$\Rightarrow \frac{\gamma(|\beta_1|(1+|b_1^1|)+|\beta_2|_-|b_1^1|)}{\sum_{n=2}^{\infty} n(1+\gamma|\beta_1)} \leq \frac{\lambda\sum_{n=2}^{\infty} n(1+\gamma|\beta_1)}{\gamma\sum_{n=2}^{\infty} n(1+\lambda|\beta_1)}$$

$\Rightarrow \gamma^2 \sum_{n=2}^{\infty} n(1+\lambda|\beta_1|)\left[|\beta_1|(1+|b_1|)+\beta_2-|b_1|\right] \leq \lambda\left(\sum_{n=2}^{\infty} n(1+\gamma|\beta_1|)\right)^2$
$\Rightarrow \frac{\gamma^2\sum_{n=2}^{\infty} n[|\beta_1|(1+|b_1|)+\beta_1-|b_1|]}{\left(\sum_{n=2}^{\infty} n(1+\gamma|\beta_1|)\right)^2-\gamma^2\sum_{n=2}^{\infty} n|\beta_1[|\beta|(1|b_1|)+\beta_2-|b_1|]} \leq \lambda$

Which shows that $f_1(z) * f_2(z) = (f_1 * f_2)(z)$ belong to the class $H(\beta_1, \beta_2, \lambda)$.
This completes the proof of the theorem.
Theorem 2.3 Let $f \in A$, then the integral

$$H(z) = \int_0^z \prod_{i=1}^{k} \left(\frac{f_i(s)}{s}\right)^{\frac{1}{\alpha}} ds, \alpha \in C, |\alpha| > 1. \tag{13}$$

is starlike in U if

$$\sum_{n=2}^{\infty}(n-1)(|a_n^i|+|b_n^i|) < 1$$

Proof
By differetiating (12), we obtain:

$$H'(z) = \prod_{i=1}^{k}\left(\frac{f_i(z)}{z}\right)^{\frac{1}{\alpha}}$$

Let

$$M = zH'(z), N(z) = H(z) \tag{14}$$

From (12) and (13), we have:

$$\begin{aligned}\frac{M'(z)}{N'(z)} &= 1 + \frac{zH''(z)}{H'(z)} \\ &= 1 + \sum_{i=1}^{k} \frac{1}{\alpha}\Big(\frac{zf_i'(z)}{f_i(z)} - 1\Big)\end{aligned}$$

$$\begin{aligned}\left|\frac{M'(z)}{N'(z)} - 1\right| &= \left|\frac{1}{\alpha}\sum_{i=1}^{k} \frac{zf_i'(z) - f_i(z)}{f_i(z)}\right|, |z| \to 1 \\ &\leq \frac{1}{|\alpha|}\sum_{n=1}^{k} \frac{\sum_{n=2}^{\infty}(n-1)(|a_n^i| + |b_n^i|)}{1 + |b_1^i| + \sum_{n=2}^{\infty}|b_n^i|} < 1\end{aligned}$$

Thus, Re $\frac{M'(z)}{N'(z)} > 0$ and by lemma 1, we have Re $\frac{M(z)}{N(z)} > 0 =$ Re $\frac{zH'(z)}{H(z)} > 0$.
Hence, the integral operator $H(z)$ is starlike.

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