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ON COINCIDENCE AND COMMON FIXED POINT FOR (ϕ,ψ)-WEAKLY CONTRACTIVE MAPPINGS VIA C. CLASS FUNCTIONS IN G-METRIC SPACES

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ABSTRACT. In this paper, we establish a common fixed point theorem of weakly compatible mappings under (ϕ,ψ)-weakly contractive condition involving C-class functions in G-metric spaces. Our results improve and extend the results in [10, 14]. We also provide an example to support our results.

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1. Introduction

In 2006, Mustafa and Sims [11] , introduced the new concept of generalized metric spaces, named as G-metric spaces. After that many researcher extend the known contraction in G-metric space one of these is $(\varphi-\psi)$-weak contraction see [12, 13, 14]. The definitions and motivations are found in [11]:

Definition 1.1 Let X be a non empty set and let $G : X \times X \times X \to R^+$be a function satisfying
(G_1) $G(x, y, z) = 0$ if $x = y = z$,
(G_2) $0 < G(x, x, y)$ $\forall x, y \in X$ with $x \neq y$,
(G_3) $G(x, x, y) \leq G(x, y, z)$ $\forall x, y, z \in X$ with $z \neq y$,
(G_4) $G(x, y, y) = G(x, z, y) = G(y, z, x)$ (symmetry in all three variables),
(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ $\forall a, x, y, z \in X$ (rectangle inequality).
Then the function G is called a generalized metric or more specifically a $G-$metric on X and the pair is (X, G) called $G-$metric spaces.

Definition 1.2 Let (X, G) be a $G-$metric space and let $\{x_n\}$ be a sequence in X, a point x in X is said to be the limit of the sequence $\{x_n\}$ if $\lim_{n\to\infty} G(x, x_n, x_n) = 0$ and one says the sequence $\{x_n\}$ is $G-$convergent to x.

Proposition 1.3 Let (X, G) be G-metric space, then for a sequence $\{x_n\} \subseteq X$ and point $x \in X$, the following are equivalent:
(i) $\{x_n\}$ is G-convergent to x,
(ii) $G(x_n, x_n, x) \to 0$ as $n \to \infty$,
(iii) $G(x_n, x, x) \to 0$ as $n \to \infty$,
(iv) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.

Proposition 1.4 In a G-metric space (X, G), the following are equivalent:
(i) the sequence $\{x_n\}$ is a G-Cauchy sequence.
(ii) For every $\epsilon > 0$, $\exists n \in N$, $G(x_n, x_m, x_m) < \epsilon$ $\forall n, m \in N$.

Definition 1.5 A $G-$metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Proposition 1.6 Let (X, G) be a $G-$metric space. Then for all $x, y, z, a \in X$, we have
(i) if $G(x, y, z) = 0$ then $x = y = z$;
(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
(iii) $G(x, y, y) \leq 2G(y, x, x)$,
(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,
(v) $G(x, y, z) \leq \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z))$,
(vi) $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$.

In 1922 Banach [5] established a theorem known as Banach contraction. Banach contraction principle states "A contraction mapping in a complete metric space has a unique fixed point". After that many researchers generalized this principle in many directions using different contractive type conditions. Alber and Gurre [3] gave the concept of weak contraction and studied the existence of fixed points for self maps in Hilbert spaces. The concept of weak contraction extended by Rhoades [15] to metric spaces and defined $\phi-$weak contraction as following:
A self map T on metric space (X, d) is said to be $\phi-$weak contraction if there exists a map $\phi : R^+ \to R^+$with $\Phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$ such that

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)) \ \forall x, y \in X,$$

and proved the following theorem

Theorem 1.7 Weak contractive self map in a complete metric space has a unique fixed point.

Dutta and Choudhury [6] generalized the concept of weak contractive as a $(\varphi - \psi)$ weak contractive and established the following result

Theorem 1.8 Let T be a self-map on a complete metric space (X, d) satisfying the following inequality

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)), \ \forall x, y \in X,$$

where $\varphi, \psi : R^{+} \rightarrow R^{+}$ is monotonic non-decreasing and continuous function such that $\psi(0) = 0 = \varphi(0), \psi(t) > 0$ and $\varphi(t) > 0$ for $t > 0$. Then T has a unique fixed point.

Definition 1.9 [7] Let S and T be self mappings on a non empty set X.

1-A point $x \in X$ is said to be a fixed point of T if $Tx = x$,

2-A point $x \in X$ is said to be a coincidence point of S and T if $Sx = Tx$ and we shall called $\omega = Sx = Tx$ that a point of coincidence of S and T,

3-A point $x \in X$ is said to be a common fixed point of S and T if $x = Sx = Tx$.

Definition 1.10 [10] Let X be a non-empty set and $T, f : X \rightarrow X$. The mapping T and f are said to be weakly compatible if they commute at their coincidence point (i.e. $Tfx = fTx$ whenever $Tx = fx$).

Definition 1.11 [9] The following two classes of mappings are defined as:

$\Psi = \{\psi : \psi : [0, \infty) \rightarrow [0, \infty)$ is continuous and nondecreasing with $\psi(t) = 0 \Longleftrightarrow t = 0\}$

$\Phi = \{\phi : \phi : [0, \infty) \rightarrow [0, \infty)$ is nondecreasing, $\phi(t) = 0 \Longleftrightarrow t = 0\}$

Aage and Salunke [1] proved the following result for weak contraction in $G-$metric spaces.

Theorem 1.12 Let (X, G) be a complete $G-$metric space and let $T : X \rightarrow X$ be mapping satisfying

$$G(Tx, Ty, Tz) \leq G(x, y, z) - \phi(G(x, y, z))$$

for all $x, y, z \in X$. If $\phi : [0, \infty) \rightarrow [0, \infty)$ is continuous and non-decreasing with $\phi^{-1}(0) = 0$, $\phi(t) > 0$ for all $t \in [0, \infty)$. Then T has a unique fixed point in X.

E. S. Eke [8] generalized Theorem 1.12 in the following

Theorem 1.13 Let (X, G) be $G-$metric space and Y a nonempty subset of X. Let $T, S :$

$Y \to X$ be mapping satisfying

$$G(Tx,Ty,Tz) \leq G(Sx,Sy,Sz) - \phi(G(Sx,Sy,Sz)),$$

for all $x,y,z \in X$. If $\phi : [0,\infty) \to [0,\infty)$ is continuous and nonincreasing with $\phi^{-1}(0) = 0$, $\phi(t) > 0$ for all $t \in (0,\infty)$. Suppose that S and T are weakly compatible with $T(Y) \subseteq S(Y)$. If $S(Y)$ or $T(Y)$ is a complete subspace of X, then the mappings S and T have a unique common fixed point in X.
For $(\phi - \psi)$ weak contraction, Mohanta [13], proved the following theorem for two mappings in G-metric spaces

Theorem 1.14 Let (X,G) be $G-$metric space and ϕ, ψ be altering distance functions. Let the mappings $T, f : X \to X$ satisfying

$$\psi(G(Tx,Ty,Tz)) \leq \psi(G(fx,fy,fz)) - \phi(G(fx,fy,fz)),$$

for all $x,y,z \in X$, where $T(X) \subseteq f(X)$ is a complete subspace of X. Then f and T have a unique point of coincidence. Moreover if F and T are weakly compatible. Then f and T have a unique common fixed point in X.
Recently, Khandagji et al. [10] proved the following theorem:

Theorem 1.15 Let (X,G) be $G-$metric space. Let $f : X \to X$ be a self mapping on X satisfying the following condition

$$\Psi(G(fx,fy,fz)) \leq \psi(M(x,y,z)) - \phi(M(x,y,z)),$$

where

$$M(x,y,z) = \max \left\{ \begin{array}{c} G(x,y,z), G(x,fx,fx), G(y,fy,fy), G(z,fz,fz), \\ \alpha G(x,fx,y) + (1-\alpha)G(fy,fy,z), \\ \beta G(x,fx,fx) + (1-\beta)G(y,fy,fy) \end{array} \right\},$$

for all $x,y,z \in X$, where $0 < \alpha < \beta < 1$, ψ is an altering distance function and $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0$ iff $t = 0$. Then f has a unique fixed point.
Very recently, Rashwan and Saleh [14], removed the completeness of the space in Theorem 1.14 and proved the following theorem for two weakly compatible mappings in $G-$metric space

Theorem 1.16 Let (X,G) be $G-$metric space. Let $f, g : X \to X$ be a self mapping on X satisfying the following conditions

$$\psi(G(fx,fy,fz)) \leq \psi(N(x,y,z)) - \phi(N(x,y,z)),$$

where

$$N(x,y,z) = \max\left\{\begin{array}{c} G(gx,gy,gz), G(gx,fx,fx), G(gy,fy,fy), G(gz,fz,fz), \\ \alpha G(fx,fx,gy) + (1-\alpha)G(fy,fy,gz), \\ \beta G(gx,fx,fx) + (1-\beta)G(gy,fy,fy) \end{array}\right\},$$

for all $x,y,z \in X$, where $0 < \alpha < \beta < 1$ and $\phi,\psi : [0,\infty) \to [0,\infty)$ with $\phi,\psi \in \Psi$.
If:
(i) $f(x) \subseteq g(x)$
(ii) $f(x)$ or $g(x)$ is a complete metric subspace of X.
Then f and g have a unique point of coincidence. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.
Ansari [4] defined the concept of C-class functions and presented new fixed point results which improve and extend several results in the literature.

Definition 1.17 A mapping $F : [0,\infty)^2 \to \mathbb{R}$ is called C-*class* function if it is continuous and satisfies following axioms:
$(F_1) F(s,t) \leq s$,
(F_2) $F(s,t) = s$ implies that either $s = 0$ or $t = 0$ for all $s,t \in [0,\infty)$.
Note that $F(0,0) = 0$. We denote C-class functions as $\mathcal{C}$.

Example 1.18 [4] The following functions $F : [0,\infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$, for all $s,t \in [0,\infty)$
(1) $F(s,t) = s - t$, $F(s,t) = s \Rightarrow t = 0$
(2) $F(s,t) = ks$, $0<k<1$, if $F(s,t) = s \Rightarrow s = 0$
(3) $F(s,t) = \frac{s}{(1+t)^r}$, $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$ or $t = 0$
(4) $F(s,t) = (s+l)^{\log_{t+a}(a)} - l$, $l > 1$, $a > 0$, $F(s,t) = s \Rightarrow t = 0$
(5) $F(s,t) = \sqrt[n]{\log(1+s^n)}$, $n \in N$, $F(s,t) = s \Rightarrow s = 0$
(6) $F(s,t) = (s+l)^{(\frac{1}{1+t})} - l$, $l > 1$, $r \in (0,\infty)$, $F(s,t) = s \Rightarrow t = 0$
(7) $F(s,t) = s\log_{t+a} a$, $a > 1$, $F(s,t) = s \Rightarrow s = 0$ or $t = 0$
(8) $F(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$, $F(s,t) = s \Rightarrow t = 0$
(9) $F(s,t) = s\beta(s)$, $\beta : [0,\infty) \to [0,1)$, $F(s,t) = s \Rightarrow s = 0$
(10) $F(s,t) = s - \frac{t}{k+t}$, $F(s,t) = s \Rightarrow t = 0$
(11) $F(s,t) = s - \varphi(s), F(s,t) = s \Rightarrow s = 0$, here $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$

(12) $F(s,t) = sh(s,t)$, $F(s,t) = s \Rightarrow s = 0$, here $h : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function such that $h(t,s) < 1$ for all t, s
(13) $F(s,t) = s - (\frac{2+t}{1+t})t$, $F(s,t) = s \Rightarrow t = 0$

In this paper, we establish a coincidence and a common fixed point theorems of weakly compatible mappings under (ϕ, ψ)-weakly contractive condition involving C-class functions in G-metric spaces. Our results improve and extend the main results in [10, 14].

2. Main Results

The following lemmas are fundamental in the sequel.

Lemma 2.1 [2] let f and g be weakly compatible mappings on a set X. If f and g have a unique point of coincidence $\omega = fx = gx$, $x \in X$, then ω is the unique common fixed point of f and g.

Lemma 2.2 Let (X, G) be a $G-$metric space and $f, g : (G, X) \to (G, X)$ be two mappings such that

$$\psi(G(fx, fy, fz)) \leq F(\psi(N(x,y,z)), \phi(N(x,y,z))), \tag{2.1}$$

where

$$N(x,y,z) = \max \left\{ \begin{array}{c} G(gx,gy,gz), G(gx,fx,fx), G(gy,fy,fy), G(gz,fz,fz), \\ \alpha G(fx,fx,gy) + (1-\alpha)G(fy,fy,gz), \\ \beta G(gx,fx,fx) + (1-\beta)G(gy,fy,fy) \end{array} \right\},$$

for all $x, y, z \in X$, $0 < \alpha < \beta < 1$ and $\phi \in \Phi$, $\psi \in \Psi$. Then f, g have at most a point of coincidence.

Proof. Suppose that $u = fp = gp$ and $v = fq = gq$. Then by (2.1), we obtain

$$\psi(G(fp, fp, fq)) \leq F(\psi(N(p,p,q)), \phi(N(p,p,q))),$$

where

$$N(p,p,q) = \max \left\{ \begin{array}{c} G(gp,gp,gq), G(gp,fp,fp), G(gp,fp,fp), G(gq,fq,fq), \\ \alpha G(fp,fp,gp) + (1-\alpha)G(fp,fp,gq), \\ \beta G(gp,fp,fp) + (1-\beta)G(gp,fp,fp) \end{array} \right\}.$$

This yields

$$N(p,p,q) = \max \left\{ G(u,u,v), 0, 0, 0, (1-\alpha)G(u,u,v), 0 \right\}.$$

Then

$$\begin{aligned}\psi(G(u,u,v)) &\leq F(\psi(G(u,u,v)) - \phi(G(u,u,v))) \\ &< \psi(G(u,u,v)),\end{aligned}$$

since ψ is nondecreasing, we obtain $\psi(G(u,u,v)) = 0$. Hence $u = v$.

The following lemma holds in every G-metric space, which similar to be used in prove of several fixed point theorems in the metric space setting.

Lemma 2.3 Let (X,G) be a $G-$metric space and let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$. If $\{x_n\}$ is not a Cauchy sequence in (X,G), then there exists $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tends to ϵ when $k \to \infty$

(2.2) $\quad G(x_{m_k}, x_{n_k}, x_{n_k}), G(x_{m_k}, x_{n_k-1}, x_{n_k-1}), G(x_{m_k+1}, x_{n_k}, x_{n_k}), G(x_{n_k-1}, x_{m_k+1}, x_{m_k+1}).$

Now, we are ready to state and prove our main theorem.

Theorem 2.4 Let (X,G) be a $G-$metric space and $f, g : (G,X) \to (G,X)$ be two mappings satisfying inequality (2.1). If

(i) $f(x) \subseteq g(x)$

(ii) $f(x)$ or $g(x)$ is a complete G-metric subspace of X.

Then f and g have a unique point of coincidence. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Let $x_\circ$ be an arbitrary point of X, since $f(x) \subseteq g(x)$, we choose $x_1 \in X$ such that $y_\circ = fx_\circ = gx_1$. Continuing this process, we get

$$y_n = fx_n = gx_{n+1}, \text{ for all } n = 0, 1, 2, ...$$

If $y_n = y_{n+1}$ for some n, then $y_{n+1} = fx_{n+1} = gx_{n+1}$.

This yields f and g have a coincidence point.

We may assume that $y_n \neq y_{n+1}$ for each n.

Now we will prove the following:

(1) $G(y_n, y_{n+1}, y_{n+1}) \to 0$ as $n \to \infty$,

(2) $\{y_n\}$ is a G-Cauchy sequence.

Indeed by (2.1), we have

(2.3)

$$\psi(G(y_n, y_{n+1}, y_{n+1})) = \psi(G(fx_n, fx_{n+1}, fx_{n+1})) \leq F(\psi(N(x_n, x_{n+1}, x_{n+1})), \phi(N(x_n, x_{n+1}, x_{n+1}))),$$

where,

$$N(x_n,x_{n+1},x_{n+1}) = \max\left\{\begin{array}{c} G(gx_n,gx_{n+1},gx_{n+1}),G(gx_n,fx_n,fx_n),G(gx_{n+1},fx_{n+1},fx_{n+1}),\\ G(gx_{n+1},fx_{n+1},fx_{n+1}),\\ \alpha G(fx_n,fx_n,gx_{n+1})+(1-\alpha)G(fx_{n+1},fx_{n+1},gx_{n+1}),\\ \beta G(gx_n,fx_n,fx_n)+(1-\beta)G(gx_{n+1},fx_{n+1},fx_{n+1}) \end{array}\right\}.$$

This implies that

$$\begin{aligned} N(x_n,x_{n+1},x_{n+1}) &= \max\left\{\begin{array}{c} G(y_{n-1},y_n,y_n),G(y_{n-1},y_n,y_n),G(y_n,y_{n+1},y_{n+1}),G(y_n,y_{n+1},y_{n+1}),\\ \alpha G(y_n,y_n,y_n)+(1-\alpha)G(y_{n+1},y_{n+1},y_n),\\ \beta G(y_{n-1},y_n,y_n)+(1-\beta)G(y_n,y_{n+1},y_{n+1}) \end{array}\right\}\\ &= \max\left\{G(y_{n-1},y_n,y_n),G(y_n,y_{n+1},y_{n+1})\right\}. \end{aligned}$$

If $N(x_n,x_{n+1},x_{n+1}) = G(y_n,y_{n+1},y_{n+1})$ for some $n \in N$, then from (2.3), we obtain

$$\begin{aligned} \psi(G(y_n,y_{n+1},y_{n+1})) &\le F(\psi(G(y_n,y_{n+1},y_{n+1})),\phi(G(y_n,y_{n+1},y_{n+1})))\\ &< \psi(G(y_n,y_{n+1},y_{n+1}), \end{aligned}$$

and since the function ψ is nondecreasing, it follows that

$$G(y_n,y_{n+1},y_{n+1}) \le G(y_{n-1},y_n,y_n),$$

that is, there exists

$$\lim_{n\to\infty} G(y_n,y_{n+1},y_{n+1}) = \delta \ge 0.$$

If $\delta > 0$, we get $\psi(\delta) \le \psi(\delta) - \phi(\delta)$, that is $\delta = 0$ which is a contradiction. Hence, we obtain that

$$\lim_{n\to\infty} G(y_n,y_{n+1},y_{n+1}) = 0.$$

Further using Lemma 2.3, we shall prove that $\{y_n\}$ is a G-Cauchy sequence. Suppose this is not true. Then by Lemma 2.3 there exists $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following sequences tends to ϵ when $k \to \infty$

$$G(x_{m_k},x_{n_k},x_{n_k}),G(x_{m_k},x_{n_k-1},x_{n_k-1}),G(x_{m_k+1},x_{n_k},x_{n_k}),G(x_{n_k-1},x_{m_k+1},x_{m_k+1}).$$

Putting $x = x_{m(k)}$, $y = x_{m(k)}$ and $z = x_{n(k)}$ in (2.1) we conclude that

$$\begin{aligned} \psi(G(y_{m(k)},y_{m(k)},y_{n(k)})) &= \psi(G(fx_{m(k)},fx_{m(k)},fx_{m(k)+1}))\\ &\le F(\psi(N(x_{m(k)},x_{m(k)},x_{n(k)})),\phi(N(x_{m(k)},x_{m(k)},x_{n(k)}))). \end{aligned}$$

Since

$$\lim_{k\to\infty} G(x_{m(k)},x_{m(k)},x_{n(k)}) = \epsilon, \quad \lim_{k\to\infty} G(y_{n(k)},y_{m(k)},y_{n(k)}) = \epsilon.$$

Therefore we find that

$$\psi(\epsilon) \le F(\psi(N(x_{m(k)},x_{m(k)},x_{n(k)})),\phi(N(x_{m(k)},x_{m(k)},x_{n(k)}))). \tag{2.4}$$

Letting $k \to \infty$, we get $\psi(\epsilon) \leq F(\psi(\epsilon), \phi(\epsilon))$, that is $\phi(\epsilon) = 0$, $\phi \in \Phi$, we get $\epsilon = 0$, which is a contradiction, we have proved that $\{y_n\}$ is a Cauchy sequence in (X, G).
Suppose that $g(X)$ is a G-complete subspace of X, so that there exists a point $q \in g(X)$ such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1} = q.$$

Also, we prove that $fp = q$, by (2.1), we have

$$\psi(G(fx_n, fp, fp)) \leq F(\psi(N(x_n, p, p)), \phi(N(x_n, p, p))), \tag{2.5}$$

where

$$\lim_{n\to\infty} N(x_n, p, p) = G(q, fp, fp). \tag{2.6}$$

Letting $n \to \infty$ and using (2.6), one gets

$$\psi(G(q, fp, fp)) \leq F(\psi(G(q, fp, fp)), \phi(G(q, fp, fp))), \tag{2.7}$$

hence $\psi(G(q, fp, fp)) = 0$ or $\phi(G(q, fp, fp)) = 0$ and then $fp = q$. Then q is a point of coincidence of f and g. Moreover if f and g are weakly compatible then from Lemma 2.1, q is the unique common fixed point of f and g. The proof is similar when we assume that $f(X)$ is complete since $f(X) \subseteq g(X)$.

If we take $g = I_x$ (where I_x is the identity mapping) in Theorem 2.4 we have the following corollary.

Corollary 2.1 Let (X, G) be a $G-$complete metric space and $f : (G, X) \to (G, X)$ be a mapping such that

$$\psi(G(fx, fy, fz)) \leq F(\psi(M(x, y, z)), \phi(M(x, y, z))),$$

where

$$M(x, y, z) = \max \left\{ \begin{array}{c} G(x, y, z), G(x, fx, fx), G(y, fy, fy), G(z, fz, fz), \\ \alpha G(fx, fx, y) + (1-\alpha) G(fy, fy, z), \\ \beta G(x, fx, fx) + (1-\beta) G(y, fy, fy) \end{array} \right\},$$

for all $x, y, z \in X$, $0 < \alpha < \beta < 1$ and $\phi \in \Phi$, $\psi \in \Psi$, $F \in \mathcal{C}$. Then f has a unique fixed point in X.

If we take $F(s, t) = ks$, $k \in (0, 1)$ in Theorem 2.4 we have the following corollary.

Corollary 2.2 Let (X, G) be a $G-$metric space and $f, g : (G, X) \to (G, X)$ be a self mappings such that $f(x) \subseteq g(x)$ and $f(x)$ or $g(x)$ is complete subspace of X and the following condition holds

$$\psi(G(fx, fy, fz)) \leq k\psi(N(x, y, z)),$$

for all $x, y, z \in X$. Then f and g have a unique point of coincidence. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

If we take $F(s, t) = s - t$ in Theorem 2.4 we have the following corollary.

Corollary 2.3 Let (X, G) be a $G-$metric space and $f, g : X \to X$ be a self mappings such that $f(x) \subseteq g(x)$ and $f(x)$ or $g(x)$ is complete subspace of X and the following condition holds

$$\psi(G(fx, fy, fz)) \leq \psi(N(x, y, z)) - \phi(N(x, y, z)),$$

for all $x, y, z \in X$, where $0 < \alpha < \beta < 1$ and $\phi \in \Phi$, $\psi \in \Psi$. Then f and g have a unique point of coincidence. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

If we take $F(s, t) = \frac{s}{1+t}$ in Theorem 2.4 we have the following corollary.

Corollary 2.4 Let (X, G) be a $G-$metric space and $f, g : X \to X$ be a self mappings such that $f(x) \subseteq g(x)$ and $f(x)$ or $g(x)$ is complete subspace of X and the following condition holds

$$\psi(G(fx, fy, fz)) \leq \frac{\psi(N(x, y, z))}{1 + \psi(N(x, y, z))},$$

for all $x, y, z \in X$, where $0 < \alpha < \beta$ and $\phi \in \Phi$, $\psi \in \Psi$. Then f and g have a unique point of coincidence. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Now, we give an example to verify the conditions of Theorem 2.4.

Example Let $X = \mathbb{R}$ with G-metric $G(x, y, z) = |x - y| + |y - z| + |z - x|$, $x, y, z \in \mathbb{R}$ and f, g are mappings on X defined by $f(x) = 2$ and $g(x) = 2x - 2$.
If we take $\psi(t) = t$, $\phi(t) = 1$ for $t \in [0, \infty)$ and $\alpha, \beta \in (0, 1]$, $F(s, t) = \frac{s}{1+t}$, $s, t \in [0, \infty)$.
So we have

$$F(s, t) = \frac{s}{1 + t} = \frac{\psi(N(x, y, z))}{1 + \psi(N(x, y, z))} = \frac{1}{2}N(x, y, z).$$

It is clear that $f(x) \subseteq g(x)$ and (f,g) is commuting and hence weakly compatible.
Also,

$$0 = \psi(G(fx, fy, fz)) \leq \frac{1}{2}N(x, y, z),\ \ x, y, z \in X.$$

Therefore condition (2.1) holds for all $x, y, z \in X$ and the hypothesis of Theorem 4.2 are satisfied and 2 is a unique common fixed point of f and g.

3. Conclusions

(i) Our results improve and extend Theorems 1.15 and 1.16 by using new (ϕ, ψ)-weakly contraction mappings involving C-class function in the G-metric space setting
(ii) In (ϕ, ψ)-weakly contraction mappings, the continuity of a function ϕ need not be continuous.

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