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SOME OSCILLATION RESULTS FOR SECOND ORDER DIFFERENCE EQUATION WITH A SUBLINEAR NEUTRAL TERM

B.KAMARAJ*, AND R.VASUKI

Department of Mathematics,
SIVET College, Chennai-600 073, India
*Corresponding author: kamaraj1928@yahoo.co.in
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Abstract. This paper deals with oscillation of second order difference equation with a sublinear neutral term. Some new oscillation theorems for such equations are obtained and examples are inserted to illustrate the results.
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1. Introduction

Consider a second order difference equation with sublinear neutral term of the form

$$\Delta(a_n\Delta(x_n + p_n x^{\alpha}_{\tau(n)})) + q_n x^{\beta}_{\sigma(n+1)} = 0,\ n \geq n_0 \tag{1.1}$$

where n_0 is a nonnegative integer, subject to the following conditions:

(H_1) $0 < \alpha \leq 1$ and β are ratios of odd positive integers;
(H_2) $\{a_n\}$, $\{p_n\}$ and $\{q_n\}$ are positive real sequences with $p_n \geq 1$ for all $n \geq n_0$;
(H_3) $\{\sigma(n)\}$ and $\{\tau(n)\}$ are increasing sequence of integers for all $n \geq n_0$ with $\lim_{n\to\infty} \sigma(n) = \lim_{n\to\infty} \tau(n) = \infty$.

By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ defined and satisfies equation (1.1) for all $n \geq n_0$. We consider only those solutions $\{x_n\}$ of equation (1.1) which satisfy $\sup\{|x_n| : n \geq N\} > 0$ for all $N \geq n_0$. A solution of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative and otherwise it is called nonoscillatory.

In recent years, there has been a great interest in investigating the oscillatory and asymptotic behavior of neutral type difference equations, see [1–3, 6, 8] and the references cited therein.

In [4, 5, 7, 9, 10], the authors studied the oscillatory behavior of solutions of equation (1.1) when $\alpha \geq 1$ or $0 < \alpha < 1$ and either

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty, \tag{1.2}$$

or

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} < \infty. \tag{1.3}$$

However in [5, 7, 9, 10] one requires that $p_n \to 0$ as $n \to \infty$ which is quite restrictive and now the problem is how to derive new oscillation criteria for equation (1.1) without such condition. Motivated by this observation, in this paper we obtain some new oscillation criteria for equation (1.1). In Section 2, we present oscillation results for the equation (1.1) and in Section 3, we provide some examples to illustrate the main results.

2. Oscillation Results

In this section, we established some new oscillation results for equation (1.1). Due to the form of the equation, we only need to give proofs for the case of eventually positive solutions since the proofs for the eventually negative solutions would be similar. Define

$$\begin{aligned}
R_n &= \sum_{s=n_0}^{n-1} \frac{1}{a_s}, \quad A_n = \sum_{s=n}^{\infty} \frac{1}{a_s}, \\
B_n &= \frac{1}{p_{\tau^{-1}(n)}} \left(1 - \frac{M^{\frac{1-\alpha}{\alpha}}}{p^{1/\alpha}_{\tau^{-1}(\tau^{-1}(n))}} R^{\frac{1-\alpha}{\alpha}}_{\tau^{-1}(n)} \right) > 0 \text{ for all constants } M > 0, \\
C_n &= \frac{1}{p_{\tau^{-1}(n)}} \left(1 - \frac{K^{\frac{1-\alpha}{\alpha}}}{p^{1/\alpha}_{\tau^{-1}(\tau^{-1}(n))}} \frac{A^{1/\alpha}_{\tau^{-1}(\tau^{-1}(n))}}{A^{\frac{1}{\alpha}}_{\tau^{-1}(n)}} \right) > 0 \text{ for all constants } K > 0,
\end{aligned}$$

$$D_n = \frac{1}{p_{\tau^{-1}(n)}} \left(1 - \frac{K_1^{\frac{1}{\alpha}-1}}{p^{1/\alpha}_{\tau^{-1}(\tau^{-1}(n))}} \right) > 0 \text{ for all constants } K_1 > 0,$$

and

$$E_n = \frac{1}{p_{\tau^{-1}(n)}} \left(1 - \frac{M^{\frac{1-\alpha}{\alpha}} R^{1/\alpha}_{\tau^{-1}(\tau^{-1}(n))}}{p^{1/\alpha}_{\tau^{-1}(\tau^{-1}(n))} R_{\tau^{-1}(n)}} \right) > 0 \text{ for all constants } M > 0,$$

where τ^{-1} is the inverse function of τ. Set

$$z_n = x_n + p_n x^{\alpha}_{\tau(n)}. \tag{2.1}$$

We begin with the following lemma.

Lemma 2.1. *Let z_n be defined by (2.1) with $z_n > 0$, $\Delta z_n > 0$ and $\Delta(a_n \Delta z_n) \leq 0$ for all $n \geq n_1 \geq n_0$. Then for $\tau(n) \leq n$ and $E_n > 0$ for all $n \geq n_1$, we have*

$$x_n^\alpha \geq E_n z_{\tau^{-1}(n)}$$

for all $n \geq n_1$.

Proof. From (2.1), we have $z_n = x_n + p_n x_{\tau(n)}$ or

$$x_n^\alpha = \frac{1}{p_{\tau^{-1}(n)}} \left(z_{\tau^{-1}(n)} - x_{\tau^{-1}(n)} \right), \ n \geq n_1 \geq n_0. \tag{2.2}$$

On the other hand,

$$z_n = z_{n_1} + \sum_{s=n_0}^{n-1} \frac{a_s \Delta z_s}{a_s} \geq R_n a_n \Delta z_n.$$

Hence

$$\Delta \left(\frac{z_n}{R_n} \right) \leq 0 \text{ for } n \geq n_1.$$

Further

$$\begin{aligned} x_{\tau^{-1}(n)} &\leq \frac{1}{p_{\tau^{-1}(\tau^{-1}(n))}} z^{1/\alpha}_{\tau^{-1}(\tau^{-1}(n))} \\ &\leq \frac{R^{1/\alpha}_{\tau^{-1}(\tau^{-1}(n))}}{p^{1/\alpha}_{\tau^{-1}(\tau^{-1}(n))}} \frac{z^{1/\alpha}_{\tau^{-1}(n)}}{R^{1/\alpha}_{\tau^{-1}(n)}} \end{aligned}$$

where we have used $\frac{z_n}{R_n}$ is nonincreasing. Since $\frac{z_n}{R_n} \leq M$ for $M > 0$ and for all $n \geq n_1$, we have from (2.2) that

$$x_n^\alpha \geq \frac{1}{p^{1/\alpha}_{\tau^{-1}(n)}} \left(1 - \frac{M^{\frac{1-\alpha}{\alpha}}}{p^{1/\alpha}_{\tau^{-1}(\tau^{-1}(n))}} \frac{R^{1/\alpha}_{\tau^{-1}(\tau^{-1}(n))}}{R_{\tau^{-1}(n)}} \right) z_{\tau^{-1}(n)}$$

or

$$x_n^\alpha \geq E_n z_{\tau^{-1}(n)}.$$

This completes the proof. □

Lemma 2.2. *Let z_n be defined by (2.1) with $z_n > 0$, $\Delta z_n > 0$ and $\Delta(a_n \Delta z_n) \leq 0$ for all $n \geq n_1 \geq n_0$. Then for $\tau(n) \geq n$ and $B_n > 0$ for all $n \geq n_1$, we have*

$$x_n^\alpha \geq B_n z_{\tau^{-1}(n)}$$

for all $n \geq n_1$.

Proof. From the proof of Lemma 2.1, we have

$$x_n^\alpha = \frac{1}{p_{\tau^{-1}(n)}}\left(z_{\tau^{-1}(n)} - x_{\tau^{-1}(n)}\right).$$

Since $\{z_n\}$ is nondecreasing and $x_{\tau(n)} < \frac{1}{p_n^{1/\alpha}} z_n^{1/\alpha}$, we obtain

$$x_n^\alpha \;\geq\; \left(1 - \frac{M^{\frac{1-\alpha}{\alpha}} R_{\tau^{-1}(n)}^{\frac{1-\alpha}{\alpha}}}{p_{\tau^{-1}(\tau^{-1}(n))}^{1/\alpha}}\right) z_{\tau^{-1}(n)}$$

where we have used $\frac{z_n}{R_n} \leq M$ for all $n \geq n_1$. The proof is now complete. □

Lemma 2.3. *Let z_n be defined by (2.1) with $z_n > 0$, $\Delta z_n < 0$ and $\Delta(a_n \Delta z_n) \leq 0$ for all $n \geq n_1 \geq n_0$. Then for $A_{n_0} < \infty$, $\tau(n) \geq n$ and $C_n > 0$ for all $n \geq n_1$, we have*

$$x_n^\alpha \geq C_n z_{\tau^{-1}(n)}$$

for all $n \geq n_1$.

Proof. From $\Delta(a_n \Delta z_n) < 0$ for all $n \geq n_1$, we have

$$\Delta z_s \leq \frac{a_n \Delta z_n}{a_n} \;\text{ for }\; s \geq n.$$

Summing the last inequality from j to n and then letting $j \to \infty$, we obtain

$$0 \leq z_n + A_n a_n \Delta z_n$$

or

$$\Delta\left(\frac{z_n}{A_n}\right) \geq 0 \text{ for all } n \geq n_1.$$

From (2.1), we have

$$\begin{aligned} x_n^\alpha &= \frac{1}{p_{\tau^{-1}(n)}}\left(z_{\tau^{-1}(n)} - x_{\tau^{-1}(n)}\right) \\ &\geq \frac{1}{p_{\tau^{-1}(n)}}\left(z_{\tau^{-1}(n)} - \frac{1}{p_{\tau^{-1}(\tau^{-1}(n))}^{1/\alpha}} \frac{A_{\tau^{-1}(\tau^{-1}(n))}^{1/\alpha}}{A_{\tau^{-1}(n)}^{1/\alpha}} z_{\tau^{-1}(n)}^{1/\alpha}\right) \\ &\geq C_n z_{\tau^{-1}(n)} \end{aligned}$$

where we have used $z_n \leq K$ for $K > 0$ and all $n \geq n_1$. This completes the proof. □

First we establish oscillation criteria for equation (1.1) when condition (1.2) holds.

Theorem 2.4. *Assume that condition (1.2), $\beta \geq \alpha$, $\tau(n) \geq n$ and $\sigma(n+1) \leq n$ for all $n \geq n_0$. If there exists a positive nodecreasing sequence $\{\rho_n\}$ such that*

$$\limsup_{n\to\infty} \sum_{s=n_1}^{n} \left[\rho_s q_s B_{\sigma(s+1)}^{\beta/\alpha} - \frac{\alpha(\Delta\rho_s)^2 a_{\tau^{-1}(\sigma(s))}}{4\beta M_1^{\frac{\beta}{\alpha}-1}\rho_s}\right] = \infty \tag{2.3}$$

for all constants $M_1 > 0$, then every solution of equation (1.1) *is oscillatory.*

Proof. Assume the contrary that equation (1.1) has an eventually positive solution, say $x_n > 0$, $x_{\tau(n)} > 0$, and $x_{\sigma(n)} > 0$ for all $n \geq n_1$ for some $n_1 \geq n_0$. Then it follows from equation (1.1) that

$$\Delta(a_n \Delta z_n) = -q_n x^{\beta}_{\sigma(n+1)} \leq 0, \; n \geq n_1. \tag{2.4}$$

In view of condition (1.2), it is easy to see that $\Delta z_n > 0$ for all $n \geq n_1$. Then from Lemma 2.2, that

$$x_n^{\alpha} \geq B_n z_{\tau^{-1}(n)}, \quad n \geq n_1. \tag{2.5}$$

From (2.4) and (2.5) it follows that

$$\Delta(a_n \Delta z_n) + q_n B^{\beta/\alpha}_{\sigma(n+1)} z^{\beta/\alpha}_{\tau^{-1}(\sigma(n+1))} \leq 0, \; n \geq n_1. \tag{2.6}$$

Define

$$w_n = \rho_n \frac{a_n \Delta z_n}{z^{\beta/\alpha}_{\tau^{-1}(\sigma(n))}}, \; n \geq n_1,$$

then $w_n > 0$ for all $n \geq n_1$, and

$$\Delta w_n \;\; \leq \;\; -\rho_n q_n B^{\beta/\alpha}_{\sigma(n+1)} + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n}{\rho_{n+1}} w_{n+1} \frac{\Delta z^{\beta/\alpha}_{\sigma(n)}}{z^{\beta/\alpha}_{\tau^{-1}(\sigma(n))}}, \; n \geq n_1. \tag{2.7}$$

By Mean value theorem

$$\Delta z^{\frac{\beta}{\alpha}}_{\tau^{-1}(\sigma(n))} \geq \begin{cases} \frac{\beta}{\alpha} z^{\frac{\beta}{\alpha}-1}_{\tau^{-1}(\sigma(n+1))} \Delta z_{\tau^{-1}(\sigma(n))}, & \text{if } \frac{\beta}{\alpha} \leq 1; \\ \frac{\beta}{\alpha} z^{\frac{\beta}{\alpha}-1}_{\tau^{-1}(\sigma(n))} \Delta z_{\tau^{-1}(\sigma(n))}, & \text{if } \frac{\beta}{\alpha} \geq 1. \end{cases}$$

Using the last inequality in (2.7) and then using the fact that of $a_{\tau^{-1}(\sigma(n))} \Delta z_{\tau^{-1}(\sigma(n))}$ is nonincreasing and of $z_{\tau^{-1}(n)}$ is nondecreasing, we obtain

$$\Delta w_n \;\; \leq \;\; -\rho_n q_n B^{\beta/\alpha}_{\sigma(n+1)} + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\beta}{\alpha} M_1^{\frac{\beta}{\alpha}-1} \frac{\rho_n}{\rho^2_{n+1}} \frac{w^2_{n+1}}{a_{\tau^{-1}(\sigma(n))}}, \; n \geq n_1$$

where we have $z_n \geq M_1 > 0$ for all $n \geq n_1$. Summing the last inequality from n_1 to n, and using completing the square we obtain

$$\sum_{s=n_1}^{n} \left[\rho_s q_s B^{\beta/\alpha}_{\sigma(s+1)} - \frac{\alpha (\Delta \rho_s)^2 a_{\tau^{-1}(\sigma(s))}}{4 \beta M_1^{\frac{\beta}{\alpha}-1} \rho_s} \right] \leq w_{n_1}$$

which contradicts (2.3) as $n \to \infty$. The proof is now complete. ∎

Next we derive a oscillation results without the condition $\beta \geq \alpha$ or $\beta \leq \alpha$.

Theorem 2.5. *Assume that condition* (1.2), $\tau(n) \geq n$, *and* $\sigma(n+1) \leq n$ *for all* $n \geq n_0$. *If*

$$\sum_{n=n_1}^{\infty} q_n B_{\sigma(n+1)}^{\frac{\beta}{\alpha}} = \infty \tag{2.8}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a positive solution of equation (1.1). Proceeding as in the proof of Theorem 2.4, we have (2.6). Summing (2.6) from n_1 to n, we obtain

$$\sum_{s=n_1}^{n} q_s B_{\sigma(s+1)}^{\frac{\beta}{\alpha}} z_{\tau^{-1}(\sigma(s+1))}^{\frac{\beta}{\alpha}} \leq a_{n_1} \Delta z_{n_1}. \tag{2.9}$$

Since $\{z_n\}$ is nondecreasing we have $z_n \geq M_1 > 0$ for all $n \geq n_1$, and hence from (2.9) we obtain a contradiction to (2.8) as $n \to \infty$. This completes the proof. □

Theorem 2.6. *Assume that condition* (1.2), $\beta \geq \alpha$, *and* $\sigma(n) \geq \tau(n) \geq n$ *for all* $n \geq n_0$. *If there exists a positive nodecreasing sequence* $\{\rho_n\}$ *such that*

$$\limsup_{n\to\infty} \sum_{s=n_1}^{n} \left[\rho_s q_s B_{\sigma(s+1)}^{\beta/\alpha} - \frac{\alpha(\Delta\rho_s)^2}{4\beta M_1^{\frac{\beta}{\alpha}-1}\rho_s} a_s \right] = \infty \tag{2.10}$$

for all constants $M_1 > 0$, *then every solution of equation* (1.1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a positive solution of equation (1.1). Proceeding as in the proof of Theorem 2.4, we have (2.6). Define

$$w_n = \rho_n \frac{a_n(\Delta z_n)}{z_n^{\frac{\beta}{\alpha}}},\ n \geq n_1.$$

Similar as in the proof of Theorem 2.4, we obtain a contradiction to (2.10). This completes the proof. □

Theorem 2.7. *Assume that condition* (1.2), $\beta \geq \alpha$, *and* $\sigma(n+1) \leq \tau(n) \leq n$ *for all* $n \geq n_0$. *If there exists a positive nodecreasing sequence* $\{\rho_n\}$ *such that*

$$\limsup_{n\to\infty} \sum_{s=n_1}^{n} \left[\rho_s q_s E_{\sigma(s+1)}^{\beta/\alpha} - \frac{\alpha(\Delta\rho_s)^2}{4\beta M_1^{\frac{\beta}{\alpha}-1}\rho_s} a_{\tau^{-1}(\sigma(s))} \right] = \infty \tag{2.11}$$

for all constants $M_1 > 0$, *then every solution of equation* (1.1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a positive solution of equation (1.1). Proceeding as in Theorem 2.4, we have (2.4). From Lemma 2.2, we obtain

$$x_n^{\alpha} \geq E_n z_{\tau^{-1}(n)}, \quad n \geq n_1. \tag{2.12}$$

From (2.4) and (2.12), we have

$$\Delta(a_n \Delta z_n) + q_n E_{\sigma(n+1)}^{\beta/\alpha} z_{\tau^{-1}(\sigma(n+1))}^{\beta/\alpha} \leq 0,\ n \geq n_1.$$

Define

$$w_n = \rho_n \frac{a_n \Delta z_n}{z^{\beta/\alpha}_{\tau^{-1}(\sigma(n))}},\ n \geq n_1.$$

Similar to that proof of Theorem 2.4, we obtain a contradiction to (2.11). This completes the proof. □

Our next results are for the case where (1.3) holds in place of (1.2).

Theorem 2.8. *Assume that condition* (1.3), $\beta \geq \alpha$, $\tau(n) \geq n$ *and* $\sigma(n) \leq n$ *for all* $n \geq n_0$. *If there exists a positive nondecreasing sequence* $\{\rho_n\}$ *such that* (2.3) *holds for all constants* $M_1 > 0$, *and*

$$\limsup_{n\to\infty} \sum_{s=n_1}^{n-1} \left[A^{\beta/\alpha}_{s+1} B^{\beta/\alpha}_{\sigma(s+1)} q_s - \frac{\beta K_1^{\frac{\beta}{\alpha}-1} A_s^{\frac{\beta}{\alpha}-1}}{4\alpha a_s A_{s+1}^{\frac{\beta}{\alpha}}} \right] = \infty \tag{2.13}$$

for all constants $K_1 > 0$, *then every solution of equation* (1.1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a positive solution of equation (1.1), say, $x_n > 0$, $x_{\tau(n)} > 0$, and $x_{\sigma(n)} > 0$ for all $n \geq n_1$. From equation (1.1) that (2.4) holds. We then have either $\Delta z_n > 0$ or $\Delta z_n < 0$ eventually. If $\Delta z_n > 0$ holds, then we can proceed as in the proof of Theorem 2.4, we can obtain a contradiction to (2.3). Next, consider the case $\Delta z_n < 0$ for all $n \geq n_1$. Define

$$u_n = \frac{a_n \Delta z_n}{z_n^{\frac{\beta}{\alpha}}},\ n \geq n_1. \tag{2.14}$$

Then $u_n < 0$ for $n \geq n_1$. From the proof of Lemma 2.3, one obtains

$$\frac{a_n \Delta z_n A_n}{z_n} \geq -1,\ n \geq n_1. \tag{2.15}$$

Thus

$$\frac{-a_n \Delta z_n (-a_n \Delta z_n)^{\frac{\beta}{\alpha}-1} A_n^{\frac{\beta}{\alpha}}}{z_n^{\frac{\beta}{\alpha}}} \leq 1$$

for $n \geq n_1$. Since $-a_n \Delta z_n > 0$, and from (2.14), and the last inequality, we obtain

$$-\frac{1}{L^{\frac{\beta}{\alpha}-1}} \leq u_n a_n^{\frac{\beta}{\alpha}} \leq 0 \tag{2.16}$$

where $L = -a_{n_1} \Delta z_{n_1}$. On the other hand from Lemma 2.2 and (2.4), we have

$$\Delta(a_n \Delta z_n) + q_n C^{\frac{\beta}{\alpha}}_{\sigma(n+1)} z^{\frac{\beta}{\alpha}}_{\tau^{-1}(\sigma(n+1))} \leq 0,\ n \geq n_1. \tag{2.17}$$

From (2.14) and (2.17), we have

$$\Delta u_n \leq -q_n C^{\frac{\beta}{\alpha}}_{\sigma(n+1)} - \frac{\beta}{\alpha} K_1^{\frac{\beta}{\alpha}-1} A_n^{\frac{\beta}{\alpha}-1} \frac{u_n^2}{a_n},\ n \geq n_1 \tag{2.18}$$

where we have used $\frac{z_n}{A_n} \geq K_1 > 0$ for all $n \geq n_1$. Multiplying (2.18) by $A_{n+1}^{\frac{\beta}{\alpha}}$ and summing it from n_1 to $n-1$ and then using the summation by parts formula, we obtain

$$\sum_{s=n_1}^{n-1} q_s A_{s+1}^{\frac{\beta}{\alpha}} C_{\sigma(s+1)}^{\frac{\beta}{\alpha}} \leq A_{n_1}^{\frac{\beta}{\alpha}} u_{n_1} - A_n^{\frac{\beta}{\alpha}} u_n - \sum_{s=n_1}^{n-1} \left[\frac{\beta}{\alpha} A_s^{\frac{\beta}{\alpha}-1} \frac{u_s}{a_s} + \frac{\beta}{\alpha} K_1^{\frac{\beta}{\alpha}-1} A_{s+1}^{\frac{\beta}{\alpha}} A_s^{\frac{\beta}{\alpha}-1} \frac{u_s^2}{a_s}\right]$$

which on using completing the square yields

$$\sum_{s=n_1}^{n-1} \left[q_s A_{s+1}^{\beta/\alpha} C_{\sigma(s+1)}^{\beta/\alpha} - \frac{\beta K_1^{\frac{\beta}{\alpha}-1} A_s^{\frac{\beta}{\alpha}-1}}{4\alpha a_s A_{s+1}^{\frac{\beta}{\alpha}}}\right] \leq \frac{1}{L^{\frac{\beta}{\alpha}-1}} + A_{n_1}^{\frac{\beta}{\alpha}} u_{n_1}$$

when using (2.16). This contradicts (2.13) as $n \to \infty$, and the proof is completed. □

Theorem 2.9. *Assume that condition* (1.3), $\beta \geq \alpha$, *and* $\sigma(n+1) \leq \tau(n) \leq n$ *for all* $n \geq n_0$. *If there exists a positive nondecreasing sequence* $\{\rho_n\}$ *such that* (2.11) *holds for all constants* $M_1 > 0$, *and*

$$\limsup_{n\to\infty} \sum_{s=n_1}^{n-1} \left[q_s A_{s+1}^{\beta/\alpha} D_{\sigma(s+1)}^{\beta/\alpha} - \frac{\beta K_1^{\frac{\beta}{\alpha}-1} A_s^{\frac{\beta}{\alpha}-1}}{4\alpha a_s A_{s+1}^{\frac{\beta}{\alpha}}}\right] = \infty \tag{2.19}$$

hold for all constants $K_1 > 0$, *then every solution of equation* (1.1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a positive solution of equation (1.1), say, $x_n > 0$, $x_{\tau(n)} > 0$, and $x_{\sigma(n)} > 0$ for all $n \geq n_1$. From equation (1.1) that (2.4) holds. We then have either $\Delta z_n > 0$ or $\Delta z_n < 0$ eventually. If $\Delta z_n > 0$ holds, then one can proceed as in the proof of Theorem 2.7 and obtain a contradiction to (2.11). Next, consider the case $\Delta z_n < 0$ for all $n \geq n_1$. From the definition of z_n, we have

$$x_n^\alpha \geq \frac{1}{p_{\tau^{-1}(n)}} \left[z_{\tau^{-1}(n)} - \frac{z_{\tau^{-1}(\tau^{-1}(n))}^{\frac{1}{\alpha}}}{p_{\tau^{-1}(\tau^{-1}(n))}^{\frac{1}{\alpha}}}\right] \geq \frac{1}{p_{\tau^{-1}(n)}} \left(1 - \frac{K_1^{\frac{1}{\alpha}-1}}{p_{\tau^{-1}(\tau^{-1}(n))}^{\frac{1}{\alpha}}}\right) z_{\tau^{-1}(n)}$$

where we have used z_n is decreasing and $z_n \leq K_1$ for all $n \geq n_1$. From the last inequality and (2.4), we obtain

$$\Delta(a_n \Delta z_n) + q_n D_{\sigma(n+1)}^{\frac{\beta}{\alpha}} z_{\tau^{-1}(\sigma(n+1))}^{\frac{\beta}{\alpha}} \leq 0,\ n \geq n_1.$$

The remainder of the proof is similar to that of Theorem 2.8, and hence is omitted. The proof is now complete. □

3. Examples

In this section, we present some examples to illustrate the main results.

Example 3.1. *Consider the difference equation*

$$\Delta\left(\frac{1}{n}\Delta\left(x_n+\frac{n^2}{2}x_{n+2}^{\frac{1}{3}}\right)\right)+n^6x_{n-1}=0,\ n\geq 1 \tag{3.1}$$

Here $a_n=\frac{1}{n},\ p_n=\frac{n^2}{2},\ \tau(n)=n+2,\ q_n=n^6,\ \sigma(n+1)=n-1,\ \beta=1$, and $\alpha=\frac{1}{3}$. Simple calculation shows that $R_n=\frac{n(n-1)}{2},\ B_n=\frac{2}{(n-2)^2}\left(1-\frac{M^2(n-3)^2}{8(n-2)^{10}}\right)$, and by taking $\rho_n=1$, we see that all conditions of Theorem 2.4 are satisfied. Hence every solution of equation (3.1) is oscillatory.

Example 3.2. *Consider the second order difference equation*

$$\Delta\left(\frac{1}{n}\Delta\left(x_n+n^{\frac{3}{2}}x_{n+1}^{\frac{1}{3}}\right)\right)+nx_{n^2+1}^{\frac{1}{3}}=0,\ n\geq 1 \tag{3.2}$$

Here $a_n=\frac{1}{n},\ p_n=n,\ \tau(n)=n+1,\ \sigma(n+1)=n^2+1,\ \alpha=\frac{1}{3}$, and $\beta=\frac{1}{3}$. Simple calculation shows that $R_n=\frac{n(n-1)}{2},\ B_n=\frac{1}{(n-1)^{\frac{3}{2}}}\left(1-\frac{M^2(n-2)^2}{4(n-1)^7}\right)$, and it is easy to see that all conditions of Theorem 2.6 are satisfied. Hence every solution of equation (3.2) is oscillatory.

Example 3.3. *Consider the difference equation*

$$\Delta\left(2^n\Delta\left(x_n+nx_{n+2}^{\frac{1}{3}}\right)\right)+3^nx_{n-1}=0,\ n\geq 1 \tag{3.3}$$

Here $a_n=2^n,\ p_n=n,\ \tau(n)=n+2,\ \sigma(n+1)=n-1,\ \alpha=\frac{1}{3}$, and $\beta=1$. Simple calculation shows that $\frac{\beta}{\alpha}=3,\ R_n=\left(1-\frac{1}{2^{n-1}}\right),\ R_n=\left(1-\frac{1}{2^{n-1}}\right)$, and it is easy to see that all conditions of Theorem 2.8 are satisfied. Hence every solution of equation (3.2) is oscillatory.

We conclude this paper with the following remark.

Remark 3.1. *In this paper, we obtain criteria for the oscillation of all solutions of equation* (1.1) *which involves α and β. Further the results appeared in [4, 5, 7, 9, 10] cannot be applied to our examples since $\{p_n\}$ does not tend to zero as $n\to\infty$. Hence our results are new and complement to the existing results.*

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