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GENERALIZATION OF TITCHMARSH'S THEOREM FOR THE FOURIER TRANSFORM

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Abstract. Using a spherical mean operator, we obtain a generalization of Titchmarsh's Theorem for the Fourier transform for functions satisfying the ψ-Fourier Lipschitz condition in $L^2(\mathbb{R}^n)$.
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1. Introduction and preliminaries

It is well know that integral Fourier transforms are widely used in mathematical physics. Titchmarsh's [5, Therem 85] characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform:

Theorem 1.1. [5] *Let $\alpha \in (0,1)$ and assume that $f \in L^2(\mathbb{R})$. Then the following are eguivalent:*

(1) $\|f(t+h)-f(t)\|_{L^2(\mathbb{R})} = O(h^\alpha)$ *as* $h \longrightarrow 0$,
(2) $\int_{|\lambda|\geq r} |g(\lambda)|^2 d\lambda = O(r^{-2\alpha})$ *as* $r \longrightarrow \infty$.

where g stands for the Fourier transform of f.

In this paper, we prove the generalization of this Theorem 1.1 for the Fourier transform in $\mathbb{R}^n$ for functions satisfying the ψ-Fourier Lipschitz class in the space $L^2(\mathbb{R}^n)$. For this purpose, we use the spherical mean operator.

Recall that $L^2(\mathbb{R}^n)$ is the space of integrable function f with the norm

$$\|f\|_2 = \left(\int_{\mathbb{R}^n} |f(x)|^2 dx\right)^{1/2}$$

The Fourier transform for a function $f \in \mathrm{L}^2(\mathbb{R}^n)$ defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix.\xi} dx.$$

The inverse Fourier transform defined by formula

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{ix.\xi} d\xi.$$

The Plencherel equality [4] holds

$$\int_{\mathbb{R}^n} |f(x)|^2 dx = \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 d\xi.$$

We denote $j_p(x)$ for the Bessel normalized function of the first kind, i.e

$$j_p(x) = \frac{2^p \Gamma(p+1) J_p(x)}{x^p}$$

where $J_p(x)$ is the Bessel function of the first kind and $\Gamma(x)$ is the gamma-function(see[3]).

For $p \geq -\frac{1}{2}$, we introduce the Bessel normalized function of the first kind j_p defined also by

$$j_p(x) = \Gamma(p+1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n} \tag{1}$$

Moreover, from (1) we see that

$$\lim_{x \longrightarrow 0} \frac{j_p(x) - 1}{x^2} \neq 0$$

consequently, there exist $c > 0$ and $\eta > 0$ satisfying

$$|x| \leq \eta \Longrightarrow |j_p(x) - 1| \geq c|x|^2 \tag{2}$$

In [1], we have

$$|j_p(x)| \leq 1. \tag{3}$$

and we see that

$$|1 - j_{\frac{n-2}{2}}(|x|)| \leq |x|, \ \forall x \in \mathbb{R}^n. \tag{4}$$

In $\mathrm{L}^2(\mathbb{R}^n)$ consider the spherical mean operator (see [2])

$$\mathrm{M}_h f(x) = \frac{1}{w_{n-1}} \int_{\mathbb{S}^{n-1}} f(x+hw)dw,$$

where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$, w_{n-1} its total surface measure with respect to the usual induced measure dw.

Proposition 1.2. *Let $f \in \mathrm{L}^2(\mathbb{R}^n)$, then*

$$\widehat{(\mathrm{M}_h f)}(\xi) = j_{\frac{n-2}{2}}(h|\xi|)\widehat{f}(\xi)$$

Proof. (See [2, Proposition 4]). □

2. Main Result

In this section we give the main result of this paper, We need first to define ψ-Fourier Lipschitz class.

Definition 2.1. *A function $f \in \mathrm{L}^2(\mathbb{R}^n)$ is said to be in the ψ-Fourier Lipschitz class, denoted by $Lip(\psi, n, 2)$, if*

$$\|\mathrm{M}_h f(x) - f(x)\|_2 = O(\psi(h)), \ as \ h \longrightarrow 0,$$

where

1. $\psi(t)$ *is a continuous increasing function on* $[0, \infty)$,
2. $\psi(0) = 0$ *and* $\psi(ts) \leq \psi(t)\psi(s)$ *for all* $t, s \in [0, \infty)$
3. $\int_0^{1/h} s\psi(s^{-2})ds = O(\frac{1}{h^2}\psi(h^2))$ *as* $h \longrightarrow 0$

Theorem 2.2. *Let $f \in \mathrm{L}^2(\mathbb{R}^n)$. Then the following are equivalent*

1. $f \in Lip(\psi, n, 2)$.
2. $\int_{|\xi| \geq r} |\widehat{f}(\xi)|^2 d\xi = O(\psi(r^{-2}))$ *as* $r \longrightarrow +\infty$.

Proof. $1 \Longrightarrow 2$: Suppose that $f \in Lip(\psi, n, 2)$. Then we obtain

$$\|\mathrm{M}_h f(x) - f(x)\|_2 = O(\psi(h)), \ as \ h \longrightarrow 0.$$

Plancherel's identity and Proposition 1.2 give

$$\|\mathrm{M}_h f(x) - f(x)\|_2^2 = \int_{\mathbb{R}^n} |1 - j_{\frac{n-2}{2}}(|\xi|h)|^2 |\widehat{f}(\xi)|^2 d\xi.$$

From formula (2), we have

$$\int_{\frac{\eta}{2h}\leq|\xi|\leq\frac{\eta}{h}} |1-j_{\frac{n-2}{2}}(|\xi|h)|^2|\widehat{f}(\xi)|^2d\xi \geq \frac{c^2\eta^4}{16}\int_{\frac{\eta}{2h}\leq|\xi|\leq\frac{\eta}{h}} |\widehat{f}(\xi)|^2d\xi.$$

There exists then a positive constant C such that

$$\begin{aligned}\int_{\frac{\eta}{2h}\leq|\xi|\leq\frac{\eta}{h}} |\widehat{f}(\xi)|^2d\xi &\leq C\int_{\mathbb{R}^n} |1-j_{\frac{n-2}{2}}(|\xi|h)|^2|\widehat{f}(\xi)|^2d\xi \\ &\leq C\psi(h^2).\end{aligned}$$

We obtain

$$\int_{r\leq|\xi|\leq 2r} |\widehat{f}(\xi)|^2d\xi \leq C\psi(2^{-2}\eta^2r^{-2}).$$

Thus there exists then a positive constant K such that

$$\int_{r\leq|\xi|\leq 2r} |\widehat{f}(\xi)|^2d\xi \leq K\psi(r^{-2}), \quad for\ all\ r>0.$$

So that

$$\begin{aligned}\int_{|\xi|\geq r} |\widehat{f}(\xi)|^2d\xi &= \left[\int_{r\leq|\xi|\leq 2r} + \int_{2r\leq|\xi|\leq 4r} + \int_{4r\leq|\xi|\leq 8r} ....\right]|\widehat{f}(\xi)|^2d\xi \\ &= O(\psi(r^{-2})+\psi(2^{-2}r^{-2})......) \\ &\leq K(1+\psi(2^{-1})+\psi^2(2^{-1})+....)\psi(r^{-2})\end{aligned}$$

We have

$$\int_{|\xi|\geq r} |\widehat{f}(\xi)|^2d\xi \leq C\psi(r^{-2}),$$

where $C = K(1-\psi(2^{-1})^{-1}$

This proves that

$$\int_{|\xi|\geq r} |\widehat{f}(\xi)|^2d\xi = O(\psi(r^{-2})) \ \ as\ r \longrightarrow +\infty$$

$2 \Longrightarrow 1$: Suppose now that

$$\int_{|\xi|\leq r} |\widehat{f}(\xi)|^2d\xi = O(\psi(r^{-2})) \ \ as\ r \longrightarrow +\infty$$

We have to show that

$$\int_0^{\infty} s^{n-1}|1-j_{\frac{n-2}{2}}(sh)|^2\phi(s)ds = O(\psi(h^2)) \ as\ h \longrightarrow 0,$$

where

$$\phi(s) = \int_{\mathbb{S}^{n-1}} |\widehat{f}(sy)|^2 dy.$$

We write

$$\int_0^\infty s^{n-1}|1 - j_{\frac{n-2}{2}}(sh)|^2 \phi(s) ds = \mathrm{I}_1 + \mathrm{I}_2,$$

where

$$\mathrm{I}_1 = \int_0^{\frac{1}{h}} s^{n-1}|1 - j_{\frac{n-2}{2}}(sh)|^2 \phi(s) ds$$

and

$$\mathrm{I}_2 = \int_{\frac{1}{h}}^\infty s^{n-1}|1 - j_{\frac{n-2}{2}}(sh)|^2 \phi(s) ds$$

We now estimate the summands I_1 and I_2.

Firstly, from (3) we have

$$\mathrm{I}_2 \leq 4 \int_{\frac{1}{h}}^\infty s^{n-1} \phi(s) ds = O(\psi(h^2) \quad as \ h \longrightarrow 0$$

Set

$$\varphi(r) = \int_r^\infty s^{n-1} \phi(s) ds$$

From (4), an integration by parts yields

$$\begin{aligned}
\mathrm{I}_1 &\leq -h^2 \int_0^{\frac{1}{h}} r^2 \varphi'(r) dr \\
&\leq -\varphi(\frac{1}{h}) + 2h^2 \int_0^{\frac{1}{h}} r\varphi(r) dr \\
&\leq Ch^2 \int_0^{\frac{1}{h}} r\psi(r^{-2}) dx \\
&\leq Ch^2 \frac{1}{h^2} \psi(h^2) \\
&\leq C\psi(h^2).
\end{aligned}$$

and this end the proof. □

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