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THE ECCENTRIC WEIGHT OF GRAPHS

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Abstract. Let $G = (V, E)$ be a connected graph. The eccentricity $e(u)$ of a vertex $u \in V(G)$, is the maximum distance from it to another vertex of G. The eccentric wight of G, $ew(G)$, is defined as $\sum_{x \in E} f(x)$, where $f : E \longrightarrow \{0, 1\}$ is a function and $f(uv) = |e(u) - e(v)|$ for all $uv \in E(G)$. In this paper, we compute the eccentric weight of several classes of graphs. Some bounds for $ew(G)$ are established. The eccentric wight of Join and Cartesian product of graphs are obtained.

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1. Introduction

Let $G = (V, E)$ be a simple connected graph with n vertices and m edges. For a vertex $u \in V$, $deg(u)$ denotes the degree of u. For vertices $u, v \in V$, the distance $d(u, v)$ is defined as the length of a shortest path between them. The eccentricity $e(u)$ of a vertex u in G is the maximum distance between u and v for all v in G. The radius $r(G)$ is the minimum eccentricity of the vertices. A vertex u is a central vertex if $e(u) = r(G)$, and center of G is the set of all central vertices and is denoted by $Z(G)$. The diameter $diam(G)$, in G is the maximum distance between any pair of vertices of the graph. If G is disconnected, $diam(G) = \infty$. For graph theoretic terminology, we refer to [4].

A graph G is said to be self-centred if $e(u) = e(v)$ for all $u, v \in V(G)$ [2].

A graph is called a weighted graph if each edge e is assigned a non-negative number $w(e)$, called the weight of e [1].

A graph without cycles is called an acyclic graph. A connected graph without cycles is said to be a tree. A double star is the tree obtained from two disjoint stars $K_{1,n}$ and $K_{1,m}$ by connecting their centers. A graph is called unicyclic if it is connected and contains exactly one cycle. A graph is unicyclic if and only if it is connected and has size equal to its order.

Consider a communication network modelled by a graph with vertices representing the nodes of the network and edges representing the links between them. One might want to minimise the average, taken over all the nodes in the system, of the maximum time delay of a message emanating from it. This is the average eccentricity of the corresponding graph.

For disjoint graphs G_1 and G_2, the join $G = G_1 + G_2$ is the graph G with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}$. Let us denote by $G - v$ the graph obtained from G by removing the vertex $v \in ?V(G)$ and all edges incident to v.

A connected subgraph B of G is called a block if B has no cut-vertex and every subgraph $B' \subseteq G$ with $B \subset B'$ has at least one cut-vertex. A connected graph G is called a block graph if every block in G is complete.

2. The eccentric weight of graphs

Definition. Let G be a connected graph. The eccentric wight of G, $ew(G)$, is defined as $\sum_{x \in E} f(x)$, where $f : E \longrightarrow \{0, 1\}$ is a function and $f(uv) = |e(u) - e(v)|$ for all $uv \in E(G)$.

Since every pair of adjacent vertices of G have eccentricity difference zero or one, we obtain the obvious bound, $0 \leq ew(G) \leq m$. The lower bound attains for complete graph K_n and the upper bound attains for a star $K_{1,m}$.

Theorem 2.1. *For any connected graph G, $ew(G) = 0$, if and only if G is self-centred.*

Proof. *If $ew(G) = 0$ for a connected graph G, then $f(uv) = 0$ for all $uv \in E(G)$ and hence $e(u) = e(v)$ for all $u, v \in V(G)$. Thus, G is self-centred.*
Conversely, since G is self-centred graph, it follows that $e(u) = e(v)$ for all $u, v \in V(G)$ and so $f(uv) = 0$ for all $uv \in E(G)$. Thus, $ew(G) = 0$.

We now proceed to compute $ew(G)$ for some standard graphs.

Remark 2.2.

(1) For any cycle C_n, $ew(C_n) = 0$.
(2) For any path P_n,
$$ew(P_n) = \begin{cases} n-1, & \text{if n is odd ;} \\ n-2, & \text{if n is even.} \end{cases}$$
(3) For any double star $S_{r,s}$, $ew(S_{r,s}) = r + s$.
(4) For any complete bipartite graph $K_{r,s}$, $ew(K_{r,s}) = 0$

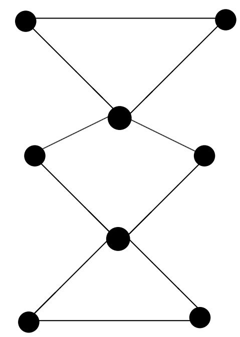
(5) For the wheel $W_{1,n}$, $n \geq 3$, $ew(W_{1,n}) = n$.

Theorem 2.3. *For any connected graph G, $ew(G) \leq m - |E(\langle Z(G) \rangle)|$.*

Proof. *Since $\langle Z(G) \rangle$ is self-centred, $ew(\langle Z(G) \rangle) = 0$. Thus $ew(G) \leq m - |E(\langle Z(G) \rangle)|$.*

Corollary 2.4. *For any $G(n, m)$ graph. If $ew(G) = m$, then $\langle Z(G) \rangle$ is totally disconnected.*

The converse of Corollary 2.4 is not true. For example, consider the graph G shown in Figure 1 $\langle Z(G) \rangle = 2K_1$, but $ew(G) \neq m$.



G

Fig.1

Observation 2.5. *Let $G \cong K_{m_1,m_2,...,m_k}$ be the complete k-partite graph, then $ew(G) = 0$*

Theorem 2.6. *If G_1 and G_2 are disjoint connected graphs having no full degree vertex, then, $ew(G_1 + G_2) = 0$.*

Proof. *Suppose that G_1 and G_2 have no full degree vertex. Then $e(u_i) \geq 2$ for all $u_i \in V(G_1)$ and $e(v_i) \geq 2$ for all $v_i \in V(G_2)$. But in $G_1 + G_2$, $e(u_i) = e(v_i) = 2$ for all $u_i \in V(G_1)$ and $v_i \in V(G_2)$. Since $d(u_i, v_i) = 1$, $d(u_i, u_j) = 2$ and $d(v_i, v_j) = 2$ for all $u_i, u_j \in V(G_1)$ and $v_i, v_j \in V(G_2)$, $f(uv) = 0$ for all $uv \in E(G_1 + G_2)$. Thus, $ew(G_1 + G_2) = 0$.*

In general, if $G = G_1 + G_2 + ... + G_k$ such that G_i, $1 \leq i \leq k$ has no full degree vertex, then $ew(G) = 0$.

3. Composite graphs

The Cartesian product $G_1 \square G_2 \square ... \square G_k$ of graphs $G_1, G_2, ..., G_k$ has the vertex set $V(G_1) \times V(G_2) \times ... \times V(G_k)$, two vertices $(u_1, u_2, ..., u_k)$ and $(v_1, v_2, ..., v_k)$ being adjacent if they differ in exactly one position, say in i-th, and $u_i v_i$ is an edge of G_i. It is well known (see [5]) that for $G = G_1 \square G_2 \square ... \square G_k$ and its two vertices $u = (u_1, u_2, ..., u_k)$ and $v = (v_1, v_2, ..., v_k)$ we have

$$e_{G1 \square G2}(u_1, u_2) = e_{G_1}(u_1) + e_{G_2}(u_2).$$

Theorem 3.1. *For graphs G and H, we have*

$$ew(G \Box H) = |H|ew(G) + |G|ew(H).$$

Proof. *For graphs G and H, we have*

$$\begin{aligned} ew(G \Box H) &= \sum_{(u_1,v_1),(u_2,v_2) \in (G \Box H)} |e_{_{G\Box H}}(u_1, v_1) - e_{_{G\Box H}}(u_2, v_2)| \\ &= |H|ew(G) + |G|ew(H). \end{aligned}$$

From Theorem 3.1, we have the following Remark.

Remark 3.2.

(1) $ew(C_r \Box C_s) = 0$
(2) $ew(P_r \Box P_s) = r.ew(P_s) + s.ew(P_r)$
(3) $ew(K_{1,r} \Box K_{1,s}) = 2rs.$
(4) $ew(W_{1,r} \Box W_{1,s}) = 2rs.$

Theorem 3.3. *If $G(n, m)$ is a graph with $\Delta(G) < m - 1$, then $ew(G + K_1) = n$.*

Proof. *Let G be a graph with $\Delta(G) < m - 1$. By the definition of $G + K_1$, in $G + K_1$, $e(u) = 1$, $u \in K_1$ and $e(v_i) = 2$ for all $v_i \in V(G)$, $1 \leq i \leq n$, Therefore, $f(uv_i) = 1$ for all $uv_i \in E(G + K_1)$, $1 \leq i \leq n$ and $f(v_iv_j) = 0$ for all $v_iv_j \in E(G + K_1)$, $1 \leq i, j \leq n$. Thus, $ew(G + K_1) = n$.*

Theorem 3.4. *If G has k full degree vertices, then $ew(G) = k(n - k)$.*

Proof. *Let $V(G) = \{v_i\colon 1 \leq i \leq n\}$ and let k be the number of full degree vertices in G. If v_j, $1 \leq j \leq k$ is the set of full degree vertices in G, then $e(v_j) = 1$, for all $v_j \in V(G)$, $1 \leq j \leq k$ and $e(v_i) = 2$, for all $v_i \in V(G)$, $k+1 \leq i \leq n$. Therefore, $f(v_iv_j) = 0$ for all $v_iv_j \in E(G)$, $1 \leq i, j \leq k$ and $f(v_iv_j) = 1$ for all $v_iv_j \in E(G)$, $k + 1 \leq i, j \leq n$. Thus, $ew(G) = k(n - k)$.*

Corollary 3.5. *If $G \cong K_n - e$, $ew(G) = 2(n - 2)$.*

Theorem 3.6. *If G_1 is a graph with n_1 vertices having k_1 full degree vertices and G_2 is a graph with n_2 vertices having k_2 full degree vertices, then*

$$ew(G_1 + G_2) = (k_1 + k_2)(n - (k_1 + k_2))$$

.

Proof. *Let G_1 having k_1 full degree vertices and G_2 having k_2 full degree vertices. If $V(G_1) = \{1 \leq u_i \leq n_1\}$ and $V(G_2) = \{1 \leq v_i \leq n_2\}$, then $e(u_i) = 1$, $1 \leq i \leq k_1$, $u_i \in V(G_1)$ and $e(v_i) = 1$, $1 \leq i \leq k_2$, $v_i \in V(G_2)$. Therefore, $e(u_i) = 2$, $k_1 + 1 \leq i \leq n_1$, $u_i \in V(G_1)$ and $e(v_i) = 2$, $k_2 + 1 \leq i \leq n_2$, $v_i \in V(G_2)$. Since $G_1 + G_2$ is a graph of order $n_1 + n_2$ with $k_1 + k_2$ full degree vertices, it follows that, by Theorem 3.3, $ew(G_1 + G_2) = (k_1 + k_2)(n - (k_1 + k_2))$.*

In general, if $G = G_1 + G_2 + ... + G_p$ such that G_i, $1 \leq i \leq p$ is a graph with n_i vertices having k_i full degree vertices, then

$$ew(G) = \sum_{i=1}^{p} k_i \left(\sum_{i=1}^{p} n_i - \sum_{i=1}^{p} k_i) \right).$$

We observe that, for any positive integer n, there are at least two non isomorphic graphs G_1 and G_2 such that $ew(G_1) = ew(G_2)$. For example $ew(K_{1,r}) = ew(W_{1,r}) = r$.

Proposition 3.7. *For any positive integer n, $n \geq 3$, there exists a graph $G(2n, m)$ such that $ew(G) = n = m$.*

Proof. *Take C_n with vertex set $\{v_1, v_2, ..., v_n\}$ and take a set of vertices $v_{n+1}, v_{n+2}, ..., v_{2n}$. Add the edges $v_1v_{n+1}, v_2v_{n+2}, ..., v_nv_{2n}$. In $G(2n, m)$, $e(v_i) = \lfloor \frac{n}{2} \rfloor + 1$ for all i, $1 \leq i \leq n$ and $e(v_i) = \lfloor \frac{n}{2} \rfloor + 2$ for all i, $n + 1 \leq i \leq 2n$. Therefore, $f(v_iv_j) = 0$ for all $v_iv_j \in E(C_n)$ and $f(v_iv_j) = 1$ for all $v_iv_j \notin E(C_n)$. Thus, $ew(G) = n = m$.*

Theorem 3.8. *For any unicyclic graph G, $m - p \leq ew(G)$, where C_p is the only cycle in G.*

Proof. *Since $G(n, m)$ is a unicyclic graph, $n = m$. There are $m - p + 1$ blocks in G say $B_1 = C_p, B_2, ..., B_{m-p+1}$. We consider the following cases.*
Case 1: *$Z(G) = \{v\}$. We consider the following subcases.*
Case 1.1: *$v \in V(G - C_p)$. Then $e(x) \neq e(y)$ for any adjacent vertices $x, y \in V(B_i)$, $i \neq 1$. Therefore, $f(xy) = 1$ for all $xy \in E(G - C_p)$. Now, if $f(xy) = 0$ for all $xy \in E(C_p)$, then $ew(G) = m - p$ and if C_p contains at least one edge xy with $f(xy) = 1$, then $m - p < ew(G)$.*
Case 1.2: *$v \in V(C_p)$. Then $f(xy) = 1$ for all $xy \in E(B_i)$, $i \neq 1$ and there exists two vertices $w, w' \in V(C_p)$ such that v adjacent to both w and w'. Therefore, $f(vw) = f(vw') = 1$. Thus, $ew(G) \geq m - p + 2 > m - p$.*
Case 2: *$Z(G) = 2$. Let $u, v \in Z(G)$. We consider the following subcases.*
Case 2.1: *$u, v \in V(B_i)$, $i \neq 1, 2$ (say) and $uv \in E(G)$. Then $f(uv) = 0$ and $f(xy) = 1$ for all $xy \in E(B_i)$, $i \neq 1, 2$. If $f(xy) = 0$ for all $xy \in E(B_1)$, then $e(x) = e(y)$ for all $x, y \in V(B_1)$, which is a contradiction to the fact that G is unicyclic. Therefore, $f(xy) = 1$ for at least one edge $xy \in E(C_p)$. Thus, $ew(G) \geq m - p$.*
Case 2.2: *$u, v \in V(C_p)$. We consider the following subcases.*

Case 2.2.1: *$uv \in E(C_p)$. Then $f(uv) = 0$ and $f(xy) = 1$ for all $xy \in E(B_i)$, $i \neq 1$. Since G is unicyclic, it follows that there are at least two edges $xx', yy' \in E(C_p)$ such that $f(xx') = 1 = f(yy')$. Thus, $ew(G) \geq m - p$.*

Case 2.2.2: *$uv \notin E(C_p)$. Then $f(xy) = 1$ for all $xy \in E(B_i)$, $i \neq 1$ and there are two vertices x, y in C_p such that $f(ux) = 1 = f(vy)$. Therefore, $ew(G) \geq m - p$.*

Case 2.3: *$u, v \in Z(G)$ and $v \notin V(C_p)$. Then $uv \in E(B_2)$. Therefore, there exists at least one edge $ww' \in E(C_p)$ such that $f(ww') = 1$ and for every $xy \in E(B_i)$, $i \neq 1, 2$, $f(xy) = 1$. Thus, $ew(G) \geq m - p$.*

Case 3: *$|Z(G)| \geq 3$. We consider the following subcases.*

Case 3.1: *$|Z(G)| = p$. Then $V(C_p)$ is center in G. Therefore, $f(ww') = 1$ for all $ww' \in E(B_i)$, $i \neq 1$ and $f(uv) = 0$ for all $uv \in E(B_1)$. Hence, $ew(G) = m - p$.*

Case 3.2: *$|Z(G)| < p$. Then $f(xy) = 1$ for all $xy \in E(B_i)$, $i \neq 1$ and for at least one edge $uv \in E(B_1)$, we have $f(uv) = 1$. Therefore, $ew(G) \geq m - p$. Hence, from all the above cases, $ew(G) \geq m - p$. Thus, $m - p \leq ew(G)$.*

Theorem 3.9. *If G is self-centred graph, then $ew(G - v) \geq ew(G)$.*

Proof. *Let G be a self-centred graph. Then $ew(G) = 0$. If $G - v$ is still self-centred, then $ew(G-v) = ew(G) = 0$ and if $G-v$ is not self-centred, then $ew(G-v) > ew(G)$. Therefore, $ew(G - v) \geq ew(G)$.*

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