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SOME LINEAR GENERATING RELATIONS ASSOCIATED WITH EVEN AND ODD DEGREE POLYNOMIALS OF LAGRANGE AND SUBUHI

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Abstract. In this paper, we obtain some linear generating relations for even and odd degree polynomials of J. L. Lagrange $g_n^{(\alpha,\beta)}(x,y)$ and Subuhi Khan $H_{n,\alpha,\beta}(x)$, by using series decomposition technique, series rearrangement method and De Moivre's theorem.

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1. Introduction and Preliminaries

The name generating function was introduced by Laplace [14] in the year 1812. Since then the theory of generating functions has been developed into various directions and found wide applications in various branches of science and technology.

Most of the generating functions derived are the extensions and generalizations of the results known in one form or another in the theory of special functions. There is a vast literature on generating functions.

A generating functions may be used to define a set of functions, to determine a differential recurrence relation or a pure recurrence relation, to evaluate certain integrals, et-cetera.

A brief discussion of Laplace's work [14] on generating functions is also found in Doetsch [6-7]. Laplace [14] used not only generating series, but also generating integrals.

The Hermite, Laguerre, Legendre, Gegenbauer and Jacobi polynomials arise from the investigation of certain linear differential equations of the Sturm-Liouville type.

Throughout in present paper, we use the following standard notations:
$\mathbb{N} := \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 := \{0, 1, 2, 3, \ldots\} = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}^- := \{-1, -2, -3, \ldots\} = \mathbb{Z}_0^- \backslash \{0\}$.

Here, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ denotes the set of positive real numbers and $\mathbb{C}$ denotes the set of complex numbers.
The Pochhammer symbol (or the shifted factorial) $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ is defined, in terms of the familiar Gamma function, by

$$(1.1) \qquad (\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C}\backslash\{0\}) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}) \end{cases},$$

it being understood *conventionally* that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.

The generalized hypergeometric function of one variable with p numerator parameters and q denominator parameters is defined by [29, p.42 Eq.(1)]

$$(1.2) \qquad {}_pF_q\left[\begin{array}{l} \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q; \end{array} z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \ldots (\alpha_p)_n}{(\beta_1)_n (\beta_2)_n \ldots (\beta_q)_n} \frac{z^n}{n!}.$$

Here p and q are positive integers or zero (interpreting an empty product as 1), and we assume that the variable z, the numerator parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ and the denominator parameters $\beta_1, \beta_2, \ldots, \beta_q$ take on complex values, provided that $\beta_j \neq 0, -1, -2, \ldots;\ j = 1, 2, \ldots, q$.

Supposing that none of the numerator parameters is zero or a negative integer (otherwise the question of convergence will not arise), and with the usual restriction on β_j, the ${}_pF_q$ series in (1.2):
(i) converges for $|z| < \infty$ if $p \leq q$,
(ii) converges for $|z| < 1$, if $p = q + 1$,
(iii) diverges for all z, $z \neq 0$, if $p > q + 1$.

Furthermore, if we set

$$(1.3) \qquad \omega = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j,$$

it is known that the ${}_pF_q$ series, with $p = q + 1$, is
(I) absolutely convergent for $|z| = 1$, if $\Re(\omega) > 0$,
(II) conditionally convergent for $|z| = 1$, $|z| \neq 1$, if $-1 < \Re(\omega) \leq 0$,
(III) divergent for $|z| = 1$, if $\Re(\omega) \leq -1$.

The idea of separation of a power series into its even and odd terms, exhibited by the elementary identity

$$(1.4) \qquad \sum_{n=0}^{\infty} \Phi(n) = \sum_{n=0}^{\infty} \Phi(2n) + \sum_{n=0}^{\infty} \Phi(2n+1)\ ,$$

is at least as old as the series themselves. Indeed, when (1.4) is applied to the generalized hypergeometric series (1.2), we are led rather immediately to the elegant result of Barr[1, p.591 Eq.(1)] subject to the suitable convergence condition,

$$ {}_pF_q\left[\begin{array}{c} \alpha_1,\alpha_2,\ldots,\alpha_p; \\ \beta_1,\beta_2,\ldots,\beta_q; \end{array} z\right] = {}_{2p}F_{2q+1}\left[\begin{array}{c} \frac{\alpha_1}{2},\frac{1+\alpha_1}{2},\ldots,\frac{\alpha_p}{2},\frac{1+\alpha_p}{2}; \\ \frac{1}{2},\frac{\beta_1}{2},\frac{1+\beta_1}{2},\ldots,\frac{\beta_q}{2},\frac{1+\beta_q}{2}; \end{array} \frac{z^2}{4^{1-p+q}}\right] + $$

$$ +z\frac{\alpha_1\ldots\alpha_p}{\beta_1\ldots\beta_q}{}_{2p}F_{2q+1}\left[\begin{array}{c} \frac{1+\alpha_1}{2},\frac{2+\alpha_1}{2},\ldots,\frac{1+\alpha_p}{2},\frac{2+\alpha_p}{2}; \\ \frac{3}{2},\frac{1+\beta_1}{2},\frac{2+\beta_1}{2},\ldots,\frac{1+\beta_q}{2},\frac{2+\beta_q}{2}; \end{array} \frac{z^2}{4^{1-p+q}}\right]. \tag{1.5} $$

The classical Hermite's polynomials $H_n(x)$ are defined by means of the following linear generating relation [23, p.187 eq(1)]

$$ \exp(2xt-t^2) = \sum_{n=0}^{\infty}\frac{H_n(x)t^n}{n!}, \tag{1.6} $$

which is valid for all finite values of x and t.
The series expansion and hypergeometric forms of classical Hermite polynomials are given below

$$ H_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k n!(2x)^{n-2k}}{k!(n-2k)!}\ , \tag{1.7} $$

$$ H_n(x) = (2x)^n{}_2F_0\left[\begin{array}{c} -\frac{n}{2},-\frac{n}{2}+\frac{1}{2}; \\ -\ ; \end{array} -\frac{1}{x^2}\right]\ , \tag{1.8} $$

$$ H_{2n}(x) = (2x)^{2n}{}_2F_0\left[\begin{array}{c} -n,\ \frac{1}{2}-n\ ; \\ -\ ; \end{array} -\frac{1}{x^2}\right]\ , \tag{1.9} $$

$$ H_{2n+1}(x) = (2x)^{2n+1}{}_2F_0\left[\begin{array}{c} -n,-\frac{1}{2}-n; \\ -\ ; \end{array} -\frac{1}{x^2}\right]\ . \tag{1.10} $$

The hypergeometric form of Binomial expansion $(1-z)^{-a}$ is given by

$$ (1-z)^{-a} = {}_1F_0\left[\begin{array}{c} a\ ; \\ -\ ; \end{array} z\right] = \sum_{r=0}^{\infty}\frac{(a)_r z^r}{r!}\ , \tag{1.11} $$

where $|z|<1$, and $a\neq$ 0,-1,-2,-3,...
Modulus and amplitude form of a complex number:
Suppose

$$ A+iB = \lambda\exp(i\mu) = \lambda\cos\mu + i\lambda\sin\mu\ , \tag{1.12} $$

then modulus

$$\lambda = \sqrt{(A^2 + B^2)} \tag{1.13}$$

and the amplitude μ is given by any one of the following

$$\mu := \begin{cases} \arctan\left(\frac{B}{A}\right) & (A, B \in \mathbb{R}_+) \\ -\arctan\left(\frac{|B|}{A}\right) & (A \in \mathbb{R}_+; B \in \mathbb{R}_-) \\ \pi - \arctan\left(\frac{B}{|A|}\right) & (A \in \mathbb{R}_-; B \in \mathbb{R}_+) \\ -\pi + \arctan\left(\frac{|B|}{|A|}\right) & (A, B \in \mathbb{R}_-) \ . \end{cases} \tag{1.14}$$

Motivated by the work of Carlson [2], MacRobert [15-16], Manocha [17], Mohd. et al. [18], Qureshi and Ahmad [20], Qureshi, Quraishi and Pal [21], Qureshi, Yasmeen and Pathan [22], Sharma [24-26] and Srivastava [27], we shall obtain some linear generating relations associated with polynomials of even and odd degree.

Some series iteration formulas [29, pp.100-102] are given below

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \Phi(n,m) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \Phi(n-m,m) \ , \tag{1.15}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\left[\frac{n}{2}\right]} \Phi(n-2m,m) \ , \tag{1.16}$$

where $[x]$ denotes the greatest integer function and series involved are absolutely convergent.

2. Hypergeometric Forms of Subuhi Khan Polynomials

$$H_{n,\alpha,\beta}(x) = \left(\frac{x}{\alpha}\right)^n {}_2F_0\left[\begin{matrix} \frac{-n}{2}, \frac{-n+1}{2}; & \frac{-4\alpha^2\beta}{x^2} \\ - \quad ; & \end{matrix}\right] \ , \tag{2.1}$$

$$H_{2n,\alpha,\beta}(x) = \left(\frac{x}{\alpha}\right)^{2n} {}_2F_0\left[\begin{matrix} -n, -n+\frac{1}{2}; & \frac{-4\alpha^2\beta}{x^2} \\ - \quad ; & \end{matrix}\right] \ , \tag{2.2}$$

$$H_{2n+1,\alpha,\beta}(x) = \left(\frac{x}{\alpha}\right)^{2n+1} {}_2F_0\left[\begin{matrix} -n, -n-\frac{1}{2}; & \frac{-4\alpha^2\beta}{x^2} \\ - \quad ; & \end{matrix}\right] \ . \tag{2.3}$$

Proof:

Subuhi Khan polynomials [12, p.84 (4.2.8, 4.2.9); see also 19, p.60 (2.9)] are generated by the following generating relation

$$\sum_{n=0}^{\infty} H_{n,\alpha,\beta}(x)\frac{t^n}{n!} = exp\left(\frac{xt}{\alpha}\right) exp(-\beta t^2), \tag{2.4}$$

which is valid for all finite values of α, β, x and t.

$$\begin{aligned} e^{\left(\frac{xt}{\alpha}\right)} e^{(-\beta t^2)} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\left(\frac{xt}{\alpha}\right)^n}{n!}\frac{(-\beta\ t^2)^m}{m!}, \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\left(\frac{x}{\alpha}\right)^n}{n!}\frac{(-\beta)^m}{m!} t^{n+2m}. \end{aligned} \tag{2.5}$$

Replacing n by $n-2m$ and using series iteration formula (1.16), we get

$$e^{\left(\frac{xt}{\alpha}\right)} e^{(-\beta t^2)} = \sum_{n=0}^{\infty}\sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{\left(\frac{x}{\alpha}\right)^{n-2m}}{(n-2m)!}\frac{(-\beta)^m}{m!} t^n\ , \tag{2.6}$$

$$\sum_{n=0}^{\infty} H_{n,\alpha,\beta}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left\{ \left(\frac{x}{\alpha}\right)^n \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{2^{2m}\left(\frac{-n}{2}\right)_m \left(\frac{-n+1}{2}\right)_m \left(\frac{-\alpha^2\beta}{x^2}\right)^m}{m!} \right\}\frac{t^n}{n!}\ . \tag{2.7}$$

Comparing the coefficient of t^n in both sides of equation (2.7) , we get hypergeometric form (2.1).

Put $\alpha = \frac{1}{2}, \beta = 1$ in (2.1), (2.2) and (2.3), we get Classical Hermite's polynomials, given by the equations (1.8), (1.9) and (1.10) respectively.

3. Hypergeometric Form of Lagrange's Polynomials in Two Variables

$$g_n^{(\alpha,\beta)}(x,y) = \frac{x^n(\alpha)_n}{n!}\,{}_2F_1\left[\begin{array}{rr} -n, & \beta;\ \frac{y}{x} \\ 1-\alpha-n\ ; & \end{array}\right]\ , \tag{3.1}$$

$$= \frac{y^n(\beta)_n}{n!}\,{}_2F_1\left[\begin{array}{rr} -n, & \alpha;\ \frac{x}{y} \\ 1-\beta-n\ ; & \end{array}\right]\ , \tag{3.2}$$

$$= g_n^{(\beta,\alpha)}(y,x)\ . \tag{3.3}$$

Proof:

The familiar (classical two-variable) polynomials $g_n^{(\alpha,\beta)}(x,y)$ generated by

$$(1-xt)^{-\alpha}(1-yt)^{-\beta}=\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n \ , \tag{3.4}$$

$$(|t|<min\{|x|^{-1},|y|^{-1}\}) \ ,$$

are known as J. L. Lagrange's polynomials [13; see also 8, p. 267 (19.11.1, 19.11.2)] in two variables which occur in certain problems of statistics.

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n=(1-xt)^{-\alpha}(1-yt)^{-\beta} \quad ; \quad |xt|<1, |yt|<1 \ , \tag{3.5}$$

$$= {}_1F_0\left[\begin{matrix}\alpha; \\ -;\end{matrix} \ xt\right] {}_1F_0\left[\begin{matrix}\beta; \\ -;\end{matrix} \ yt\right] \ , \tag{3.6}$$

$$=\sum_{n=0}^{\infty}\frac{(\alpha)_n x^n t^n}{n!}\sum_{m=0}^{\infty}\frac{(\beta)_m y^m t^m}{m!} \ , \tag{3.7}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha)_n x^n}{n!}\frac{(\beta)_m y^m t^{n+m}}{m!} \ . \tag{3.8}$$

Replacing n by $n-m$ and using series iteration formula (1.15), we get

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(\alpha)_{n-m}x^{n-m}}{(n-m)!}\frac{(\beta)_m y^m t^n}{(m)!} \ , \tag{3.9}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(\alpha)_n(\alpha+n)_{-m}x^n}{n!}\frac{(\beta)_m(-n)_m}{(-1)^m m!}\left(\frac{y}{x}\right)^m t^n \ , \tag{3.10}$$

$$=\sum_{n=0}^{\infty}\left[\frac{x^n(\alpha)_n}{n!}\sum_{m=0}^{n}\frac{(-1)^m(\beta)_m(-n)_m}{(-1)^m m!(1-\alpha-n)_m}\left(\frac{y^m}{x^m}\right)\right]t^n \ , \tag{3.11}$$

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n=\sum_{n=0}^{\infty}\left\{\frac{x^n(\alpha)_n}{n!}{}_2F_1\left[\begin{matrix}-n, & \beta; \\ 1-\alpha-n \ ; & \end{matrix} \ \frac{y}{x}\right]\right\}t^n \ . \tag{3.12}$$

Now equating the coefficient of t^n in equation (3.12), we get the hypergeometric form (3.1).
Again consider same generating relation:

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n=(1-xt)^{-\alpha}(1-yt)^{-\beta} \ , \tag{3.13}$$

$$= {}_1F_0\left[\begin{array}{c}\beta;\\ -;\end{array} yt\right] {}_1F_0\left[\begin{array}{c}\alpha;\\ -;\end{array} xt\right] , \tag{3.14}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\alpha)_m x^m t^m}{m!}\frac{(\beta)_n y^n t^n}{n!} . \tag{3.15}$$

Replacing n by $n-m$ and using series iteration formula (1.15), we get

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(\alpha)_m x^m}{m!}\frac{(\beta)_{n-m} y^{n-m} t^n}{(n-m)!} , \tag{3.16}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(\alpha)_m x^m}{m!}\frac{(\beta)_n(\beta+n)_{-m} y^n y^{-m} t^n(-n)_m}{n!(-1)^m} , \tag{3.17}$$

$$= \sum_{n=0}^{\infty}\left[\frac{y^n(\beta)_n}{n!}\sum_{m=0}^{n}\frac{(\alpha)_m(-n)_m(-1)^m}{(-1)^m m!(1-\beta-n)_m}\left(\frac{x^m}{y^m}\right)\right] t^n , \tag{3.18}$$

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n = \sum_{n=0}^{\infty}\left\{\frac{y^n(\beta)_n}{n!}{}_2F_1\left[\begin{array}{cc}-n, & \alpha;\\ 1-\beta-n\ ; & \end{array}\frac{x}{y}\right]\right\} t^n . \tag{3.19}$$

Now equating the coefficient of t^n in equation (3.19), we get the hypergeometric form (3.2). For detailed study on Lagrange's polynomials of several variables, we can refer the papers [3, 4, 5, 9, 10, 11].

The relationship can be used in order to reduce numerous properties and characteristics of the (two-variable) Lagrange's polynomials from those of the classical Jacobi polynomials [29, p.442 (8.5.17)]

$$g_n^{(\alpha,\beta)}(x,y) = (y-x)^n P_n^{(-\alpha-n,-\beta-n)}\left(\frac{x+y}{x-y}\right) \tag{3.20}$$

A transformation formula for Jacobi polynomial [29, p.441 (8.5.16)] is given by

$$P_n^{(\alpha-n,\beta-n)}(x) = g_n^{(-\alpha,-\beta)}\left(-\frac{x+1}{2}, -\frac{x-1}{2}\right) \tag{3.21}$$

A transformation formula [29, p.452 Q.25; see also 28, p.318 (103)] is given by

$$g_n^{(\alpha,\beta)}(x,y) = y^n P_n^{(\alpha+\beta-1,-\beta-n)}\left(\frac{2x-y}{y}\right) \tag{3.22}$$

Put $y = \frac{2x}{x+1}$ in above equation, we get

$$P_n^{(\alpha+\beta-1,-\beta-n)}(x) = \left(\frac{x+1}{2x}\right)^n g_n^{(\alpha,\beta)}\left(x, \frac{2x}{x+1}\right) \tag{3.23}$$

The relationships [4, p.254 (3.5), p.255 (3.9)] for two-variable Lagrange polynomials $g_n^{(\alpha,\beta)}(x,y)$ are given by

$$g_n^{(\alpha,\beta-n)}(x,y) = g_n^{(\alpha,-\alpha-\beta+1)}(x-y,-y) \tag{3.24}$$

$$g_n^{(\alpha,\beta-n)}(x,y) = g_n^{(\alpha,-\alpha-\beta+1)}(x-y,-y) = (x)^n P_n^{(-\alpha-n,\alpha+\beta-n-1)}\left(\frac{x-2y}{x}\right)$$

$$= (x)^n P_n^{(\alpha+\beta-n-1,-\alpha-n)}\left(\frac{2y-x}{x}\right) \tag{3.25}$$

$$P_n^{(\alpha+\beta-n-1,-\alpha-n)}(x) = x^{-n} g_n^{(\alpha,\beta-n)}\left(x, \frac{x(x+1)}{2}\right) \tag{3.26}$$

4. Linear Generating Relations Involving Subuhi Khan Polynomials

Any values of parameters and variables leading to the results which do not make sense, are tacitly excluded (suppose α, β and x are real numbers), then

$$\sum_{n=0}^{\infty} H_{2n,\alpha,\beta}(x)\frac{t^n}{(2n)!} = \cosh\left(\frac{x\sqrt{t}}{\alpha}\right)\left\{ {}_0F_1\left[\begin{matrix} - ; & \frac{\beta^2 t^2}{4} \\ \frac{1}{2} ; & \end{matrix}\right] - \beta t\ {}_0F_1\left[\begin{matrix} - ; & \frac{\beta^2 t^2}{4} \\ \frac{3}{2} ; & \end{matrix}\right]\right\}, \tag{4.1}$$

(4.2)
$$\sum_{n=0}^{\infty} H_{2n+1,\alpha,\beta}(x)\frac{t^n}{(2n+1)!} = \frac{1}{\sqrt{t}}\sinh\left(\frac{x\sqrt{t}}{\alpha}\right)\left\{ {}_0F_1\left[\begin{matrix} - ; & \frac{\beta^2 t^2}{4} \\ \frac{1}{2} ; & \end{matrix}\right] - \beta t\ {}_0F_1\left[\begin{matrix} - ; & \frac{\beta^2 t^2}{4} \\ \frac{3}{2} ; & \end{matrix}\right]\right\},$$

$$\sum_{n=0}^{\infty} H_{n,\alpha,\beta}(x)\frac{\cos(n\theta)}{n!} = exp\left(\frac{x\cos\theta}{\alpha} - \beta\cos(2\theta)\right)\cos\left(\frac{x\sin\theta}{\alpha} - \beta\sin(2\theta)\right), \tag{4.3}$$

$$\sum_{n=0}^{\infty} H_{n,\alpha,\beta}(x)\frac{\sin(n\theta)}{n!} = exp\left(\frac{x\cos\theta}{\alpha} - \beta\cos(2\theta)\right)\sin\left(\frac{x\sin\theta}{\alpha} - \beta\sin(2\theta)\right). \tag{4.4}$$

Proof:

Consider the generating relation (2.4) given by Subuhi

$$\sum_{n=0}^{\infty} H_{n,\alpha,\beta}(x)\frac{t^n}{n!} = \exp\left(\frac{xt}{\alpha}\right) exp(-\beta t^2), \tag{4.5}$$

$$= \exp\left(\frac{xt}{\alpha}\right)\sum_{r=o}^{\infty}\frac{(-\beta t^2)^r}{r!}. \tag{4.6}$$

Apply series decomposition formula (1.4) in equation (4.6), we get

(4.7)

$$\sum_{n=0}^{\infty} H_{2n,\alpha,\beta}(x)\frac{t^{2n}}{(2n)!}+\sum_{n=0}^{\infty} H_{2n+1,\alpha,\beta}(x)\frac{t^{2n+1}}{(2n+1)!} = exp\left(\frac{xt}{\alpha}\right)\left[\sum_{r=o}^{\infty}\frac{(-\beta t^2)^{2r}}{(2r)!}+\sum_{r=o}^{\infty}\frac{(-\beta t^2)^{2r+1}}{(2r+1)!}\right] .$$

Put $t = iT$, or $t^2 = -T^2$, we get

$$\sum_{n=0}^{\infty} H_{2n,\alpha,\beta}(x)\frac{(-T^2)^n}{(2n)!} + iT\sum_{n=0}^{\infty} H_{2n+1,\alpha,\beta}(x)\frac{(-T^2)^n}{(2n+1)!}$$

$$= exp\left(\frac{ixT}{\alpha}\right)\left[\sum_{r=o}^{\infty}\frac{(\beta T^2)^{2r}}{(2r)!}+\sum_{r=o}^{\infty}\frac{(\beta T^2)^{2r+1}}{(2r+1)!}\right] . \tag{4.8}$$

Suppose α, β and x are real numbers then equating real and imaginary parts in equation (4.8), we get

$$\sum_{n=0}^{\infty} H_{2n,\alpha,\beta}(x)\frac{(-T^2)^n}{(2n)!} = \cos\left(\frac{xT}{\alpha}\right)\left[\sum_{r=o}^{\infty}\frac{(\beta^2T^4)^r}{2^{2r}r!(\frac{1}{2})_r} + (\beta T^2)\sum_{r=o}^{\infty}\frac{(\beta^2T^4)^r}{2^{2r}r!(\frac{3}{2})_r}\right] , \tag{4.9}$$

$$\sum_{n=0}^{\infty} H_{2n+1,\alpha,\beta}(x)\frac{(-T^2)^n}{(2n+1)!} = \frac{1}{T}\sin\left(\frac{xT}{\alpha}\right)\left[\sum_{r=o}^{\infty}\frac{(\beta^2T^4)^r}{2^{2r}r!(\frac{1}{2})_r} + (\beta T^2)\sum_{r=o}^{\infty}\frac{(\beta^2T^4)^r}{2^{2r}r!(\frac{3}{2})_r}\right] . \tag{4.10}$$

Put $T = i\sqrt{t}$ or $T^2 = -t$, we get the generating relations (4.1) and (4.2).

Now consider same generating relation (2.4)

$$\sum_{n=0}^{\infty} H_{n,\alpha,\beta}(x)\frac{t^n}{n!} = exp\left(\frac{xt}{\alpha}\right) exp(-\beta t^2) , \tag{4.11}$$

$$= exp\left(\frac{xt}{\alpha} - \beta t^2\right) . \tag{4.12}$$

Put $t = \cos\theta + i\sin\theta = e^{i\theta}$ in both sides of equation (4.12) and apply De Moivre's theorem, we get

$$\sum_{n=0}^{\infty} H_{n,\alpha,\beta}(x)\frac{\cos(n\theta)+i\sin(n\theta)}{n!} = exp\left(\frac{x(\cos\theta+i\sin\theta)}{\alpha} - \beta\{\cos(2\theta)+i\sin(2\theta)\}\right) \tag{4.13}$$

$$\sum_{n=0}^{\infty} H_{n,\alpha,\beta}(x)\frac{\cos(n\theta)}{n!} + i\sum_{n=0}^{\infty} H_{n,\alpha,\beta}(x)\frac{\sin(n\theta)}{n!}$$

$$= \exp\left(\frac{x\cos\theta}{\alpha} - \beta\cos(2\theta)\right)\exp\left[i\left(\frac{x\sin\theta}{\alpha} - \beta\sin(2\theta)\right)\right] \tag{4.14}$$

Suppose α, β and x are real numbers then equating real and imaginary parts in equation (4.14), we get generating relations (4.3) and (4.4).

5. Linear Generating Relations Associated with Lagrange's Polynomials in Two Variables

Any values of parameters and variables leading to the results which do not make sense, are tacitly excluded (suppose α, β, x and y are real numbers), then

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)\cos\,(n\theta) = \frac{1}{\left(\sqrt{(1+x^2-2x\cos\theta)}\right)^{\alpha}\left(\sqrt{(1+y^2-2y\cos\theta)}\right)^{\beta}}\times$$

$$\times\cos\left\{\alpha\tan^{-1}\left(\frac{x\sin\theta}{x\cos\theta-1}\right)+\beta\tan^{-1}\left(\frac{y\sin\theta}{y\cos\theta-1}\right)\right\}\ , \tag{5.1}$$

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)\sin\,(n\theta) = \frac{-1}{\left(\sqrt{(1+x^2-2x\cos\theta)}\right)^{\alpha}\left(\sqrt{(1+y^2-2y\cos\theta)}\right)^{\beta}}\times$$

$$\times\sin\left\{\alpha\tan^{-1}\left(\frac{x\sin\theta}{x\cos\theta-1}\right)+\beta\tan^{-1}\left(\frac{y\sin\theta}{y\cos\theta-1}\right)\right\}\ , \tag{5.2}$$

$$\sum_{n=0}^{\infty} g_{2n}^{(\alpha,\beta)}(x,y)t^n$$

$$= {}_2F_1\left[\begin{array}{c}\frac{\alpha}{2},\frac{\alpha+1}{2};\\ \frac{1}{2}\quad ;\end{array} x^2t\right]{}_2F_1\left[\begin{array}{c}\frac{\beta}{2},\frac{\beta+1}{2};\\ \frac{1}{2}\quad ;\end{array} y^2t\right]+\alpha\,\beta\,x\,y\,t\,{}_2F_1\left[\begin{array}{c}\frac{\alpha+1}{2},\frac{\alpha+2}{2};\\ \frac{3}{2}\quad ;\end{array} x^2t\right]{}_2F_1\left[\begin{array}{c}\frac{\beta+1}{2},\frac{\beta+2}{2};\\ \frac{3}{2}\quad ;\end{array} y^2t\right] \tag{5.3}$$

,

$$\sum_{n=0}^{\infty} g_{2n+1}^{(\alpha,\beta)}(x,y)t^n$$

$$= \beta y\ {}_2F_1\left[\begin{array}{c}\frac{\alpha}{2},\frac{\alpha+1}{2};\\ \frac{1}{2}\quad ;\end{array} x^2t\right]{}_2F_1\left[\begin{array}{c}\frac{\beta+1}{2},\frac{\beta+2}{2};\\ \frac{3}{2}\quad ;\end{array} y^2t\right]+\alpha x\ {}_2F_1\left[\begin{array}{c}\frac{\alpha+1}{2},\frac{\alpha+2}{2};\\ \frac{3}{2}\quad ;\end{array} x^2t\right]{}_2F_1\left[\begin{array}{c}\frac{\beta}{2},\frac{\beta+1}{2};\\ \frac{1}{2}\quad ;\end{array} y^2t\right] \tag{5.4}$$

.

Making suitable adjustment of parameters and variables in the relationships from (3.20) to (3.26) all generating relations from (5.1) to (5.4), can be written in terms of restricted classical Jacobi polynomials.

Proof:

Consider the generating relation for Lagrange's polynomials of two variables:-

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n = (1-xt)^{-\alpha}(1-yt)^{-\beta}\ . \tag{5.5}$$

Put $t=\cos\theta+i\sin\theta$ in both sides of equation (5.5) and apply De Moivre's theorem, we get

(5.6) $\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)(\cos\ n\theta + i \sin\ n\theta) = \{(1-x\cos\theta)-ix\sin\theta\}^{-\alpha}\{(1-y\cos\theta)-iy\sin\theta\}^{-\beta}$.

Let $(1 - x\cos\theta = r\cos\phi)$, $(-x\sin\theta = r\sin\phi)$; $(1 - y\cos\theta = R\cos\Psi)$, $(-y\sin\theta = R\sin\Psi)$, and assume that $1 - x\cos\theta$, $-x\sin\theta$, $1 - y\cos\theta$ and $-y\sin\theta$ are positive real numbers, then we have

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)(\cos\ n\theta + i\sin\ n\theta) = \{r\exp(i\phi)\}^{-\alpha}\{R\exp(i\Psi)\}^{-\beta} \ , \tag{5.7}$$

$$= r^{-\alpha}R^{-\beta}\{\cos(\alpha\phi + \beta\Psi) - i\sin(\alpha\phi + \beta\Psi)\} \ , \tag{5.8}$$

where r, ϕ, R and Ψ are given by the following equations

$$r = \sqrt{(1 + x^2 - 2x\cos\theta)} \ , \tag{5.9}$$

$$\phi = \arctan\left(\frac{-x\sin\theta}{1 - x\cos\theta}\right) \ , \tag{5.10}$$

$$R = \sqrt{(1 + y^2 - 2y\cos\theta)} \ , \tag{5.11}$$

$$\Psi = \arctan\left(\frac{-y\sin\theta}{1 - y\cos\theta}\right) \ . \tag{5.12}$$

Suppose α, β, x and y are real numbers then equating real, imaginary parts in equation (5.8) and using (5.9), (5.10), (5.11) and (5.12), we get generating relations (5.1) and (5.2) respectively.

Again consider the generating relation for Lagrange's polynomial of two variable:-

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)t^n = (1 - xt)^{-\alpha}(1 - yt)^{-\beta} \ , \tag{5.13}$$

$$= {}_1F_0\left[\begin{array}{c}\alpha; \\ -;\end{array} xt\right]{}_1F_0\left[\begin{array}{c}\beta; \\ -;\end{array} yt\right] \ , \tag{5.14}$$

$$= \sum_{p=0}^{\infty}(\alpha)_p\frac{(xt)^p}{p!}\sum_{q=0}^{\infty}(\beta)_q\frac{(yt)^q}{q!} \ . \tag{5.15}$$

Now applying series decomposition formula (1.4) in equation (5.15), we get

$$\sum_{n=0}^{\infty} g_{2n}^{(\alpha,\beta)}(x,y)t^{2n} + \sum_{n=0}^{\infty} g_{2n+1}^{(\alpha,\beta)}(x,y)t^{2n+1}$$

(5.16)

$$= \left\{ \sum_{p=0}^{\infty} (\alpha)_{2p} \frac{(xt)^{2p}}{(2p)!} + \sum_{p=0}^{\infty} (\alpha)_{2p+1} \frac{(xt)^{2p+1}}{(2p+1)!} \right\} \left\{ \sum_{q=0}^{\infty} (\beta)_{2q} \frac{(yt)^{2q}}{(2q)!} + \sum_{q=0}^{\infty} (\beta)_{2q+1} \frac{(yt)^{2q+1}}{(2q+1)!} \right\} .$$

Put $t = iT$ or $t^2 = -T^2$, we have

$$\sum_{n=0}^{\infty} g_{2n}^{(\alpha,\beta)}(x,y)(-T^2)^n + iT \sum_{n=0}^{\infty} g_{2n+1}^{(\alpha,\beta)}(x,y)(-T^2)^n$$

$$= \left\{ \sum_{p=0}^{\infty} \frac{2^{2p} \left(\frac{\alpha}{2}\right)_p \left(\frac{\alpha+1}{2}\right)_p}{2^{2p} \left(\frac{1}{2}\right)_p} \frac{x^{2p}(-T^2)^p}{p!} + iT\alpha x \sum_{p=0}^{\infty} \frac{2^{2p} \left(\frac{\alpha+1}{2}\right)_p \left(\frac{\alpha+2}{2}\right)_p}{2^{2p} \left(\frac{3}{2}\right)_p} \frac{x^{2p}(-T^2)^p}{p!} \right\} \times$$

$$\times \left\{ \sum_{q=0}^{\infty} \frac{2^{2q} \left(\frac{\beta}{2}\right)_q \left(\frac{\beta+1}{2}\right)_q}{2^{2q} \left(\frac{1}{2}\right)_q} \frac{y^{2q}(-T^2)^q}{q!} + iT\beta y \sum_{q=0}^{\infty} \frac{2^{2q} \left(\frac{\beta+1}{2}\right)_q \left(\frac{\beta+2}{2}\right)_q}{2^{2q} \left(\frac{3}{2}\right)_q} \frac{y^{2q}(-T^2)^q}{q!} \right\} , \tag{5.17}$$

$$= \left\{ {}_2F_1 \left[\begin{array}{c} \frac{\alpha}{2}, \frac{\alpha+1}{2}; \\ \frac{1}{2} \quad ; \end{array} -x^2T^2 \right] + iT\alpha x {}_2F_1 \left[\begin{array}{c} \frac{\alpha+1}{2}, \frac{\alpha+2}{2}; \\ \frac{3}{2} \quad ; \end{array} -x^2T^2 \right] \right\} \times$$

$$\times \left\{ {}_2F_1 \left[\begin{array}{c} \frac{\beta}{2}, \frac{\beta+1}{2}; \\ \frac{1}{2} \quad ; \end{array} -y^2T^2 \right] + iT\beta y {}_2F_1 \left[\begin{array}{c} \frac{\beta+1}{2}, \frac{\beta+2}{2}; \\ \frac{3}{2} \quad ; \end{array} -y^2T^2 \right] \right\} , \tag{5.18}$$

$$= {}_2F_1 \left[\begin{array}{c} \frac{\alpha}{2}, \frac{\alpha+1}{2}; \\ \frac{1}{2} \quad ; \end{array} -x^2T^2 \right] {}_2F_1 \left[\begin{array}{c} \frac{\beta}{2}, \frac{\beta+1}{2}; \\ \frac{1}{2} \quad ; \end{array} -y^2T^2 \right]$$

$$-\alpha\beta xy \, T^2 {}_2F_1 \left[\begin{array}{c} \frac{\alpha+1}{2}, \frac{\alpha+2}{2}; \\ \frac{3}{2} \, ; \end{array} -x^2T^2 \right] {}_2F_1 \left[\begin{array}{c} \frac{\beta+1}{2}, \frac{\beta+2}{2}; \\ \frac{3}{2} \, ; \end{array} -y^2T^2 \right]$$

$$+iT\beta y \, {}_2F_1 \left[\begin{array}{c} \frac{\alpha}{2}, \frac{\alpha+1}{2}; \\ \frac{1}{2} \quad ; \end{array} -x^2T^2 \right] {}_2F_1 \left[\begin{array}{c} \frac{\beta+1}{2}, \frac{\beta+2}{2}; \\ \frac{3}{2} \quad ; \end{array} -y^2T^2 \right]$$

$$+iT\alpha x {}_2F_1 \left[\begin{array}{c} \frac{\alpha+1}{2}, \frac{\alpha+2}{2}; \\ \frac{3}{2} \quad ; \end{array} -x^2T^2 \right] {}_2F_1 \left[\begin{array}{c} \frac{\beta}{2}, \frac{\beta+1}{2}; \\ \frac{1}{2} \quad ; \end{array} -y^2T^2 \right] . \tag{5.19}$$

Suppose α, β, x and y are real numbers then equating real and imaginary parts in equation (5.19), we have

$$\sum_{n=0}^{\infty} g_{2n}^{(\alpha,\beta)}(x,y)(-T^2)^n$$

$$= {}_2F_1 \left[\begin{array}{c} \frac{\alpha}{2}, \frac{\alpha+1}{2}; \\ \frac{1}{2} \quad ; \end{array} -x^2T^2 \right] {}_2F_1 \left[\begin{array}{c} \frac{\beta}{2}, \frac{\beta+1}{2}; \\ \frac{1}{2} \quad ; \end{array} -y^2T^2 \right]$$

$$-\alpha\beta xyT^2 \, {}_2F_1 \left[\begin{array}{c} \frac{\alpha+1}{2}, \frac{\alpha+2}{2}; \\ \frac{3}{2} \quad ; \end{array} -x^2T^2 \right] {}_2F_1 \left[\begin{array}{c} \frac{\beta+1}{2}, \frac{\beta+2}{2}; \\ \frac{3}{2} \quad ; \end{array} -y^2T^2 \right] , \tag{5.20}$$

$$\sum_{n=0}^{\infty} g_{2n+1}^{(\alpha,\beta)}(x,y)(-T^2)^n$$

$$= \beta y\, {}_2F_1\left[\begin{matrix} \frac{\alpha}{2}, \frac{\alpha+1}{2}; \\ \frac{1}{2} \quad ; \end{matrix} -x^2T^2\right] {}_2F_1\left[\begin{matrix} \frac{\beta+1}{2}, \frac{\beta+2}{2}; \\ \frac{3}{2} \quad ; \end{matrix} -y^2T^2\right]$$

$$+\alpha x\, {}_2F_1\left[\begin{matrix} \frac{\alpha+1}{2}, \frac{\alpha+2}{2}; \\ \frac{3}{2} \quad ; \end{matrix} -x^2T^2\right] {}_2F_1\left[\begin{matrix} \frac{\beta}{2}, \frac{\beta+1}{2}; \\ \frac{1}{2} \quad ; \end{matrix} -y^2T^2\right]. \tag{5.21}$$

Put $T = i\sqrt{t}$ or $T^2 = -t$ in equations (5.20) and (5.21), we get generating relations (5.3) and (5.4) respectively.

6. Special cases

: Put $\alpha = \frac{1}{2}, \beta = 1$ in (4.1), (4.2), (4.3), (4.4) and using hypergeometric series decomposition formula (1.5), we get

$$\sum_{n=0}^{\infty} H_{2n}(x)\frac{t^n}{(2n)!} = e^{-t}{}_0F_1\left[\begin{matrix} - \; ; \\ \frac{1}{2} \; ; \end{matrix} x^2t\right] = e^{-t}\cosh(2x\sqrt{t}) \;, \tag{6.1}$$

$$\sum_{n=0}^{\infty} H_{2n+1}(x)\frac{t^n}{(2n+1)!} = 2xe^{-t}{}_0F_1\left[\begin{matrix} - \; ; \\ \frac{3}{2} \; ; \end{matrix} x^2t\right] = \frac{e^{-t}}{\sqrt{t}}\sinh(2x\sqrt{t}) \;. \tag{6.2}$$

$$\sum_{n=0}^{\infty} H_n(x)\frac{\cos(n\theta)}{n!} = \exp\{2x\cos\theta - \cos(2\theta)\}\cos\{2x\sin\theta - \sin(2\theta)\} \;, \tag{6.3}$$

$$\sum_{n=0}^{\infty} H_n(x)\frac{\sin(n\theta)}{n!} = \exp\{2x\cos\theta - \cos(2\theta)\}\sin\{2x\sin\theta - \sin(2\theta)\} \;. \tag{6.4}$$

The generating relations (6.1) and (6.2) are the known results of Sharma [26, p.133 Eq.(18-19)].

We conclude our present investigation, by observing that several linear generating relations can be obtained from known generating relations, in analogous manner.

References

[1] Barr, G. E.; The integration of generalized hypergeometric functions, *Math. Proc. Cambridge Philos. Soc.*, 65(3)(1969), 591-595.

[2] Carlson, B. C.; Some extensions of Lardner's relations between ${}_0F_3$ and Bessel functions, *SIAM J. Math. Anal.*, 1(1970), 232-242.

[3] Chan, W.-C. C., Chyan, C.-J. and Srivastava, H. M.; The Lagrange polynomials in several variables, *Integral Transforms and Spec. Funct.*, 12(2)(2001), 139-148.

[4] Chen, K.-Y., Liu, S.-J. and Srivastava, H. M.; Some new results for the Lagrange polynomials in several variables, *ANZIAM J.*, 49(2)(2007), 243-258.

[5] Chen, K.-Y. and Srivastava, H. M.; A new result for hypergeometric polynomials, *Proc.Amer. Math. Soc.*, 133(11)(2005), 3295-3302.

[6] Doetsch, G.; *Einfuhrung in Theorie und Anwendung der Laplace-Transformation*, Springer-Verlag Berlin, 1937.

[7] Doetsch, G.; *Introduction to the Theory and Application of the Laplace Transformation*, Translation of the German Original by Walter Nader, Springer-Verlag Berlin Heidelberg New York 1974.

[8] Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G.; *Higher Transcendental Functions*, Vol. III, McGraw -Hill Book Company, New York, Toronto and London, 1955.

[9] Erkus, E. and Altin, A.; A note on the Lagrange polynomials in several variables, *J. Math. Anal. Appl.*, 310(2005), 338-341.

[10] Erkus, E. and Srivastava, H. M.; A unified presentation of some families of multivariable polynomials, *Integral Transforms and Spec. Funct.*, 17(4)(2006), 267-273.

[11] Khan, M. A. and Shukla, A. K.; On Lagrange's polynomials of three variables, *Proyecciones Journal of Mathematics*, 17(2)(1998), 227-235.

[12] Khan, S.; *Lie theory of hypergeometric functions and integral operators*, Ph.D. Thesis, Department of Mathematics, Aligarh Muslim University, Aligarh, U.P. (India), 1993.

[13] Lagrange, J. L.; Mémoiresur l'utilité de la méthode de prendre le milieu entre les résultats de plusieurs observations, Miscellanea Taurinensia 5 (1770-1773): 167-232. Reprinted in his Oeuvres, *Gauthier-Villars, Paris*, Vol.2 (1868), 173-234.

[14] Laplace, P. S.; Théorie analytique des Probabilitiés, Courcier, Paris, 1812. Third edition with an introduction and three supplements, Courcier, Paris, 1820. Reprinted in his Oeuvres Complètes, Vol. 7, Imprimerie Royale, Paris, 1847 and Gauthier-Villars et Fils, Paris, 1886.

[15] MacRobert, T. M.; Integrals involving a modified Bessel function of the second kind and an E-function, *Proc. Glasgow Math. Assoc.*, 2(1954), 93-96.

[16] MacRobert, T. M.; Multiplication formulae for the E-functions regarded as functions of their parameters, *Pacific J. Math.*, 9(1959), 759-761.

[17] Manocha, H. L.; Generating functions for Jacobi and Laguerre polynomials, Mat. Vesnik $11(26)(1974), 43-47$.

[18] Mohd, C. W., Qureshi, M. I. and Pathan, M. A.; New generating functions for hypergeometric polynomials, Indian J. Pure Appl. Math. $16(2)(1985), 157-164$.

[19] Pathan, M. A. and Khan, S.; Representation of a Lie algebra K_5 and Hermite polynomials, *Proceedings of Second International Conference of SSFA, Held at Lucknow, Feb 2-4*, 2(2001), 57-64.

[20] Qureshi, M. I. and Ahmad Nadeem ; Linear generating relations associated with Konhauser, Madhekar-Thakare, Jacobi and generalized Laguerre polynomials, South East Asian J. Math. & Math. Sc. $4(3)(2006), 43-53$.

[21] Qureshi, M. I., Quraishi, Kaleem A. and Pal, Ram,; Some Results Involving Gould-Hopper Polynomials, *Elixir Appl. Math.*, 62(2013), 17838-17841.

[22] Qureshi, M. I., Yasmeen and Pathan, M. A.; Linear and bilinear generating functions involving Gould-Hopper polynomials, *Math. Sci. Res. J.*, 6(9)(2002), 449-456.

[23] Rainville, E.D. *Special Functions*. Macmillan, New York(1960); Reprinted by Chelsea Publ. Co., Bronx, New York, 1971.

[24] Sharma, B. L.; A formula for hypergeometric series and its application, An. Univ. Timisoara Ser. Sti. Mat. $12(1974), 145 - 154$.

[25] Sharma B. L.; Some new formulae for hypergeometric series, *Kyungpook Math. J.*, 16(1)(1976), 95-99.

[26] Sharma, B. L.; New generating functions for the Gegenbauer and the Hermite polynomials, Simon Stevin (A quarterly Journal of Pure and Applied Mathematics) $54(3-4)$(1980), 129-134.

[27] Srivastava, H. M.; The integration of generalized hypergeometric functions, *Proc. Cambridge Philos. Soc.*, 62(1966), 761-764.

[28] Srivastava, H. M. and Lavoie, J. L.; A certain method of obtaining bilateral generating functions, *Indag. Math.*, 78(1975), 304-320.

[29] Srivastava, H. M. and Manocha, H. L.; *"A Treatise on Generating functions"*, Halsted Press (Ellis Horwood Limited, Chichester, U. K.) John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.