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ON SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH λ - SPIRALLIKE GENERALIZED SAKAGUCHI TYPE FUNCTIONS

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Abstract. This paper introduces a new subclass $\mathcal{L}_s(\alpha, \beta, \lambda, s, t)$ of analytic functions associated with λ - Spirallike generalized Sakaguchi type functions. The results investigated in this paper include Characterization and Subordination properties for functions belonging to this class and we also discuss several interesting consequences of our results.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open unit disc $\mathcal{U} = \{z : z \in \mathcal{C} \text{ and } |z| < 1\}$.

Definition 1.1. *(Convolution) Given two functions f and g in the class $\mathcal{A}$, where f is given by (1.1) and g is given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ the Hadamard product (or convolution) $f * g$ is defined by the power series*

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z) \quad (z \in \mathcal{U}).$$

Definition 1.2. *For two functions f and g analytic in $\mathcal{U}$, we say that the function f is subordinate to g in $\mathcal{U}$ and write $f \prec g$, if there exists a Schwarz function ω, which is analytic in $\mathcal{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $f(z) = g(\omega(z))$, $z \in \mathcal{U}$.*

Frasin [3] introduced and studied the generalized sakaguchi classes $\mathcal{S}(\alpha, s, t)$ and $\mathcal{T}(\alpha, s, t)$, a function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{S}(\alpha, s, t)$ if it satisfying the condition

$$\Re\left\{\frac{(s-t)zf'(z)}{f(sz)-f(tz)}\right\} > \alpha,$$

for some $\alpha(0 \leq \alpha < 1)$, $s, t \in \mathcal{U}$, $|t| \leq 1$, $s \neq t$, $z \in \mathcal{U}$.
$\mathcal{T}(\alpha, s, t)$ [3] is the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ such that $zf'(z) \in \mathcal{S}(\alpha, s, t)$. The class $\mathcal{S}(\alpha, 1, t)$ was introduced and studied by Owa et al [7] and the class $\mathcal{S}(\alpha, 1, -1)$ was introduced and studied by Sakaguchi [10]. Also $\mathcal{S}(\alpha, 1, 0)$ and $\mathcal{T}(\alpha, 1, 0)$, the usual classes of starlike and convex functions of order $\alpha, (0 \leq \alpha < 1)$ and let $\mathcal{K}$ be the familiar class of functions that are convex in $\mathcal{U}$.
Let $\mathcal{S}^{\lambda}(\alpha)$ denote the class of λ-spirallike functions of order α. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{S}^{\lambda}(\alpha)$ [5] if

$$\Re\left\{e^{i\lambda}\frac{zf'(z)}{f(z)}\right\} > \alpha cos\lambda, \quad (z \in \mathcal{U},\ |\lambda| < \pi/2,\ 0 \leq \alpha < 1)$$

Note that $\mathcal{S}^{\lambda}(0) = \mathcal{S}^{\lambda}$ is the class of λ-spirallike functions introduced by Spacek [14].
Further, a function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{C}^{\lambda}(\alpha)$ if

$$\Re\left\{e^{i\lambda}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \alpha cos\lambda, \quad (z \in \mathcal{U},\ |\lambda| < \pi/2,\ 0 \leq \alpha < 1)$$

Note that $\mathcal{C}^{\lambda}(0) = \mathcal{C}^{\lambda}$ is the class of functions for which $zf'(z)$ is λ-spirallike in $\mathcal{U}$ introduced by Roberstson [9] and the class $\mathcal{C}^{\lambda}(\alpha)$ was introduced and studied by Chichra [1]. A function $f(z) \in \mathcal{C}^{\lambda}(\alpha)$ if and only if $zf'(z) \in \mathcal{S}^{\lambda}(\alpha)$.
Now we introduce a new subclass $\mathcal{L}_s(\alpha, \beta, \lambda, s, t)$ defined using Sakaguchi type functions and generalized λ-Spirallike functions as follows.

Definition 1.3. *A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{L}_s(\alpha, \beta, \lambda, s, t)$ if it satisfies*

$$\Re\left\{e^{i\lambda}\frac{(s-t)zf'(z)+\beta(s-t)z^2f''(z)}{(1-\beta)[f(sz)-f(tz)]+\beta z[f(sz)-f(tz)]'}\right\} > \alpha cos\lambda, \tag{1.2}$$

for some α, $0 \leq \alpha < 1$, $0 \leq \beta < 1$, $t \neq s$, $|t| \leq 1, |\lambda| < \pi/2$, and $z \in \mathcal{U}$.

By setting different values to $\alpha, \beta, \lambda, s$, and t we get the following subclasses

- $\mathcal{R}^{\lambda}(\alpha, s, t)$ and $\mathcal{P}^{\lambda}(\alpha, s, t)$ introduced and studied by T.Mathur et. al. [6], for $\beta = 0$ and $\beta = 1$.
- $\mathcal{S}(\alpha, s, t)$ and $\mathcal{T}(\alpha, s, t)$ introduced and studied by Frasin [3] and Owa et. al. [7], for $\lambda = 0$, $\beta = 0$ and $\lambda = 0$, $\beta = 1$.
- $\mathcal{S}_P^{\alpha}(\lambda)$ studied in [8], for $\beta = 0$, $s = 1$ and $t = 0$.

- $\mathcal{P}^{\lambda}(\alpha, t)$ and $\mathcal{M}^{\lambda}(\alpha, t)$ introduced and studied by Goyal and Goswami [4], for $s = 1$, $\beta = 0$ and $\beta = 1$.
- $\mathcal{L}_S(\alpha, \beta, t)$ studied in [11], for $\lambda = 0$, and $s = 1$.
- $\mathcal{L}_S(\alpha, \beta, s, t)$ studied in [12], for $\lambda = 0$.
- $\mathcal{L}_S(\alpha, \beta, \lambda, t)$ studied in [13], for $s = 1$.

In our present investigation we need the following definition and also a related Lemma due to Wilf [15].

Definition 1.4. *(Subordinating factor sequence) A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever f of the form (1.1) is analytic, univalent and convex in $\mathcal{U}$, we have the subordination given by*

$$\sum_{n=1}^{\infty} a_n b_n z^n \prec f(z) \quad (z \in \mathcal{U}, a_1 = 1).$$

Lemma 1.5. [15] *The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\Re\left\{1 + 2\sum_{n=1}^{\infty} b_n z^n\right\} > 0, \quad (z \in \mathcal{U}).$$

2. Characterization results

In the present section first we prove the Characterization results for the functions in the class $\mathcal{L}_s(\alpha, \beta, \lambda, s, t)$.

Theorem 2.1. *A function $f(z)$ of the form (1.1) is in the class $\mathcal{L}_s(\alpha, \beta, \lambda, s, t)$ if*

$$\left|\frac{(s-t)zf'(z) + \beta(s-t)z^2f''(z)}{(1-\beta)[f(sz) - f(tz)] + \beta z[f(sz) - f(tz)]'} - 1\right| < 1 - \gamma \tag{2.1}$$

where $0 \le \gamma \le 1$, $0 \le \beta < 1$, $|t| \le 1$, $t \ne s$, , provided that

$$|\lambda| \le cos^{-1}(\frac{1-\gamma}{1-\alpha}) \tag{2.2}$$

for some α, $0 \le \alpha < 1$ and $z \in \mathcal{U}$.

Proof. Suppose

$$\frac{(s-t)zf'(z) + \beta(s-t)z^2f''(z)}{(1-\beta)[f(sz) - f(tz)] + \beta z[f(sz) - f(tz)]'} - 1 < (1-\gamma)\omega(z),$$

$|\omega(z)| < 1$ for all $z \in \mathcal{U}$. Now

$$\begin{aligned}\Re\left\{e^{i\lambda}\frac{(s-t)zf'(z) + \beta(s-t)z^2f''(z)}{(1-\beta)[f(sz) - f(tz)] + \beta z[f(sz) - f(tz)]'}\right\} &= cos\lambda + (1-\gamma)\Re\{e^{i\lambda}\omega(z)\} \\ &\ge cos\lambda - (1-\gamma)|e^{i\lambda}\omega(z)| \\ &\ge cos\lambda - (1-\gamma) \ \ge \alpha cos\lambda\end{aligned}$$

provided that $|\lambda| \leq cos^{-1}(\frac{1-\gamma}{1-\alpha})$.
Hence the proof. □

Theorem 2.2. *If*

$$\left|\frac{(s-t)zf'(z)+\beta(s-t)z^2f''(z)}{(1-\beta)[f(sz)-f(tz)]+\beta z[f(sz)-f(tz)]'}-1\right| < (1-\alpha)cos\lambda \tag{2.3}$$

for some $0 \leq \alpha < 1$, $0 \leq \beta < 1$, $t \neq s$, $|t| \leq 1, |\lambda| < \pi/2$, *and* $z \in \mathcal{U}$. *Then* $f(z)$ *belongs to the class* $\mathcal{L}_s(\alpha, \beta, \lambda, s, t)$

Proof. Set $\gamma = 1-(1-\alpha)cos\lambda$, in the above Theorem. □

Theorem 2.3. *If the function* $f(z) \in \mathcal{A}$, *satisfies the inequality*

$$\sum_{n=2}^{\infty}[1+(n-1)\beta][|n-u_n|sec\lambda+(1-\alpha)|u_n|]|a_n| \leq (1-\alpha), \tag{2.4}$$

where $u_n = \sum_{k=1}^{n} s^{n-k}t^{k-1}$, $(t \neq s, |t| \leq 1, 0 \leq \alpha < 1, 0 \leq \beta < 1, |\lambda| < \pi/2)$, *then* $f(z) \in \mathcal{L}_s(\alpha, \beta, \lambda, s, t)$.

Proof. By Theorem (2.2) it suffices to show that

$$\left|\frac{(s-t)zf'(z)+\beta(s-t)z^2f''(z)}{(1-\beta)[f(sz)-f(tz)]+\beta z[f(sz)-f(tz)]'}-1\right| < (1-\alpha)cos\lambda$$

Since

$$\left|\frac{(s-t)zf'(z)+\beta(s-t)z^2f''(z)}{(1-\beta)[f(sz)-f(tz)]+\beta z[f(sz)-f(tz)]'}-1\right|$$

$$= \left|\frac{\sum_{n=2}^{\infty}[1+(n-1)\beta](n-u_n)a_nz^n}{z+\sum_{n=2}^{\infty}[1+(n-1)\beta]u_na_nz^n}\right| < \frac{\sum_{n=2}^{\infty}[1+(n-1)\beta]|n-u_n||a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty}[1+(n-1)\beta]|u_n||a_n||z|^{n-1}}$$

$$< \frac{\sum_{n=2}^{\infty}[1+(n-1)\beta]|n-u_n||a_n|}{1-\sum_{n=2}^{\infty}[1+(n-1)\beta]|u_n||a_n|}.$$

The last expression is bounded above by $(1-\alpha)cos\lambda$, if

$$\frac{\sum_{n=2}^{\infty}[1+(n-1)\beta]|n-u_n||a_n|}{1-\sum_{n=2}^{\infty}[1+(n-1)\beta]|u_n||a_n|} < (1-\alpha)cos\lambda$$

which is equivalent to

$$\sum_{n=2}^{\infty}[1+(n-1)\beta][|n-u_n|sec\lambda+(1-\alpha)|u_n|]|a_n| < (1-\alpha)$$

□

Remark:Upon setting suitable choices for $\alpha, \lambda, \beta, s$ and t we get the characterization results obtained earlier in [4], [7], [8] , [11], [12], [13].

3. Subordination results

In this section we prove the Subordination results for the functions in the class $\mathcal{L}_s(\alpha,\beta,\lambda,s,t)$.

Theorem 3.1. *Let $f \in \mathcal{A}$ satisfies the inequality (2.4) and suppose that $g \in \mathcal{K}$. Then*

$$\frac{(1+\beta)[|s-t|sec\lambda+(1-\alpha)|1+t|]}{2[(1+\beta)[|s-t|sec\lambda+(1-\alpha)|1+t|]+(1-\alpha)]}(f*g)(z) \prec g(z) \tag{3.1}$$

$z \in \mathcal{U}$, $|t| \leq 1, t \neq s, 0 \leq \beta < 1, 0 \leq \alpha < 1$, $|\lambda| < \pi/2$ and

$$\Re\{f(z)\} > -\frac{(1+\beta)[|s-t|sec\lambda+(1-\alpha)|1+t|]+(1-\alpha)}{(1+\beta)[|s-t|sec\lambda+(1-\alpha)|1+t|]}, \quad (z \in \mathcal{U}). \tag{3.2}$$

The constant factor $\frac{(1+\beta)[|s-t|sec\lambda+(1-\alpha)|1+t|]}{2[(1+\beta)[|s-t|sec\lambda+(1-\alpha)|1+t|]+(1-\alpha)]}$ in the subordination result (3.1) cannot be replaced by any larger one.

Proof. Let $f \in \mathcal{A}$ satisfy the inequality (2.4) and suppose that $g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{K}$. Then we have

$$\frac{(1+\beta)[|s-t|sec\lambda+(1-\alpha)|1+t|]}{2[(1+\beta)[|s-t|sec\lambda+(1-\alpha)|1+t|]+(1-\alpha)]}(f*g)(z) \tag{3.3}$$

$$= \frac{(1+\beta)[|s-t|sec\lambda+(1-\alpha)|1+t|]}{2[(1+\beta)[|s-t|sec\lambda+(1-\alpha)|1+t|]+(1-\alpha)]}(z+\sum_{n=2}^{\infty} a_n c_n z^n)$$

By definition (1.4) the subordination result (3.1) holds true if the sequence

$$\left\{\frac{(1+\beta)[|s-t|sec\lambda+(1-\alpha)|1+t|]}{2[(1+\beta)[|s-t|sec\lambda+(1-\alpha)|1+t|]+(1-\alpha)]}a_n\right\}_{n=1}^{\infty} \tag{3.4}$$

is a subordinating factor sequence with $a_1 = 1$.
In view of lemma (1.5) it is enough to prove the inequality:

$$\Re\left\{1+\sum_{n=1}^{\infty}\frac{(1+\beta)[|s-t|sec\lambda+(1-\alpha)|1+t|]}{[(1+\beta)[|s-t|sec\lambda+(1-\alpha)|1+t|]+(1-\alpha)]}a_n z^n\right\} > 0, \quad (z \in \mathcal{U}). \tag{3.5}$$

Now,

$$\Re\left\{1+\frac{(1+\beta)[|s-t|sec\lambda+(1-\alpha)|1+t|]}{[(1+\beta)[|s-t|sec\lambda+(1-\alpha)|1+t|]+(1-\alpha)]}\sum_{n=1}^{\infty} a_n z^n\right\}$$
$$= \Re\left\{1+\frac{(1+\beta)[|s-t|sec\lambda+(1-\alpha)|1+t|]}{[(1+\beta)[|s-t|sec\lambda+(1-\alpha)|1+t|]+(1-\alpha)]}z\right.$$
$$+\frac{1}{(1+\beta)[|s-t|sec\lambda+(1-\alpha)|1+t|]+(1-\alpha)}$$
$$\left.\sum_{n=2}^{\infty}(1+\beta)[|s-t|sec\lambda+(1-\alpha)|1+t|]a_n z^n\right\}.$$

when $|z| = r, (0 < r < 1)$,

$$\geq 1 - \frac{(1+\beta)[|s-t|sec\lambda + (1-\alpha)|1+t|]}{[(1+\beta)[|s-t|sec\lambda + (1-\alpha)|1+t|] + (1-\alpha)]} r$$
$$- \frac{1}{(1+\beta)[|s-t|sec\lambda + (1-\alpha)|1+t|] + (1-\alpha)}$$
$$\sum_{n=2}^{\infty} [1+(n-1)\beta][|n-u_n|sec\lambda + (1-\alpha)|u_n|]|a_n| r^n$$

$$> 1 - \frac{(1+\beta)[|s-t|sec\lambda + (1-\alpha)|1+t|]}{[(1+\beta)[|s-t|sec\lambda + (1-\alpha)|1+t|] + (1-\alpha)]} r$$
$$- \frac{1-\alpha}{(1+\beta)[|s-t|sec\lambda + (1-\alpha)|1+t|] + (1-\alpha)} r$$
$$> 0$$

Then (3.5) holds in $\mathcal{U}$. This proves the inequality (3.1). The inequality (3.2) follows from (3.1), by taking convex function $g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n$. To prove the sharpness of the constant $\frac{(1+\beta)[|s-t|sec\lambda + (1-\alpha)|1+t|]}{2[(1+\beta)[|s-t|sec\lambda + (1-\alpha)|1+t|] + (1-\alpha)]}$ we consider the function $f_0(z) \in \mathcal{L}_S(\alpha, \beta, \lambda, t)$ given by

$$f_0(z) = z - \frac{(1-\alpha)sec\lambda}{(1+\beta)[|s-t|sec\lambda + (1-\alpha)|1+t|]} z^2 \tag{3.6}$$

From (3.1),

$$\frac{(1+\beta)[|s-t|sec\lambda + (1-\alpha)|1+t|]}{2[(1+\beta)[|s-t|sec\lambda + (1-\alpha)|1+t|] + (1-\alpha)]} f_0(z) \prec \frac{z}{1-z}, \quad (z \in \mathcal{U}) \tag{3.7}$$

For the function f_0, it is easy to verify that

$$\min \left\{ \Re \left\{ \frac{(1+\beta)[|s-t|sec\lambda + (1-\alpha)|1+t|]}{2[(1+\beta)[|s-t|sec\lambda + (1-\alpha)|1+t|] + (1-\alpha)]} f_0(z) \right\} \right\} = -\frac{1}{2}. \quad (|z| \leq 1)$$

This shows that the constant $\frac{(1+\beta)[|s-t|sec\lambda + (1-\alpha)|1+t|]}{2[(1+\beta)[|s-t|sec\lambda + (1-\alpha)|1+t|] + (1-\alpha)]}$ is the best possible, which completes the proof. □

Remark: Suitable choices of $\alpha, \beta, \lambda, s$ and t yield the subordination results derived in [2], [4], [8], [6], [11], [1 and [13].

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