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# ESTIMATES ON COEFFICIENTS OF A GENERAL SUBCLASS OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH SYMMETRIC $q$-DERIVATIVE OPERATOR BY MEANS OF THE CHEBYSHEV POLYNOMIALS

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Abstract. In this paper, we study a newly-constructed subclass of bi-univalent functions defined by using symmetric $q$-derivative operator. Upper bounds for the second and third coefficients, and also Fekete-Szegö inequalities of functions in this subclass are founded. Moreover, certain special cases are also indicated.

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## 1. Introduction, Definitions and Notations

Let $A$ indicate an analytic function family, which is normalized under the condition of $f(0) = f'(0) - 1 = 0$ in $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and given by the following Taylor-Maclaurin series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Also let $S$ be the subclass of $A$ consisting of the form (1) which are also univalent in $U$.

If the functions $f$ and $g$ are analytic in $U$, then $f$ is said to be subordinate to $g$, written as

$$f(z) \prec g(z), \qquad (z \in U)$$

if there exists a Schwarz function $w(z)$, analytic in $U$, with

$$w(0) = 0 \quad \text{and} \qquad |w(z)| < 1 \qquad (z \in U)$$

such that

$$f(z) = g(w(z)) \qquad (z \in U).$$

It is well known that every function $f \in S$ has an inverse $f^{-1}$, satisfying $f^{-1}(f(z)) = z,\ (z \in U)$ and $f(f^{-1}(w)) = w,\ \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right)$, where

$$f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots \tag{2}$$

(for details, see Duren [11]). A function $f \in A$ is said to be bi-univalent in $U$ if both $f$ and $f^{-1}$ are univalent in $U$. Let $\Sigma$ stand for the class of bi-univalent functions defined in the unit disk $U$. For a brief history and interesting examples in the class $\Sigma$, see [27].

Lewin [19] studied the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie [7] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Later, Netanyahu [23] showed that $max\,|a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$. Brannan and Taha [8] introduced certain subclasses of the bi-univalent function class $\Sigma$ similar to the familiar subclasses $S^{\star}(\beta)$ and $K(\beta)$ of starlike and convex function of order $\beta$ $(0 \leq \beta < 1)$ respectively (see [23]). The classes $S_{\Sigma}^{\star}(\beta)$ and $K_{\Sigma}(\beta)$ of bi-starlike functions of order $\alpha$ and bi-convex functions of order $\beta$, corresponding to the function classes $S^{\star}(\beta)$ and $K(\beta)$, were also introduced analogously. For each of the function classes $S_{\Sigma}^{\star}(\beta)$ and $K_{\Sigma}(\beta)$, they found non-sharp estimates on the initial coefficients. Recently, many authors investigated bounds for various subclasses of bi-univalent functions [2], [4], [12], [18], [20], [27], [28], [29]. Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions [1], [3], [14], [15], [17]. The coefficient estimate problem for each of $|a_n|$ $(\ n \in \mathbb{N}\backslash\{1, 2\};\ \ \mathbb{N} = \{1, 2, 3, ...\})$ is still an open problem.

Chebyshev polynomials have become increasingly important in numerical analysis, from both theoretical and practical points of view. There are four kinds of Chebyshev polynomials. The majority of books and research papers dealing with specific orthogonal polynomials of Chebyshev family, contain mainly results of Chebyshev polynomials of first and second kinds $T_n(x)$ and $U_n(x)$ and their numerous uses in different applications, see for example, Doha [10] and Mason [22].

The Chebyshev polynomials of the first and second kinds are well known. In the case of a real variable $t$ on $(-1, 1)$, they are defined by

$$T_n(t) = \cos n\theta,$$

$$U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta},$$

where the subscript $n$ denotes the polynomial degree and where $x = \cos\theta$.

In the field of Geometric Functions Theory, various subclasses of analytic functions have been studied from different viewpoints. The fractional $q$-calculus is the important tools that are used to investigate subclasses of analytic functions. For example, the extension of the theory of univalent functions can be described by using the theory of $q$-calculus. Moreover, the $q$-calculus operators, such as fractional $q$-integral and fractional $q$-derivative operators, are used to construct several subclasses of analytic functions (see, e.g., [5], [9], [24], [30]). In a recent paper Purohit and Raina [26], investigated applications of fractional $q$-calculus operators to defined certain new classes of functions which are analytic in the open disk. Later, Mohammed and Darus [21] studied approximation and geometric properties of these $q$-operators in some subclasses of analytic functions in compact disk.

For the convenience, we provide some basic definitions and concept details of $q$-calculus which are used in this paper. We suppose throughout the paper that $0 < q < 1$. We shall follow the notation and terminology in [13]. We recall the definitions of fractional $q$-calculus operators of complex valued function $f(z)$.

**Definition 1.** *Let $q \in (0,1)$ and define*

$$[n]_q = \frac{1-q^n}{1-q},$$

*for $n \in \mathbb{N}$.*

**Definition 2.** *(see [16]) The q-derivative of a function $f$, defined on a subset of $\mathbb{C}$, is given by*

$$(D_q f)(z) = \begin{cases} \dfrac{f(z)-f(qz)}{(1-q)z} & for \quad z \neq 0, \\ \\ f'(0) & for \quad z = 0. \end{cases}$$

We note that $\lim_{q \to 1^-} (D_q f)(z) = f'(z)$ if $f$ is differentiable at $z$. Additionally, in view of (1), we deduce that

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q \, a_n z^{n-1}. \tag{3}$$

**Definition 3.** *(see [6]) The symmetric q-derivative $\widetilde{D}_q f$ of a function $f$ given by (1) is defined as follows:*

$$(\widetilde{D}_q f)(z) = \begin{cases} \dfrac{f(qz)-f(q^{-1}z)}{(q-q^{-1})z} & for \quad z \neq 0, \\ \\ f'(0) & for \quad z = 0. \end{cases} \tag{4}$$

From (4), we deduce that $\widetilde{D}_q z^n = \widetilde{[n]}_q z^{n-1}$, and a power series of $\widetilde{D}_q f$ is

$$(\widetilde{D}_q f)(z) = 1 + \sum_{n=2}^{\infty} \widetilde{[n]}_q a_n z^{n-1},$$

when $f$ has the form (1) and the symbol $\widetilde{[n]}_q$ denotes the number

$$\widetilde{[n]}_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

It is easy to check that the following properties hold

$$\begin{aligned}
\widetilde{D}_q(f(z)+g(z)) &= (\widetilde{D}_q f)(z) + (\widetilde{D}_q g)(z),\\[1em]
\widetilde{D}_q(f(z)g(z)) &= g(q^{-1}z)(\widetilde{D}_q f)(z) + f(qz)(\widetilde{D}_q g)(z)\\
&= g(qz)(\widetilde{D}_q f)(z) + f(q^{-1}z)(\widetilde{D}_q g)(z),
\end{aligned}$$

and finally, we have the following relation

$$\left(\widetilde{D}_q f\right)(z) = D_{q^2} f(q^{-1}z).$$

From (2) and (4), we also deduce that

$$\begin{aligned}
(\widetilde{D}_q g)(w) &= \frac{g(qw) - g(q^{-1}w)}{(q-q^{-1})w}\\[1em]
&= 1\ - \widetilde{[2]}_q a_2 w + \widetilde{[3]}_q \left(2a_2^2 - a_3\right) w^2 \qquad (5)\\[1em]
&\quad - \widetilde{[4]}_q \left(5a_2^3 - 5a_2a_3 + a_4\right) w^3 + \cdots .
\end{aligned}$$

**Definition 4.** *A function $f \in \Sigma$ given by (1) is said to be in the class $\widetilde{H}^q_\Sigma(t)$, if the following conditions are satisfied:*

$$\left(\widetilde{D}_q f(z)\right) \prec H(z,t) := \frac{1}{1-2tz+z^2}, \qquad \left(\frac{1}{2} < t < 1,\ z \in U\right)$$

*and*

$$\left(\widetilde{D}_q g(w)\right) \prec H(w,t) := \frac{1}{1-2tw+w^2}, \qquad \left(\frac{1}{2} < t < 1,\ w \in U\right).$$

*where $g = f^{-1}$.*

We note that

$$\lim_{q\to 1^-} \widetilde{H}^q_\Sigma(t) = \left\{ f \in \Sigma : \begin{array}{ll} \lim_{q\to 1^-}\left(\widetilde{D}_q f(z)\right) > 0, & z \in U \\[1em] \lim_{q\to 1^-}\left(\widetilde{D}_q g(w)\right) > 0, & w \in U \end{array} \right\} = H_\Sigma(t).$$

The class $H_\Sigma(t)$ is defined as follows:

**Definition 5.** *A function $f \in \Sigma$ given by (1) is said to be in the class $H_\Sigma(t)$, if the following conditions are satisfied:*

$$f'(z) \prec H(z,t) := \frac{1}{1-2tz+z^2}, \qquad \left(\frac{1}{2} < t < 1,\ z \in U\right)$$

*and*

$$g'(w) \prec H(w,t) := \frac{1}{1-2tw+w^2}, \quad \left(\frac{1}{2} < t < 1,\ w \in U\right).$$

*where $g = f^{-1}$.*

We note that if $t = \cos\alpha,\ \alpha \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$, then

$$\begin{aligned} H(z,t) &= \frac{1}{1-2tz+z^2} \\ &= 1 + \sum_{n=1}^{\infty} \frac{\sin(n+1)\alpha}{\sin\alpha} z^n \quad (z \in U). \end{aligned}$$

Thus

$$H(z,t) = 1 + 2\cos\alpha z + (3\cos^2\alpha - \sin^2\alpha)z^2 + \cdots \quad (z \in U).$$

Next, we write

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + \cdots \quad (z \in U, \quad t \in (-1,1)),$$

where $U_{n-1} = \dfrac{\sin(n \arccos t)}{\sqrt{1-t^2}}$ $(n \in \mathbb{N})$ are the Chebyshev polynomials of the second kind. Also it is known that

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t),$$

and

$$U_1(t) = 2t, \quad U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t, \quad \ldots. \tag{6}$$

The Chebyshev polynomials $T_n(t),\ t \in (-1,1)$, of the first kind have the generating function of the form

$$\sum_{n=0}^{\infty} T_n(t) z^n = \frac{1-tz}{1-2tz+z^2} \quad (z \in U).$$

The aim of this work is to study the Chebyshev polynomial expansions to provide estimates for the initial coefficients of some subclasses of bi-univalent functions defined by symmetric $q$-derivative operator. Also, we establish Fekete-Szegö inequalities for the class $\widetilde{H}_\Sigma^q(t)$.

## 2. Preliminary Lemma

Let $P$ be the class of functions with positive real part consisting of all analytic functions $p : U \to \mathbb{C}$ satisfying $p(0) = 1$ and $\Re(p(z)) > 0$.

The class $P$ is called the class of Caratheodory function. The following result will be required for proving our results.

**Lemma 6.** [25] *If the function $p \in P$ is defined by*

$$p\left(z\right) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots ,$$

*then*

$$|p_n| \leq 2 \qquad \left(n \in \mathbb{N} = \{1, 2, \ldots\}\right).$$

## 3. Coefficient bounds for the function class $\widetilde{H}^q_\Sigma(t)$

**Theorem 7.** *Let $f$ given by (1) be in the class $\widetilde{H}^q_\Sigma(t)$. Then*

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{4\left(\widetilde{[3]}_q - \widetilde{[2]}_q^2\right)t^2 + 2\widetilde{[2]}_q^2 t + \widetilde{[2]}_q^2}},$$

*and*

$$|a_3| \leq \frac{4t^2}{\widetilde{[2]}_q^2} + \frac{2t}{\widetilde{[3]}_q}.$$

*Proof.* Let $f \in \widetilde{H}^q_\Sigma(t)$ and $g$ be the analytic extension of $f^{-1}$ to $U$. Then there exist two functions $u$ and $v$, analytic in $U$ with $u\left(0\right) = v\left(0\right) = 0,\ |u\left(z\right)| < 1,\ |v\left(w\right)| < 1,\ z, w \in U$ such that

$$\left(\widetilde{D}_q f\right)(z) = H(u(z), t), \tag{7}$$

$$\left(\widetilde{D}_q g\right)(w) = H(v(w), t). \tag{8}$$

Next, define the functions $p$,$q \in P$ by

$$p\left(z\right) = \frac{1 + u\left(z\right)}{1 - u\left(z\right)} = 1 + p_1 z + p_2 z^2 + \cdots$$

and

$$q\left(w\right) = \frac{1 + v\left(w\right)}{1 - v\left(w\right)} = 1 + q_1 w + q_2 w^2 + \cdots .$$

In the following, one can derive

$$u\left(z\right) = \frac{p\left(z\right) - 1}{p\left(z\right) + 1} = \frac{1}{2}p_1 z + \frac{1}{2}\left(p_2 - \frac{1}{2}p_1^2\right)z^2 + \cdots \tag{9}$$

and

$$v(w)=\frac{q(w)-1}{q(w)+1}=\frac{1}{2}q_1w+\frac{1}{2}\left(q_2-\frac{1}{2}q_1^2\right)w^2+\cdots. \tag{10}$$

Combining (7), (8), (9) and (10),

$$\left(\widetilde{D}_qf\right)(z)=1+\frac{1}{2}U_1(t)p_1z+\left(\frac{1}{4}U_2(t)p_1^2+\frac{1}{2}U_1(t)\left(p_2-\frac{1}{2}p_1^2\right)\right)z^2+\cdots \tag{11}$$

and

$$\left(\widetilde{D}_qg\right)(w)=1+\frac{1}{2}U_1(t)q_1w+\left(\frac{1}{4}U_2(t)q_1^2+\frac{1}{2}U_1(t)\left(q_2-\frac{1}{2}q_1^2\right)\right)w^2+\cdots. \tag{12}$$

It follows from (11) and (12) that

$$\widetilde{[2]}_qa_2=\frac{U_1(t)}{2}p_1, \tag{13}$$

$$\widetilde{[3]}_qa_3=\frac{U_1(t)}{2}\left(p_2-\frac{p_1^2}{2}\right)+\frac{U_2(t)}{4}p_1^2, \tag{14}$$

and

$$-\widetilde{[2]}_qa_2=\frac{U_1(t)}{2}q_1, \tag{15}$$

$$\widetilde{[3]}_q\left(2a_2^2-a_3\right)=\frac{U_1(t)}{2}\left(q_2-\frac{q_1^2}{2}\right)+\frac{U_2(t)}{4}q_1^2, \tag{16}$$

From (13) and (15) we obtain

$$p_1=-q_1, \tag{17}$$

$$2\widetilde{[2]}_q^2a_2^2=\frac{U_1^2(t)}{4}\left(p_1^2+q_1^2\right). \tag{18}$$

If we add (14) to (16), we get

$$2\widetilde{[3]}_qa_2^2=\frac{U_1(t)}{2}\left(p_2+q_2\right)+\frac{U_2(t)-U_1(t)}{4}\left(p_1^2+q_1^2\right). \tag{19}$$

Using (18) in equality (19),

$$2\left[\widetilde{[3]}_q-\frac{U_2(t)-U_1(t)}{U_1^2(t)}\widetilde{[2]}_q^2\right]a_2^2=\frac{U_1(t)}{2}\left(p_2+q_2\right). \tag{20}$$

From Lemma 5 and (20) we get

$$|a_2|\leq\frac{2t\sqrt{2t}}{\sqrt{4\left(\widetilde{[3]}_q-\widetilde{[2]}_q^2\right)t^2+2\widetilde{[2]}_q^2t+\widetilde{[2]}_q^2}}.$$

Next, if we subtract (16) from (14), we obtain

$$2\widetilde{[3]}_q\left(a_3-a_2^2\right)=\frac{U_1(t)}{2}\left(p_2-q_2\right)+\frac{U_2(t)-U_1(t)}{4}\left(p_1^2-q_1^2\right). \tag{21}$$

Then, in view of (17) and (18), also (21)

$$a_3 = \frac{U_1^2(t)}{8\widetilde{[2]}_q^2}\left(p_1^2 + q_1^2\right) + \frac{U_1(t)}{4\widetilde{[3]}_q}\left(p_2 - q_2\right).$$

Notice that from Lemma 5

$$|a_3| \leq \frac{4t^2}{\widetilde{[2]}_q^2} + \frac{2t}{\widetilde{[3]}_q}.$$

□

**Corollary 8.** *Let $f$ given by (1) be in the class $H_\Sigma(t)$. Then*

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{1+2t-1t^2}},$$

*and*

$$|a_3| \leq t^2 + \frac{2t}{3}.$$

## 4. Fekete-Szegö inequalities for the function class $\widetilde{H}_\Sigma^q(t)$

**Theorem 9.** *Let $f$ given by (1) be in the class $\widetilde{H}_\Sigma^q(t)$ and $\eta \in \mathbb{R}$. Then*

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \dfrac{2t}{\widetilde{[3]}_q}; & for \quad |\eta - 1| \leq \dfrac{4\left(\widetilde{[3]}_q - \widetilde{[2]}_q^2\right)t^2 + 2\widetilde{[2]}_q^2 t + \widetilde{[2]}_q^2}{4\widetilde{[3]}_q t^2} \\ \\ \dfrac{8\,|1-\eta|\,t^3}{4\left(\widetilde{[3]}_q - \widetilde{[2]}_q^2\right)t^2 + 2\widetilde{[2]}_q^2 t + \widetilde{[2]}_q^2}; & for \quad |\eta - 1| \geq \dfrac{4\left(\widetilde{[3]}_q - \widetilde{[2]}_q^2\right)t^2 + 2\widetilde{[2]}_q^2 t + \widetilde{[2]}_q^2}{4\widetilde{[3]}_q t^2} \end{cases}.$$

*Proof.* From (20) and (21)

$$\begin{aligned} a_3 - \eta a_2^2 &= (1-\eta)\frac{U_1^3(t)\left(p_2 + q_2\right)}{4\left[\widetilde{[3]}_q U_1^2(t) - \left(U_2(t) - U_1(t)\right)\widetilde{[2]}_q^2\right]} + \frac{U_1(t)}{4\widetilde{[3]}_q}\left(p_2 - q_2\right) \\ &= U_1(t)\left[\left(h(\eta) + \frac{1}{4\widetilde{[3]}_q}\right)p_2 + \left(h(\eta) - \frac{1}{4\widetilde{[3]}_q}\right)q_2\right] \end{aligned}$$

where

$$h(\eta) = \frac{U_1^2(t)\left(1-\eta\right)}{4\left[\widetilde{[3]}_q U_1^2(t) - \left(U_2(t) - U_1(t)\right)\widetilde{[2]}_q^2\right]}.$$

Then, in view of Lemma 5, we conclude that

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \dfrac{2t}{\widetilde{[3]}_q}; & 0 \leq |h(\eta)| \leq \dfrac{1}{4\widetilde{[3]}_q}, \\ \\ 8t\,|h(\eta)|\,; & |h(\eta)| \geq \dfrac{1}{4\widetilde{[3]}_q}. \end{cases}$$

**Corollary 10.** *Let $f$ given by (1) be in the class $H_\Sigma(t)$ and $\eta \in \mathbb{R}$. Then*

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \dfrac{2t}{3}; & 0 \leq |\eta-1| \leq \frac{1+2t-t^2}{3t^2}, \\ \\ \dfrac{2\,|1-\eta|\,t^3}{1+2t-t^2}; & |\eta-1| \geq \frac{1+2t-t^2}{3t^2}. \end{cases}$$

□

## References


[1] Ş. Altınkaya, S. Yalçın, Coefficient bounds for a subclass of bi-univalent functions, TWMS J. Pure Appl. Math. **6**(2015), 180-185.

[2] Ş. Altınkaya, S. Yalçın, Coefficient Estimates for Two New Subclasses of Bi-univalent Functions with respect to Symmetric Points, J. Funct. Spaces **2015**(2015), Article ID 145242, 5 pages.

[3] Ş. Altınkaya, S. Yalçın, Faber polynomial coefficient bounds for a subclass of bi-univalent functions, C. R. Acad. Sci. Paris, Ser. I. **353**(2015), 1075–1080.

[4] Ş. Altınkaya, S. Yalçın, On the Chebyshev polynomial bounds for classes of univalent functions, Khayyam J. Math. **2**(2016), 1-5.

[5] M. Aydoğan, Y. Kahramaner, Y. Polatoğlu, Close-to-Convex Functions Defined by Fractional Operator, Appl. Math. Sci. **7**(2013), 2769–2775.

[6] K. L. Brahim , Y. Sidomou, On some symmetric $q$-special functions, Le Mat. **68**(2013), 107–122.

[7] D. A. Brannan, J. G. Clunie, Aspects of contemporary complex analysis, (Proceedings of the NATO Advanced Study Instute Held at University of Durham:July 1-20, 1979). New York: Academic Press, 1980.

[8] D. A. Brannan, T. S. Taha, On some classes of bi-univalent functions, Stud. Univ. Babeş-Bolyai Math. **32**(1986), 70-77.

[9] H. Aldweby, M. Darus, A subclass of harmonic univalent functions associated with $q$-analogue of Dziok-Srivastava operator, ISRN Math. Anal. **2013**(2013), Article ID 382312, 6 pages.

[10] E. H. Doha, The first and second kind Chebyshev coefficients of the moments of the general-order derivative of an infinitely differentiable function, Int. J. Comput. Math. **51**(1994), 21-35.

[11] P. L. Duren, Univalent Functions, Grundlehren der Mathematischen Wissenschaften, Springer, New York, NY, USA **259**, 1983.

[12] B. A. Frasin, M. K. Aouf, New subclasses of bi-univalent functions, Appl. Math. Lett. **24**(2011), 1569–1573.

[13] G. Gasper, M. Rahman, Basic Hypergeometric Series, Cambridge Univ. Press, Cambridge, MA, 1990.

[14] S. G. Hamidi, J. M. Jahangiri, Faber polynomial coefficient estimates for analytic bi-close-to-convex functions, C. R. Acad. Sci. Paris, Ser. I **352**(2014), 17–20.

[15] S. G. Hamidi, J. M. Jahangiri, Faber polynomial coefficients of bi-subordinate functions, C. R. Acad. Sci. Paris, Ser. I **354**(2016), 365–370.

[16] F. H. Jackson, On $q$-functions and a certain difference operator, Trans. Royal Soc. Edinburgh, **46**(1908), 253–281.

[17] J. M. Jahangiri, S. G. Hamidi, Coefficient estimates for certain classes of bi-univalent functions, Int. J. Math. Math. Sci. **2013**(2013), ArticleID 190560, 4 pages.

[18] S. Kanas, Seong-A Kim, S. Sivasubramanian, Verification of Brannan and Clunie's conjecture for certain subclasses of bi-univalent functions, Ann. Polinici Mathematici **113**(2015), 295-304.

[19] M. Lewin, On a coefficient problem for bi-univalent functions, Proc. Amer. Math. Soc. **18**(1967), 63–68.

[20] N. Magesh, J. Yamini, Coefficient bounds for a certain subclass of bi-univalent functions, Int. Math. Forum **8**(2013), 1337–1344.

[21] A. Mohammed, M. Darus, A generalized operator involving the $q$-hypergeometric function, Mat. Vesnik **65**(2013), 454–465.

[22] J. C. Mason, Chebyshev polynomials approximations for the L-membrane eigenvalue problem, SIAM J. Appl. Math. **15**(1967), 172-186.

[23] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, Arch. Ration. Mech. Anal. **32**(1969), 100–112.

[24] Y. Polatoğlu, Growth and distortion theorems for generalized $q$-starlike functions, Adv. Math., Sci. J. **5**(2016), 7–12.

[25] C. Pommerenke, Univalent Functions, Vandenhoeck & Ruprecht, Göttingen, 1975.

[26] S. D. Purohit, R. K. Raina, Fractional $q$ -calculus and certain subclass of univalent analytic functions, Mathematica **55**(2013), 62–74.

[27] H M. Srivastava, A. K. Mishra, P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett. **23**(2010), 1188–1192.

[28] H. M. Srivastava, D. Bansal, Coefficient estimates for a subclass of analytic and bi-univalent functions, J. Egyptian Math. Soc. **23**(2015), 242-246.

[29] H. M. Srivastava, S. B. Joshi, S. S. Joshi, H. Pawar, Coefficient estimates for certain subclasses of meromorphically bi-univalent functions, Palest. J. Math. **5**(Special Issue: 1)(2016), 250-258.

[30] H. E. Özkan Uçar, Coefficient inequalties for $q$-starlike functions, Appl. Math. Comp. **276**(2016), 122–126.