SOME PROPERTIES OF QUASI-ARMENDARIZ RINGS AND THEIR GENERALIZATIONS

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Abstract. Let $R$ be a ring and $(S, \leq)$ a strictly ordered monoid. The generalized power series ring $\left[\left[ R^S, \leq \right] \right]$ with coefficients in $R$ and exponents in $S$ is a common generalization of polynomial rings, power series rings, Laurent polynomial rings, group rings, and Malcev-Neumann Laurent series rings. We initiate the study of the $S$-quasi-Armendariz condition on $R$, a generalization of the standard quasi-Armendariz condition from polynomials to generalized power series. The class of quasi-Armendariz rings includes semiprime rings, Armendariz rings, right (left) $p.q.$-Baer rings and right (left) PP rings. The $S$-quasi-Armendariz rings are closed under direct product. Also it is shown that, if $R$ is a left APP-ring, then $R$ is $S$-quasi-Armendariz. The a necessary and sufficient condition is given for rings under which the ring $R$ is reflexive if and only if $\left[\left[ R^S, \leq \right] \right]$ is reflexive and $r_{\left[\left[ R^S, \leq \right] \right]}(f_{\left[\left[ R^S, \leq \right] \right]})$ is pure as a right ideal in $\left[\left[ R^S, \leq \right] \right]$ for any element $f \in \left[\left[ R^S, \leq \right] \right]$. We conclude some characterizations for generalized power series ring to be semiprime, quasi-Baer ring.

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1. Preliminaries

All rings considered here are associative with identity. Any concept and notation not defined here can be found in Ribenboim ([17]–[20]), Elliott and Ribenboim [5]. We will write monoids multiplicatively unless otherwise indicated. If $R$ is a ring and $X$ is a nonempty subset of $R$, then the left (right) annihilator of $X$ in $R$ is denoted by $\ell_R(X)(r_R(X))$.

Let $(S, \leq)$ be an ordered set. Recall that $(S, \leq)$ is artinian if every strictly decreasing sequence of elements of $S$ is finite, and that $(S, \leq)$ is narrow if every subset of pairwise order-incomparable elements of $S$ is finite. Thus, $(S, \leq)$ is artinian and narrow if and only if every nonempty subset of $S$ has at least one but only a finite number of minimal elements. Let
S be a commutative monoid. Unless stated otherwise, the operation of S will be denoted additively, and the neutral element by 0. The following definition is due to Elliott and Ribenboim [5].

Let \((S, \leq)\) is a strictly ordered monoid (that is, \((S, \leq)\) is an ordered monoid satisfying the condition that, if \(s, s', t \in S\) and \(s < s'\), then \(s + t < s' + t\), and \(R\) a ring. Let \([[R^S]_\leq]]\) be the set of all maps \(f : S \to R\) such that \(\text{supp}(f) = \{s \in S| f(s) \neq 0\}\) is artinian and narrow. With pointwise addition, \([[R^S]_\leq]]\) is an abelian additive group. For every \(s \in S\) and \(f, g \in [[R^S]_\leq]]\), let \(X_s(f, g) = \{(u, v) \in S \times S| u + v = s, f(u) \neq 0, g(v) \neq 0\}\). It follows from Ribenboim [20, 4.1] that \(X_s(f, g)\) is finite. This fact allows one to define the operation of convolution:

\[
(fg)(s) = \sum_{(u,v) \in X_s(f, g)} f(u)g(v).
\]

Clearly, \(\text{supp}(fg) \subseteq \text{supp}(f) + \text{supp}(g)\), thus by Ribenboim [18, 3.4] \(\text{supp}(fg)\) is artinian and narrow, hence \(fg \in [[R^S]_\leq]]\). With this operation, and pointwise addition, \([[R^S]_\leq]]\) becomes an associative ring, with identity element e, namely \(e(0) = 1, e(s) = 0\) for every \(0 \neq s \in S\). Which is called the ring of generalized power series with coefficients in \(R\) and exponents in \(S\). Many examples and results of rings of generalized power series are given in Ribenboim ([17]–[20]), Elliott and Ribenboim [5] and Varadarajan ([12], [13]). For example, if \(S = \mathbb{N} \cup \{0\}\) and \(\leq\) is the usual order, then \([[R^{\mathbb{N} \cup \{0\}}]_\leq]]\) \(\cong R[[x]]\), the usual ring of power series. If \(S\) is a commutative monoid and \(\leq\) is the trivial order, then \([[R^S]_\leq]]\) \(\cong R[S]\), the monoid ring of \(S\) over \(R\). Further examples are given in Ribenboim [18]. To any \(r \in R\) and \(s \in S\), we associate the maps \(c_r, e_s \in [[R^S]_\leq]]\) defined by

\[
c_r(x) = \begin{cases} r, & x = 0, \\ 0, & \text{otherwise}, \end{cases} \quad e_s(x) = \begin{cases} 1, & x = s, \\ 0, & \text{otherwise}. \end{cases}
\]

It is clear that \(r \mapsto c_r\) is a ring embedding of \(R\) into \([[R^S]_\leq]]\), \(s \mapsto e_s\), is a monoid embedding of \(S\) into the multiplicative monoid of the ring \([[R^S]_\leq]]\), and \(c_r e_s = e_s c_r\). Recall that a monoid \(S\) is torsion-free if the following property holds: If \(s, t \in S\), if \(k\) is an integer, \(k \geq 1\) and \(ks = kt\), then \(s = t\).

In this paper we give a new concept of \(S\)-quasi-Armendariz ring, which are a common generalization of quasi-Armendariz rings and \(S\)-Armendariz rings. We prove that, if \(R\) is a left \(APP\)-ring, then \(R\) is \(S\)-quasi-Armendariz. Moreover, a ring \(R\) is reflexive ring if and only if \([[R^S]_\leq]]\) is reflexive ring and (1) \(r_R(a)R\) is pure as a right ideal in \(R\) for any element \(a \in R\); (2) \(r_{[[R^S]_\leq]]}(f[[R^S]_\leq]])\) is pure as a right ideal in \([[R^S]_\leq]]\) for any element \(f \in [[R^S]_\leq]]\) in that case \(R\) is \(S\)-quasi-Armendariz ring, where \((S, \leq)\) be a strictly ordered monoid. Also as a Corollary, a ring \(R\) is a quasi-Baer ring if and only if \([[R^S]_\leq]]\) is quasi-Baer ring and we give a
lattice structure to the right (left) annihilators of a ring and characterize $S$-quasi-Armendariz rings as those rings $R$ for which an analogue of the Hirano map is a lattice isomorphism from the right (left) annihilators of $R$ to the right (left) annihilators of $[[R^S,\leq]]$.

2. Generalization of quasi-Armendariz rings

We start by the following definition:

**Definition 2.1.** Let $S$ be a torsion-free and cancellative monoid, $\leq$ a strict order on $S$. We say a ring $R$, $S$-quasi-Armendariz, if whenever $f, g \in [[[R^S,\leq]]$ satisfy $f[[R^S,\leq]]g = 0$, then $f(u)Rg(v) = 0$ for each $u, v \in S$.

The following result appeared in [24, Lemma 2.1].

**Lemma 2.2.** Let $S$ be a torsion-free and cancellative monoid, $\leq$ a strict order on $S$. Then $[[[R^S,\leq]]$ is reduced if and only if $R$ is reduced.

Reduced rings are semicommutative. From Proposition 2.4 reduced rings are $S$-quasi-Armendariz for any torsion free and cancellative monoid $S$. In [23, Corollary 2.3] it was claimed that all semicommutative rings are McCoy. However, Hirano's claim that, if $R$ is semicommutative then $R[x]$ is semicommutative, but this was later shown to be false in [2, Example 2]. Moreover, Nielsen [15] gave an example to show that a semicommutative ring $R$ need not be right McCoy, we also prove that the polynomial ring $R[x]$ over it actually is not semicommutative. By Liu [24], A ring $R$ is called $S$-Armendariz ring, if for each $f, g \in [[[R^S,\leq]]$ such that $fg = 0$ implies that $f(u)g(v) = 0$ for each $u, v \in S$ and it was shown that generalized power series rings over semicommutative rings are semicommutative. Here we have the following.

**Lemma 2.3.** [24, Proposition 2.7] Let $(S, \leq)$ be a strictly ordered monoid and $R$ be an $S$-Armendariz ring. Then $R$ is semicommutative if and only if $[[[R^S,\leq]]$ is semicommutative.

**Proposition 2.4.** Let $S$ be a torsion-free and cancellative monoid, $\leq$ a strict order on $S$ and $R$ a reduced ring. Then $R$ is an $S$-quasi-Armendariz.

**Proof.** Let $0 \neq f, g \in [[[R^S,\leq]]$ be such that $f[[R^S,\leq]]g = 0$. By Ribenboim [18], there exists a compatible strict total order $\leq'$ on $S$, which is finer than $\leq$. We will use transfinite induction on the strictly totally ordered set $(S, \leq)$ to show that $f(u)Rg(v) = 0$ for any $u \in supp(f)$ and $v \in supp(g)$. Let $s$ and $t$ denote the minimum elements of $supp(f)$ and $supp(g)$ in the $\leq'$ order, respectively. If $u \in supp(f)$ and $v \in supp(g)$ are such that $u + v = s + t$, then $s \leq' u$ and $t \leq' v$. If $s <' u$ then $s + t <' u + v = s + t$, a contradiction. Thus $u = s$. Similarly, $v = t$. Hence for any $r \in R, 0 = (fc_rg)(s + t) = \sum_{(u,v) \in X_{s+t}}(fc_rg)f(u)rg(v) = f(s)rg(t)$. 

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Now suppose that $w \in S$ is such that for any $u \in supp(f)$ and $v \in supp(g)$ with $u + v <' w$, $f(u)Rg(v) = 0$. We will show that $f(u)Rg(v) = 0$ for any $u \in supp(f)$ and $v \in supp(g)$ with $u + v = w$. We write $X_w(f,g) = \{(u,v) \mid u + v = w, u \in supp(f), v \in supp(g)\}$ as $\{(u_i,v_i) \mid i = 1,2,\ldots,n\}$ such that

$$u_1 <' u_2 <' \ldots <' u_n.$$  

Since $S$ is cancellative, $u_1 = u_2$ and $u_1 + v_1 = u_2 + v_2 = w$ imply $v_1 = v_2$. Since $\leq'$ is a strict order, $u_1 <' u_2$ and $u_1 + v_1 = u_2 + v_2 = w$ imply $v_2 <' v_1$. Thus we have

$$v_n <' \ldots <' v_2 <' v_1.$$  

Now, for any $r \in R$,

$$0 = (fc,g)(w) = \sum_{(u,v) \in X_w(f,c,g)} f(u)rg(v) = \sum_{i=1}^{n} f(u_i)rg(v_i). \quad (1)$$

For any $i \geq 2, u_1 + v_i <' u_i + v_i = w$, and thus, by induction hypothesis, we have $f(u_1)Rg(v_i) = 0$. Since $R$ is reduced, by Lemma 2.2 this implies $g(v_i)Rf(u_1) = 0$. Hence, multiplying (1) on the right by $f(u_1)g(v_1)$, we obtain

$$(\sum_{i=1}^{n} f(u_i)rg(v_i)) f(u_1)g(v_1) = f(u_1)g(v_1)rf(u_1)g(v_1) = 0.$$  

Then $(f(u_1)rg(v_1))^2 = 0$. Since $R$ is reduced, we have $f(u_1)rg(v_1) = 0$. Now (1) becomes

$$\sum_{i=2}^{n} f(u_i)rg(v_i) = 0. \quad (2)$$

Multiplying $f(u_2)g(v_2)$ on (2) from the right-hand side, we obtain $f(u_2)rg(v_2) = 0$ by the same way as the above. Continuing this process, we can prove $f(u_i)rg(v_i) = 0$ for any $r \in R$, for $i = 1,2,\ldots,n$. Thus $f(u)Rg(v) = 0$ for any $u \in supp(f)$ and $v \in supp(g)$ with $u + v = w$. Therefore, by transfinite induction, $f(u)Rg(v) = 0$ for any $u \in supp(f)$ and $v \in supp(g)$.

**Corollary 2.5.** [24, Lemma 3.1] Let $S$ be a torsion-free and cancellative monoid, $\leq$ a strict order on $S$, and $R$ a reduced ring. Then $R$ is $S$-Armendariz.

**Proposition 2.6.** Let $S$ be a torsion-free and cancellative monoid, $\leq$ a strict order on $S$. If $R$ is reduced semicommutative ring, then $R$ is $S$-Armendariz if and only if $R$ is $S$-quasi-Armendariz.

**Proof.** Apply Lemma 2.3 and Proposition 2.4. \qed

**Proposition 2.7.** Let $(S, \leq)$ be a strictly ordered monoid. Then every $S$-Armendariz rings are $S$-quasi-Armendariz.
An ideal $I$ of $R$ is said to be right s-unital if, for each $a \in I$ there exists an element $e \in I$ such that $ae = a$. Note that if $I$ and $J$ are right s-unital ideals, then so is $I \cap J$ (if $a \in I \cap J$, then $a \in aIJ \subseteq a(I \cap J)$).

The following result follows from Tominaga [11, Theorem 1].

Lemma 2.8. An ideal $I$ of a ring $R$ is left (resp. right) s-unital if and only if for any finitely many elements $a_1, a_2, \ldots, a_n \in I$, there exists an element $e \in I$ such that $a_i = ea_i$ (resp. $a_i = a_ie$) for each $i = 1, 2, \ldots, n$.

Clark defined quasi-Baer rings in [22]. A ring $R$ is called quasi-Baer if the left annihilator of every left ideal of $R$ is generated by an idempotent. Note that this definition is left-right symmetric. Some results of a quasi-Baer ring can be found in [16] and [22] and used them to characterize when a finite dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. As a generalization of quasi-Baer rings, Birkenmeier, Kim and Park in [10] introduced the concept of principally quasi-Baer rings. A ring $R$ is called left principally quasi-Baer (or simply left p.q.-Baer) if the left annihilator of a principal left ideal of $R$ is generated by an idempotent. Similarly, right p.q.-Baer rings can be defined. A ring is called p.q.-Baer if it is both right and left p.q.-Baer. Observe that biregular rings and quasi-Baer rings are p.q.-Baer. For more details and examples of left p.q.-Baer rings, see ([7]-[10]) and [27]. A ring $R$ is called a right (resp., left) PP-ring if every principal right (resp., left) ideal is projective (equivalently, if the right (resp., left) annihilator of an element of $R$ is generated (as a right (resp., left) ideal) by an idempotent of $R$). A ring $R$ is called a PP-ring (also called a Rickart ring [3, p. 18]) if it is both right and left PP. We say a ring $R$ is a left APP-ring if the left annihilator $l_R(Ra)$ is right s-unital as an ideal of $R$ for any element $a \in R$. This concept is a common generalization of left p.q.-Baer rings and right PP-rings.

Proposition 2.9. Let $(S, \leq)$ a strictly totally ordered monoid. If $R$ is left APP-ring, then $R$ is $S$-quasi-Armendariz.

Proof. Let $0 \neq f, g \in [[R^{S, \leq}]]$ be such that $f[[R^{S, \leq}]]g = 0$. We use the transfinite induction to show that $f(u)Rg(v) = 0$ for all $u, v \in S$. Assume that $\pi(f) = u_0, \pi(g) = v_0$. Let $(u, v) \in X_{u_0+v_0}(f, g)$. So $u_0 \leq u$ and $v_0 \leq v$. If $u_0 < u$, then $u_0 + v_0 < u + v_0 \leq u + v = u_0 + v_0$, a contradiction. Thus $u = u_0$. Similarly, $v = v_0$. So $X_{u_0+v_0}(f, g) = \{(u_0, v_0)\}$. Hence for any $r \in R$, from $f[[R^{S, \leq}]]g = 0$ we have,

$$0 = (f_{c_r}g)(u_0 + v_0) = \sum_{(u, v) \in X_{u_0+v_0}(f_{c_r}g)} f(u)rg(v) = f(u_0)rg(v_0).$$

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So $f(u_0)R g(v_0) = 0$. Now, let $\lambda \in S$ with $u_0 + v_0 \leq \lambda$ and assume that for any $u \in \text{supp}(f)$ and any $v \in \text{supp}(g)$, if $u + v < \lambda$, then $f(u)R g(v) = 0$. We claim that $f(u)R g(v) = 0$, for each $u \in \text{supp}(f)$ and each $v \in \text{supp}(g)$ with $u + v = \lambda$. For convenience, we write $X_{\lambda}(f,g) = \{(u,v) \mid u + v = \lambda, u \in \text{supp}(f), v \in \text{supp}(g)\}$ as $\{(u_i,v_i) \mid i = 1,2,\ldots,n\}$ such that

$u_1 < u_2 < \cdots < u_n,$

where $n$ is a positive integer (Note that if $u_1 = u_2$, then from $u_1 + v_1 = u_2 + v_2$ we have $v_1 = v_2$, and then $(u_1,v_1) = (u_2,v_2)$). Since $f[[R^{S,\leq}]]g = 0$, for any $r \in R$ we have:

$$0 = (fe,g)(\lambda) = \sum_{(u,v) \in X_{\lambda}(f, e g)} f(u)rg(v) = \sum_{i=1}^{n} f(u_i)rg(v_i).$$

(3)

Let $e_{u_1} \in r_R(f(u_1)R)$. So $f(u_1)Re_{u_1} = 0$ and which implies $f(u_1)Re_{u_1}g(v_1) = 0$. Let $r' \in R$ be an arbitrary element. Then we have $f(u_1)r'e_{u_1}g(v_1) = 0$. Take $r = r'e_{u_1}$ in Eq. (3). Thus,

$$\sum_{i=2}^{n} f(u_i)r'e_{u_1}g(v_i) = 0.$$

Note that $u_1 + v_i < u_i + v_i = \lambda$ for any $i \geq 2$. So by compatibility and induction hypothesis, $f(u_1)R g(v_i) = 0$ for each $i \geq 2$. Since $R$ is right $APP$, $r_R(f(u_1)R)$ is left $s$-unital. So without lose of generality and using Lemma 2.8, we can assume that $g(v_i) = e_{u_1}g(v_i)$, for each $i \geq 2$. Therefore

$$\sum_{i=2}^{n} f(u_i)r'g(v_i) = 0.$$ (4)

Let $e_{u_2} \in r_R(f(u_2)R)$. So $f(u_2)Re_{u_2} = 0$ and then $f(u_2)Re_{u_2}g(v_2) = 0$. This implies $f(u_2)Re_{u_2}g(v_2) = 0$.

Let $p \in R$ be an arbitrary element. So $f(u_2)pe_{u_2}g(v_2) = 0$. Also note that $u_2 + v_i < u_i + v_i = \lambda$ for any $i \geq 3$. So by induction hypothesis, $f(u_2)R g(v_i) = 0$. Therefore $g(v_i) \in r_R(f(u_2)R)$, for each $i \geq 3$. Since $r_R(f(u_2)R)$ is left $s$-unital, without lose of generality and using Lemma 2.8, again we can assume that $g(v_i) = e_{u_2}g(v_i)$, for each $i \geq 3$. Take $r' = pe_{u_2}$ in Eq. (4), so we have:

$$\sum_{i=2}^{n} f(u_i)pe_{u_2}g(v_i) = 0.$$ (5)

Continuing in this manner, we have $f(u_n)qg(v_n) = 0$, where $q$ is an arbitrary element of $R$. Thus $f(u_n)R g(v_n) = 0$. Hence $f(u_{n-1})R g(v_{n-1}) = 0, \ldots, f(u_2)R g(v_2) = 0, f(u_1)R g(v_1) = 0$. Therefore, by transfinite induction, $f(u)R g(v) = 0$ for any $u, v \in S$, and the proof is complete. 

**Corollary 2.10.** Let $(S, \leq)$ a strictly totally ordered monoid. If $I$ is a finitely generated left ideal of $R$ then for all $a \in l_R(I), a \in a l_R(I)$. So $R$ is $S$-quasi-Armendariz.
Proof. By Proposition 2.9 and [26, Proposition 2.6]. □

**Proposition 2.11.** Let $S$ be a torsion-free and cancellative monoid, $\leq$ a strict order on $S$ and $R$ an $S$-quasi-Armendariz ring. If $f_1, \ldots, f_n \in [[R^S]] \triangleq \Lambda$ are such that $f_1 \Lambda f_2 \Lambda \cdots \Lambda f_n = 0$, then $f_1(u_1)Rf_2(u_2)R \cdots Rf_n(u_n) = 0$ for all $u_1, u_2, \ldots, u_n \in S$.

Proof. Assume that $f_1 \Lambda f_2 \Lambda \cdots \Lambda f_n = 0$. Then for any $g_2, g_3, \ldots, g_{n-1} \in \Lambda$,

$$f_1 \Lambda (f_2 g_2 \cdots g_{n-1} f_n) = 0.$$ 

Since $R$ is $S$-quasi-Armendariz, we have

$$f_1(u_1)R((f_2 g_2 \cdots g_{n-1} f_n)(v)) = 0$$

for any $u_1, v \in S$. Thus

$$(C_{f_1(u_1)r_1} (f_2 g_2 \cdots g_{n-1} f_n))(v) = 0$$

for any $r_1 \in R$ and any $v \in S$. So $C_{f_1(u_1)r_1} f_2 g_2 \cdots g_{n-1} f_n = 0$, therefore $C_{f_1(u_1)r_1} f_2 \Lambda \cdots \Lambda f_n = 0$, for any $r_1 \in R$. Thus

$$(C_{f_1(u_1)r_1} f_2) \Lambda (f_3 g_3 \cdots g_{n-1} f_n) = 0.$$ 

By the hypothesis, we have

$$(C_{f_1(u_1)r_1} f_2)(u_2)R(f_3 g_3 \cdots g_{n-1} f_n)(z) = 0$$

for any $u_2, z \in S$. Yields

$$f_1(u_1)r_1 f_2(u_2)R(f_3 g_3 \cdots g_{n-1} f_n)(z) = 0.$$ 

So $f_1(u_1)r_1 f_2(u_2)r_2(f_3 g_3 \cdots g_{n-1} f_n)(z) = 0$, for any $r_1, r_2, \in R$. Thus

$$(C_{f_1(u_1)r_1} f_2)(u_2)R(f_3 g_3 \cdots g_{n-1} f_n)(z) = 0,$$

for any $r_1, r_2, \in R$ and any $z \in S$. So $C_{f_1(u_1)r_1} f_2 f_3 g_3 \cdots g_{n-1} f_n = 0$ for any $g_3, \ldots, g_{n-1} \in \Lambda$. Thus,

$$C_{f_1(u_1)r_1 f_2(u_2)r_2} f_3 \Lambda \cdots \Lambda f_n = 0.$$ 

Since $R$ is $S$-quasi-Armendariz. Repeating this process, we can get

$$C_{f_1(u_1)r_1 f_2(u_2)r_2 \cdots r_{n-1}} f_n(u_n) = 0.$$ 

So $f_1(u_1)r_1 f_2(u_2)r_2 \cdots r_{n-1} f_n(u_n) = 0$ for any $u_1, u_2, \ldots, u_n \in S$ and any $r_1, r_2, \ldots, r_{n-1} \in R$. Therefore $f_1(u_1)Rf_2(u_2)R \cdots Rf_n(u_n) = 0$ for any $u_1, u_2, \ldots, u_n \in S$. □

The following is a generalization of Proposition 2.4.
Corollary 2.12. Let $S$ be a torsion-free and cancellative monoid, $\leq$ a strict order on $S$ and $R$ a reduced ring. If $f_1, \ldots, f_n \in [[R^{S,\leq}]]$ are such that $f_1\Lambda f_2\Lambda \cdots \Lambda f_n = 0$, then $f_1(u_1)Rf_2(u_2)R \cdots Rf_n(u_n) = 0$ for all $u_1, u_2, \ldots, u_n \in S$.

Proposition 2.13. Let $S$ be a torsion-free and cancellative monoid, $\leq$ a strict order on $S$, and $R$ a reduced ring. Then $fRg = 0$ if and only if $f[[R^{S,\leq}]]g = 0$.

Proof. ($\Rightarrow$) Assume that $0 \neq f, g \in [[R^{S,\leq}]]$ are such that $fRg = 0$. By Corollary 2.5, $R$ is $S$-Armendariz, so for any $h \in [[R^{S,\leq}]]$ and any $s \in S$,

$$(fgh)(s) = \sum_{(u,w,v) \in X_s(f,h,g)} f(u)h(w)g(v) = 0.$$ \nonumber

Thus $fgh = 0$. This show that $f[[R^{S,\leq}]]g = 0$. The "only if part" is clear. \hfill $\Box$

According to [6], a right ideal $I$ is reflexive if $xRy \in I$ implies $yRx \in I$ for $x, y \in R$. Hence we shall call a ring $R$ a reflexive ring if 0 is a reflexive ideal (i.e., $aRb = 0$ implies $bRa = 0$ for $a, b \in R$). Moreover, a right ideal $I$ is called completely reflexive if $xy \in I$ implies $yx \in I$. A ring $R$ is completely reflexive if (0) has the corresponding property. It is clear that every completely reflexive ring is reflexive.

Proposition 2.14. Let $(S, \leq)$ be a strictly totally ordered monoid and $R$ be an $S$-quasi-Armendariz ring. Then $R$ is reflexive ring if and only if $[[R^{S,\leq}]]$ is reflexive ring.

Proof. ($\Rightarrow$) Let $R$ be reflexive ring. Suppose that $f, g \in [[R^{S,\leq}]]$ are such that $f[[R^{S,\leq}]]g = 0$. Since $R$ is $S$-quasi-Armendariz, we have $f(u)Rg(v) = 0$ for any $u \in \text{supf}(f)$ and $v \in \text{supp}(g)$. But $R$ is reflexive, so $g(v)Rf(u) = 0$ for all $u, v \in S$. Now for any $h \in [[R^{S,\leq}]]$ and any $s \in S$,

$$(ghf)(s) = \sum_{(v,w,u) \in X_s(g,h,f)} g(v)h(w)f(u) = 0.$$ \nonumber

Thus $ghf = 0$. This show that $g[[R^{S,\leq}]]f = 0$. This means that $[[R^{S,\leq}]]$ is reflexive. ($\Leftarrow$) Let $a, b \in R$ be such that $aRb = 0$. Then $C_a[[R^{S,\leq}]]C_b = 0$. Hence $C_b[[R^{S,\leq}]]C_a = 0$ by reflexive. So $bRa = 0$. Therefore $R$ is reflexive. \hfill $\Box$

Corollary 2.15. Let $(S, \leq)$ be a strictly totally ordered monoid and $R$ a reduced ring. Then $R$ is reflexive ring if and only if $[[R^{S,\leq}]]$ is reflexive.

Due to Hirano [23]. A ring $R$ is called quasi-Armendariz provided that $a_iRb_j = 0$ for all $i, j$ whenever $f(x) = a_0 + a_1 x + \cdots + a_n x^n, g(x) = b_0 + b_1 x + \cdots + b_m x^m \in R[x]$ satisfy $f(x)R[x]g(x) = 0$.

Corollary 2.16. [14, Proposition 3.2] Let $R$ be a quasi-Armendariz ring, then the following statements are equivalent:
(1) $R$ is reflexive.
(2) $R[x]$ is reflexive.
(3) $R[x; x^{-1}]$ is reflexive.

A ring $R$ is called semiprime if for any $a \in R$, $aRa = 0$, implies $a = 0$. Let $R$ be a ring and $(S, \leq)$ a strictly totally ordered monoid. A ring $R$ is called $S$-semiprime if $f[[R^{S,\leq}]]f = 0$, then $f = 0$ for each $f \in [[R^{S,\leq}]]$.

The following result appeared in [25, Lemma 2.7]

**Lemma 2.17.** Let $R$ be a ring and $(S, \leq)$ a strictly totally ordered monoid. Then $R$ is a semiprime ring if and only if $[[R^{S,\leq}]]$ is a semiprime ring.

**Proposition 2.18.** Let $(S, \leq)$ be a strictly totally ordered monoid. If $R$ is a semiprime, then $R$ is $S$-quasi-Armendariz.

*Proof.* It follows from Proposition 2.9. □

**Corollary 2.19.** If $S$ be a commutative, torsion-free, and cancellative monoid, then every semiprime ring $R$ is $S$-quasi-Armendariz.

**Corollary 2.20.** [23, Corollary 3.8] A semiprime ring is a quasi-Armendariz ring.

**Corollary 2.21.** Let $R$ be a ring and $(S, \leq)$ a strictly totally ordered monoid. If $R$ is semiprime, then $[[R^{S,\leq}]]$ is $S$-quasi-Armendariz ring.

**Corollary 2.22.** Let $R$ be a ring and $(S, \leq)$ a strictly totally ordered monoid. Assume that $R$ is semiprime. Then $R$ is reflexive ring if and only if $[[R^{S,\leq}]]$ is reflexive.

**Theorem 2.23.** Let $S$ be a torsion-free and cancellative monoid, $\leq$ a strict order on $S$. Then the following conditions are equivalent:

(1) $R$ is semiprime;

(2) $R$ is reduced $S$-quasi-Armendariz.

*Proof.* (1) $\Rightarrow$ (2) Is trivial.

(2) $\Rightarrow$ (1) Let $R$ be a reduced $S$-quasi-Armendariz. In particular for any $0 \neq f \in [[R^{S,\leq}]]$ be such that $f[[R^{S,\leq}]]f = 0$, then $f(u)Rf(u) = 0$. Thus, $(Rf(u))^2 = 0$ since $R$ is reduced. Therefore $f(u) = 0$. □

Let $I$ be an index set and $R_i$ be a ring for each $i \in I$. Let $(S, \leq)$ be a strictly ordered monoid, if there is an injective homomorphism $f : R \rightarrow \prod_{i \in I} R_i$ such that, for each $j \in I, \pi_jf : R \rightarrow R_j$ is a surjective homomorphism, where $\pi_j : \prod_{i \in I} R_i \rightarrow R_j$ is the $j$th projection. We have the following.
Proposition 2.24. Let $R_i$ be a ring, $(S, \leq)$ a strictly totally ordered monoid, for each $i$ in a finite index set $I$. If $R_i$ is $S$-quasi-Armendariz for each $i$, then $R = \prod_{i \in I} R_i$ is $S$-quasi-Armendariz.

Proof. Let $R = \prod_{i \in I} R_i$ be the direct product of rings $(R_i)_{i \in I}$ and $R_i$ is $S$-quasi-Armendariz for each $i \in I$. Denote the projection $R \to R_i$ as $\Pi_i$. Suppose that $f, g \in [[R^S,\leq]]$ are such that $f[[R^S,\leq]]g = 0$. Set $f_i = \Pi_i f$, $g_i = \Pi_i g$ and $h_i = \Pi_i h$. Then $f_i, g_i \in [[R_i^S,\leq]]$. For any $u, v \in S$, assume $f(u) = (a_i^u)_{i \in I}$, $g(v) = (b_i^v)_{i \in I}$. Now, for any $h \in [[R^S,\leq]]$, any $r \in R$ and any $s \in S$,

$$(fc,g)(s) = \sum_{(u,v) \in X_s(f,c,g)} f(u)rg(v)$$

$$= \sum_{(u,v) \in X_s(f,c,g)} (a_i^u)_{i \in I}(r_i)_{i \in I}(b_i^v)_{i \in I}$$

$$= \sum_{(u,v) \in X_s(f,c,g)} ((a_i^u)r_i(b_i^v))_{i \in I}$$

$$= \sum_{(u,v) \in X_s(f,c,g)} (f_i(u)r_ig_i(v))_{i \in I}$$

$$= \left( \sum_{(u,v) \in X_s(f,c,g)} f_i(u)r_ig_i(v) \right)_{i \in I}$$

Since $(fc,g)(s) = 0$ we have

$$(fc,c_ig_i)(s) = 0.$$ 

Thus, $f_i h_i g_i = 0$. Now it follows $f_i(u)r_ig_i(v) = 0$ for any $r \in R$, any $u, v \in S$ and any $i \in I$, since $R_i$ is $S$-quasi-Armendariz. Hence, for any $u, v \in S$,

$$f(u)r_ig_i(v) = (f_i(u)(r_i)g_i(v))_{i \in I} = 0$$

since $I$ is finite. Thus, $f(u)rg(v) = 0$. This means that $R$ is $S$-quasi-Armendariz. \qed

3. Characterizations generalized power series quasi-Armendariz rings via annihilators

In this section we give a lattice structure to the right (left) annihilators of a ring and characterize $S$-quasi-Armendariz rings as those rings $R$ for which an analogue of the Hirano [23] map is a lattice isomorphism from the right (left) annihilators of $R$ to the right (left) annihilators of $[[R^S,\leq]]$.

Let $\gamma = C(f)$ be the content of $f$, i.e., $C(f) = \{f(u) | u \in \text{supp}(f)\} \subseteq R$. Since, $R \simeq c_R$ we can identify, the content of $f$ with

$$c_{C(f)} = \{c_{f(u)} | u_i \in \text{supp}(f)\} \subseteq [[R^S,\leq]].$$

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**Lemma 3.1.** [21, Lemma 2.1] Let $R$ be a ring, $S$ a strictly ordered monoid, $[[R^{S,\leq}]]$ the generalized power series ring and $U \subseteq R$. Then

$$[[R^{S,\leq}]]\ell_{R}(U) = \ell_{[[R^{S,\leq}]]}(U), (r_{R}(U))[[R^{S,\leq}]] = r_{[[R^{S,\leq}]]}(U)).$$

By Lemma 3.1 we have two maps $\phi : rAnn_{R}(id(R)) \rightarrow rAnn_{[[R^{S,\leq}]]}(id([[R^{S,\leq}]]))$ and $\psi : lAnn_{R}(id(R)) \rightarrow lAnn_{[[R^{S,\leq}]]}(id([[R^{S,\leq}]]))$ defined by $\phi(I) = I[[R^{S,\leq}]]$ and $\psi(J) = [[R^{S,\leq}]]J$ for every $I \in rAnn_{R}(id(R)) = \{r_{R}(U)|U \text{ is an ideal of } R\}$ and $J \in lAnn_{R}(id(R)) = \{l_{R}(U)|U \text{ is an ideal of } R\}$, respectively. Obviously, $\phi$ is injective. In the following Theorem we show that $\phi$ and $\psi$ are bijective maps if and only if $R$ is $S$-quasi-Armendariz. This Theorem is a generalization of a result of Hashemi ([4, Proposition 2.1]) that generalizes a result of Hirano ([23, Proposition 3.4]).

**Theorem 3.2.** Let $R$ be a ring, $S$ a strictly ordered monoid and $[[R^{S,\leq}]]$ the generalized power series. Then the following are equivalent:

1. $R$ is generalized power series quasi-Armendariz ring.
2. The function $\phi : rAnn_{R}(id(R)) \rightarrow rAnn_{[[R^{S,\leq}]]}(id([[R^{S,\leq}]]))$ is bijective, where $\phi(I) = I[[R^{S,\leq}]].$
3. The function $\psi : lAnn_{R}(id(R)) \rightarrow lAnn_{[[R^{S,\leq}]]}(id([[R^{S,\leq}]]))$ is bijective, where $\psi(J) = [[R^{S,\leq}]]J$.

**Proof.** (1) $\Rightarrow$ (2) Let $Y \subseteq [[R^{S,\leq}]]$ and $\gamma = \bigcup_{f \in Y} C(f)$. From Lemma 3.1 it is sufficient to show that $r_{[[R^{S,\leq}]]}(f) = r_{R}C(f)[[R^{S,\leq}]]$ for all $f \in Y$. In fact, let $g \in r_{[[R^{S,\leq}]]}(f)$ and for any $h \in [[R^{S,\leq}]]$. Then $fhg = 0$ and by assumption $f(u_{i})tg(v_{j}) = 0$ for each $u_{i} \in supp(f), t \in R$ and each $v_{j} \in supp(g)$. Then for a fixed $u_{i} \in supp(f), t \in R$ and each $v_{j} \in supp(g), 0 = f(u_{i})tg(v_{j}) = (c_{f(u_{i})}t_{i}g)(v_{j})$ and it follows that $g \in r_{R} \cup_{u_{i} \in supp(f)} c_{f(u_{i})}t_{i}[[R^{S,\leq}]] = r_{R}C(f)[[R^{S,\leq}]]$. So $r_{[[R^{S,\leq}]]}(f) \subseteq r_{R}C(f)[[R^{S,\leq}]]$.

Conversely, let $g \in r_{R}C(f)[[R^{S,\leq}]]$, then $c_{f(u_{i})}t_{i}g = 0$ for each $u_{i} \in supp(f), t \in R$. Hence, $0 = (c_{f(u_{i})}t_{i}g)(v_{j}) = f(u_{i})tg(v_{j})$ for each $u_{i} \in supp(f), t \in R$ and $v_{j} \in supp(g)$. Thus,

$$(fhg)(s) = \sum_{(u_{i}, v_{j}) \in X_{s}(f, c_{i}g)} f(u_{i})tg(v_{j}) = 0$$

and it follows that $g \in r_{[[R^{S,\leq}]]}(f)$. Hence $r_{R}C(f)[[R^{S,\leq}]] \subseteq r_{[[R^{S,\leq}]]}(f)$ and it follows that $r_{R}C(f)[[R^{S,\leq}]] = r_{[[R^{S,\leq}]]}(f)$. So

$$r_{[[R^{S,\leq}]]}(Y) = \cap_{f \in Y} r_{[[R^{S,\leq}]]}(f) = \cap_{f \in Y} r_{R}C(f)[[R^{S,\leq}]] = r_{R}(\gamma)[[R^{S,\leq}]].$$

(2) $\Rightarrow$ (1) Suppose that $f, g \in [[R^{S,\leq}]]$ be such that $f[[R^{S,\leq}]]g = 0$. Then $g \in r_{[[R^{S,\leq}]]}(f)$ and by assumption $r_{[[R^{S,\leq}]]}(f) = \gamma[[R^{S,\leq}]]$ for some right ideal $\gamma$ of $R$. Consequently, $0 = fc_{i}t_{i}g(v_{j})$ and for any $u_{i} \in supp(f), 0 = (fc_{i}t_{i}g(v_{j}))(u_{i}) = f(u_{i})tg(v_{j})$ for each $u_{i} \in supp(f), t \in R$ and
$v_j \in \text{supp}(g)$. Hence, $R$ is a generalized power series quasi-Armendariz ring. The proof of (1)\(\iff\)(3) is similar to the proof of (1)\(\iff\)(2). \hfill \Box

**Definition 3.3.** A submodule $N$ of a left $R$-module $M$ is called a pure submodule if $L \otimes_R N \rightarrow L \otimes_R M$ is a monomorphism for every right $R$-module $L$. By [1, Proposition 11.3.13], for an ideal $I$, the following conditions are equivalent:

1. $I$ is right s-unital;
2. $R/I$ is flat as a left $R$-module;
3. $I$ is pure as a left ideal of $R$.

**Theorem 3.4.** Let $R$ be a ring, $(S, \leq)$ a strictly totally ordered monoid. Then the following statements are equivalent:

1. $r_R(a)R$ is pure as a right ideal in $R$ for any element $a \in R$;
2. $r_{[[R^S, \leq]]}(f[[R^{S, \leq}]])$ is pure as a right ideal in $[[R^{S, \leq}]]$ for any element $f \in [[R^{S, \leq}]]$.

In this case $R$ is an $S$-quasi-Armendariz ring.

**Proof.** Assume that the condition (1) holds. Firstly, by using the same method of the proof of Proposition 2.9 we can proved that $R$ is an $S$-quasi-Armendariz. Finally, by using Lemma 2.8 we can see that the condition (2) holds.

Conversely, suppose that the condition (2) holds. Let $a$ be an element of $R$. Then $r_{[[R^{S, \leq}]][f[[R^{S, \leq}]]]}(a[[R^{S, \leq}]])$ is left s-unital. Hence, for any $b \in r_R(aR)$, there exists an element $f \in [[R^{S, \leq}]]$ such that $bf = b$. Let $f(0)$ be the constant term of $f$. Then $f(0) \in r_R(aR)$ and $f(0)b = b$. This implies that $r_R(aR)$ is left s-unital. Therefore condition (1) holds. \hfill \Box

Let $R$ be a quasi-Baer ring and let $a \in R$. Then $l_R(Ra) = Re$ for some idempotent $e \in R$, and so $R/l_R(Ra) \cong R(1-e)$ is projective. Therefore a quasi-Baer ring satisfies the hypothesis of Theorem 3.4. Hence we have the following:

**Corollary 3.5.** Let $R$ be a ring, $(S, \leq)$ a strictly totally ordered monoid. Then a ring $R$ is a quasi-Baer ring if and only if $[[R^{S, \leq}]]$ is quasi-Baer ring.

**References**