Asia Pacific Journal of Mathematics, Vol. 5, No. 1 (2018), 85-91 ISSN 2357-2205

SUFFICIENT CONDITIONS FOR THE UNIVALENCE OF AN INTEGRAL OPERATOR

B.A. FRASIN*

Faculty of Science, Department of Mathematics, Al al-Bayt University, Mafraq, Jordan

Received Dec 26, 2017

Abstract. Let $F_{\alpha,\beta,\gamma}(z) = \left\{(1+\alpha+\beta+\gamma)\int\limits_0^z (f(t))^\alpha (g(t))^\beta (h(t))^\gamma dt\right\}^{\frac{1}{\alpha+\beta+\gamma}}$ be an integral operator defined in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ where the functions f, g, h are analytic in $\mathcal{U}$ and α, β, γ are complex numbers such that $\alpha+\beta+\gamma \neq 0$. The object of the present paper is to obtain some sufficient conditions for this integral operator.

2010 Mathematics Subject Classification.30C45.

Key words and phrases. analytic and univalent functions; univalence condition; integral operator.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of all analytic functions f in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$.

A function $f(z) \in \mathcal{A}$ is said to be a member of the class $\mathcal{B}(\mu, \delta)$ if it satisfies

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^\mu - 1\right| < 1-\delta \tag{1.1}$$

for some $\delta(0 \leq \delta < 1)$, $\mu \geq 0$ and for all $z \in \mathcal{U}$. The class $\mathcal{B}(\mu, \delta)$ were introduced and studied by Frasin and Jahangiri [12]. We observe that $\mathcal{B}(1, \delta) \subset \mathcal{S}^*(\delta)$ and $\mathcal{B}(0, \delta) \subset \mathcal{R}(\delta)$, where $\mathcal{S}^*(\delta)$ is the subclass of $\mathcal{A}$ consisting of starlike functions of order δ and satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \delta \qquad (z \in \mathcal{U})$$

for some $\delta(0 \leq \delta < 1)$ and $\mathcal{R}(\delta)$ is the subclass of $\mathcal{A}$ consisting functions and satisfying

$$\operatorname{Re}(f'(z)) > \delta, \qquad (z \in \mathcal{U})$$

for some $\delta(0 \le \delta < 1)$. Another interesting subclass is the special case $\mathcal{B}(2,\delta) \equiv \mathcal{B}(\delta)$ which has been introduced by Frasin and Darus [11] (see also, [7]) .

Ponnusamy and Sing [20] studied the subclass $\mathcal{S}_\epsilon$ of analytic functions defined as follows

$$\mathcal{S}_\epsilon = \{f \in \mathcal{A} : \left|\frac{zf'(z)}{f(z)} - 1\right| < \epsilon\,|z|\,,\ 0 < \epsilon \le 1,\ z \in \mathcal{U}\}. \tag{1.2}$$

The problem of finding sufficient conditions for univalence of various integral operators has been investigated in many recent works (see, for example, [1]-[4], [6]-[10], [15, 16]).

Pescar [17] obtained new univalence criteria for the integral operator defined by

$$F_{\alpha,\beta,\gamma}(z) = \left\{(1+\alpha+\beta+\gamma)\int_0^z (f(t))^\alpha\,(g(t))^\beta\,(h(t))^\gamma dt\right\}^{\frac{1}{\alpha+\beta+\gamma}} \tag{1.3}$$

where the functions $f, g, h \in \mathcal{A}$ and α, β, γ are complex numbers such that $\alpha+\beta+\gamma \neq 0$. Here and throughout in the sequel every many-valued function is taken with the principal branch.

In our present investigation we study some univalence conditions for the integral operator $F_{\alpha,\gamma,\beta}(z)$.

In order to derive our main results, we need the following lemmas.

Lemma 1.1. *([5])If $f(z) \in \mathcal{B}(\delta)$, then*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{(1-\delta)(1+|z|)}{1-|z|} \qquad (0 \le \delta < 1,\ z \in \mathcal{U}). \tag{1.4}$$

Lemma 1.2. *([18]) Let $\lambda \in \mathbb{C}$ with $Re(\lambda) > 0$. If $k \in \mathcal{A}$ satisfies*

$$\frac{1-|z|^{2\mathrm{Re}(\lambda)}}{\mathrm{Re}(\lambda)}\left|\frac{zk''(z)}{k'(z)}\right| \le 1,$$

for all $z \in \mathcal{U}$, then, for any complex number ζ with $\mathrm{Re}(\zeta) \ge \mathrm{Re}(\lambda)$, the integral operator

$$F_\zeta(z) = \left\{\zeta\int_0^z t^{\zeta-1}k'(t)dt\right\}^{\frac{1}{\zeta}} \tag{1.5}$$

is in the class $\mathcal{S}$.

Also, we need the following general Schwarz Lemma.

Lemma 1.3. *([13]) Let the function f be regular in the disk $\mathcal{U}_R = \{z : |z| < R\}$,with $|f(z)| < M$ for fixed M. If $f(z)$ has one zero with multiplicity order bigger than m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m}\,|z|^m \qquad (z \in \mathcal{U}_R)$$

The equality can hold only if

$$f(z) = e^{i\theta}\,(M/R^m)\,z^m$$

where θ is constant.

2. Univalence conditions for the integral operator $F_{\alpha,\beta,\gamma}$

First, we give univalence conditions for the integral operator $F_{\alpha,\beta,\gamma}$ where the analytic functions $f(z), g(z)$ and $h(z)$ are in the class $\mathcal{B}(\delta)$.

Theorem 2.1. *Let the analytic functions $f(z), g(z)$ and $h(z)$ be in the class $\mathcal{B}(\delta)$; $0 \leq \delta < 1$. Let $\alpha \in \mathbb{C}$ with $Re(\alpha) = a > 0$. If*

$$(|\alpha| + |\beta| + |\gamma|) \leq \min\left\{\frac{a}{2(1-\delta)}; \frac{1}{4(1-\delta)}\right\} \tag{2.1}$$

then the integral operator $F_{\alpha,\beta,\gamma}(z)$ defined by (1.3) is analytic and univalent in $\mathcal{U}$.

Proof. We observe that the integral operator $F_{\alpha,\beta,\gamma}(z)$ defined by (1.3) can be rewritten as

$$F_{\alpha,\beta,\gamma}(z) = \left\{(1+\alpha+\beta+\gamma)\int_0^z t^{\alpha+\beta+\gamma}\left(\frac{f(t)}{t}\right)^{\alpha}\left(\frac{g(t)}{t}\right)^{\beta}\left(\frac{h(t)}{t}\right)^{\gamma} dt\right\}^{\frac{1}{\alpha+\beta+\gamma}}.$$

Define the regular function $H(z)$ by

$$H(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\alpha}\left(\frac{g(t)}{t}\right)^{\beta}\left(\frac{h(t)}{t}\right)^{\gamma} dt.$$

Clearly $H \in \mathcal{A}$, i.e. $H(0) = H'(0) - 1 = 0$. On the other hand, it is easy to see that

$$\frac{zH''(z)}{H'(z)} = \alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + \beta\left(\frac{zg'(z)}{g(z)} - 1\right) + \gamma\left(\frac{zh'(z)}{h(z)} - 1\right). \tag{2.2}$$

Thus, we have

$$\left|\frac{zH''(z)}{H'(z)}\right| \leq |\alpha|\left|\frac{zf'(z)}{f(z)} - 1\right| + |\beta|\left|\frac{zg'(z)}{g(z)} - 1\right| + |\gamma|\left|\frac{zh'(z)}{h(z)} - 1\right|. \tag{2.3}$$

Since the analytic functions $f(z), g(z)$ and $h(z)$ are in the class $\mathcal{B}(\delta)$, from (1.4) and (2.3), we obtain

$$\left|\frac{zH''(z)}{H'(z)}\right| \leq (|\alpha| + |\beta| + |\gamma|)(1-\delta)\left(\frac{1+|z|}{1-|z|}\right). \tag{2.4}$$

Multiply both sides of (2.4) by $\frac{1-|z|^{2\mathrm{Re}(\lambda)}}{\mathrm{Re}(\lambda)}$, we get

$$\begin{aligned}\frac{1-|z|^{2\mathrm{Re}(\lambda)}}{\mathrm{Re}(\lambda)}\left|\frac{zH''(z)}{H'(z)}\right| &\leq \frac{1-|z|^{2\mathrm{Re}(\lambda)}}{\mathrm{Re}(\lambda)}(|\alpha|+|\beta|+|\gamma|)(1-\delta)\left(\frac{1+|z|}{1-|z|}\right)\\ &\leq \frac{1-|z|^{2\mathrm{Re}(\lambda)}}{1-|z|}\frac{2(|\alpha|+|\beta|+|\gamma|)(1-\delta)}{\mathrm{Re}(\lambda)} \qquad (2.5)\end{aligned}$$

for all $z \in \mathcal{U}$.

Let us denote $|z| = x$, $x \in [0,1)$, $\mathrm{Re}(\alpha) = a > 0$ and $\Phi(x) = \frac{1-x^{2a}}{1-x}$. It is easy to prove that

$$\Phi(x) \leq \begin{cases} 1, & \text{if } 0 < a < \frac{1}{2} \\ 2a, & \text{if } \frac{1}{2} < a < \infty. \end{cases} \qquad (2.6)$$

From (2.5), (2.6) and the hypothesis (2.1), we have

$$\begin{aligned}\frac{1-|z|^{2a}}{a}\left|\frac{zH''(z)}{H'(z)}\right| &\leq \begin{cases} \frac{2(|\alpha|+|\beta|+|\gamma|)(1-\delta)}{a}, & \text{if } 0 < a < \frac{1}{2} \\ 4(|\alpha|+|\beta|+|\gamma|)(1-\delta), & \text{if } \frac{1}{2} < a < \infty. \end{cases}\\ &\leq 1\end{aligned}$$

for all $z \in \mathcal{U}$. Applying Lemma 1.2 for the function $H(z)$, we prove that $F_{\alpha,\beta,\gamma}(z) \in \mathcal{S}$. Thus, the proof is complete □

Next, we prove the following theorem.

Theorem 2.2. *Let the analytic functions $f(z), g(z)$ and $h(z)$ be in the class $\mathcal{S}_\epsilon$; $0 < \epsilon \leq 1$. Let $\alpha \in \mathbb{C}$ with Re$(\alpha) = a > 0$. If*

$$(|\alpha|+|\beta|+|\gamma|) \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2\epsilon} \qquad (2.7)$$

then the integral operator $F_{\alpha,\beta,\gamma}(z)$ defined by (1.3) is analytic and univalent in $\mathcal{U}$.

Proof. Let the analytic functions $f(z), g(z)$ and $h(z)$ be in the class $\mathcal{S}_\epsilon$. From (2.2), we conclude that

$$\left|\frac{zH''(z)}{H'(z)}\right| \leq (|\alpha|+|\beta|+|\gamma|)\epsilon\,|z|\,. \qquad (2.8)$$

Thus, we have

$$\frac{1-|z|^{2a}}{a}\left|\frac{zH''(z)}{H'(z)}\right| \leq \frac{(|\alpha|+|\beta|+|\gamma|)\epsilon\,|z|\,(1-|z|^{2a})}{a} \qquad (z \in \mathcal{U}).$$

Let us denote $|z| = x$, $x \in [0,1]$, $\mathrm{Re}(\alpha) = a > 0$ and $\Psi(x) = x(1-x^{2a})$. It is easy to prove that the maximum is attained at the point $x = 1/(2a+1)^{1/2a}$ and therefore we have

$$\Psi(x) \leq \frac{2a}{(2a+1)^{\frac{2a+1}{2a}}}.$$

In view of this inequality and (2.7), we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zH''(z)}{H'(z)}\right| \le 1 \qquad (z\in\mathcal{U}).$$

Applying Lemma 1.2 for the function $H(z)$, we prove that $F_{\alpha,\beta,\gamma}(z)\in\mathcal{S}$. □

With the aid of Lemma 1.3, we prove the following theorem.

Theorem 2.3. *Let the analytic functions $f(z), g(z)$ and $h(z)$ be in the class $\mathcal{B}(\mu,\delta);\mu\ge 0,\ 0\le\delta<1$. If for all $z\in\mathcal{U}$ and $M\ge 1$, we have*

$$|f(z)|\le M\,,\ |g(z)|\le M \text{ and } |h(z)|\le M \tag{2.9}$$

and

$$\mathrm{Re}(\lambda)\ge ((2-\delta)\,M^{\mu-1}+1)(|\alpha|+|\beta|+|\gamma|) \qquad (\lambda\in\mathbb{C}), \tag{2.10}$$

then the integral operator $F_{\alpha,\beta,\gamma}(z)$ defined by (1.3) is analytic and univalent in $\mathcal{U}$.

Proof. It follows from (2.3) that

$$\begin{aligned}\left|\frac{zH''(z)}{H'(z)}\right| &\le |\alpha|\left(\left|\frac{zf'(z)}{f(z)}\right|+1\right)+|\beta|\left(\left|\frac{zg'(z)}{g(z)}\right|+1\right)+|\gamma|\left(\left|\frac{zh'(z)}{h(z)}\right|+1\right)\\ &= |\alpha|\left(\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu}\right|\left|\left(\frac{f(z)}{z}\right)^{\mu-1}\right|+1\right)\\ &\quad +|\beta|\left(\left|g'(z)\left(\frac{z}{g(z)}\right)^{\mu}\right|\left|\left(\frac{g(z)}{z}\right)^{\mu-1}\right|+1\right)\\ &\quad |\gamma|\left(\left|h'(z)\left(\frac{z}{h(z)}\right)^{\mu}\right|\left|\left(\frac{h(z)}{z}\right)^{\mu-1}\right|+1\right).\end{aligned} \tag{2.11}$$

Since for all $z\in\mathcal{U}$, $|f(z)|\le M$, $|g(z)|\le M$ and $|h(z)|\le M$, applying the Schwarz lemma for all $z\in\mathcal{U}$, we immediately obtain

$$\left|\frac{f(z)}{z}\right|\le M,\qquad \left|\frac{g(z)}{z}\right|\le M \text{ and } \left|\frac{h(z)}{z}\right|\le M\,.$$

Therefore, from (2.11) and (1.1), we see that

$$\begin{aligned}\left|\frac{zH''(z)}{H'(z)}\right| &\le |\alpha|\left(\left(\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu}-1\right|+1\right)M^{\mu-1}+1\right)\\ &\quad +|\alpha|\left(\left(\left|g'(z)\left(\frac{z}{g(z)}\right)^{\mu}-1\right|+1\right)M^{\mu-1}+1\right)\\ &\quad +|\alpha|\left(\left(\left|h'(z)\left(\frac{z}{h(z)}\right)^{\mu}-1\right|+1\right)M^{\mu-1}+1\right)\\ &\le ((2-\delta)\,M^{\mu-1}+1)(|\alpha|+|\beta|+|\gamma|).\end{aligned} \tag{2.12}$$

Multiply both sides of (2.12) by $\frac{1-|z|^{2\mathrm{Re}(\lambda)}}{\mathrm{Re}(\lambda)}$ and making use of (2.10), we get

$$\begin{aligned}\frac{1-|z|^{2\mathrm{Re}(\lambda)}}{\mathrm{Re}(\lambda)}\left|\frac{zH''(z)}{H'(z)}\right| &\leq \frac{1-|z|^{2\mathrm{Re}(\lambda)}}{\mathrm{Re}(\lambda)}((2-\delta)\,M^{\mu-1}+1)(|\alpha|+|\beta|+|\gamma|) \\ &\leq \frac{1}{\mathrm{Re}(\lambda)}((2-\delta)\,M^{\mu-1}+1)(|\alpha|+|\beta|+|\gamma|) \\ &\leq 1\end{aligned}$$

Applying Lemma 1.2 for the function $H(z)$, we prove that $F_{\alpha,\beta,\gamma}(z) \in \mathcal{S}$. □

References

[1] D. Blezu and R.N. Pascu, Univalence criteria for integral operators, Glasnik Math. 36 (56) (2001), 241-245.

[2] D. Breaz and N. Breaz, Univalence conditions for certain integral operators, Studia Universitatis Babes-Bolyai, Mathematica, Vol. XLVII No.2 (2002), 9-15.

[3] D. Breaz and V. Pescar, Univalence conditions for some general integral operators, Banach J. Math. Anal. (1) 2 (2008), 53-58.

[4] S. Bulut, Univalence preserving integral operators defined by generalized Al- Oboudi differential operators, An. Şt. Univ. Ovidius Constanţa, 17 (1), 2009, 37-50.

[5] E. Deniz and H. Orhan, An extension of the univalence criterion for a family of integral operators, Ann. Univ. Mariae Curie-Skłodowska Sect. A 64 (2010), no. 2, 29-35.

[6] B.A. Frasin, New criteria for univalence of certain integral operators, Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis, 27 (2011), 31–39.

[7] B. A. Frasin, A note on certain analytic and univalent functions, Southeast Asian J. Math. 28 (2004), 829-836.

[8] B.A. Frasin, Univalence of two general integral operator, Filomat, 23 (2009), 223–229.

[9] B.A. Frasin, Order of convexity and univalency of general integral operator, J. Franklin Inst. 348 (2011), 1013-1019.

[10] B.A. Frasin, New general integral operator, Comput. Math. Appl. 62 (2011) 4272–4276.

[11] B. A. Frasin and M. Darus, On certain analytic univalent functions, Int. J. Math. Math. Sci.25 (5) (2001), 305-310.

[12] B.A. Frasin and J. M. Jahangiri, A new and comprehensive class of analytic functions, Anal. Univ. Oradea, fac. math. Tom XV (2008), 59-62.

[13] Z. Nehari, Conformal Mapping, McGraw-Hill Book Comp., New York, 1952 (Dover. Publ. Inc., 1975).

[14] V. Pescar, Univalence of certain integral operators, Acta Univ. Apul. 12 (2006), 43-48.

[15] V. Pescar, On some integral classes of integral operators, General Math. 16 (1) (2008), 11-17.

[16] V. Pescar, Univalence conditions for certain integral operators, J. Inequal. Pure Appl. Math. 7 (2006), Article 147.

[17] V. Pescar, On an integral operator, Acta Universitatis Apulensis, 48 (2016), 119-125.

[18] N. Pascu, An improvement of Backer's univalence criterion, Proceedings of the Commemorative Session Simion Stoilow, Brasov, (1987), 43-48.

[19] N. Pascu and V. Pescar, On the integral operators of Kim-Merkis and Pfaltzgraff, Mathematica, 32 (55) (1990), 185-192.

[20] S. Ponnusamy and V. Singh, Critreria for univalent, starlike and convex functions, Bull. Belg. Math. Soc. 9 (2002), 511-531.