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CERTAIN HYPERGEOMETRIC GENERATING RELATIONS USING GOULD'S IDENTITY AND THEIR GENERALIZATIONS

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Abstract. In the present paper, we have obtained hypergeometric generating relations associated with two hypergeometric polynomials of one variable $H_n^{(\alpha,\beta)}(x;m)$ and $\mathscr{B}_n^{(\alpha,\beta)}(x;m,\lambda,\mu)$ with their independent demonstrations via Gould's identity.As applications,some well known and new generating relations are deduced.Using bounded sequences, further generalizations of two main hypergeometric generating relations have also been given for two generalized polynomials $S_n^{(\alpha,\beta)}(x;m)$ and $T_n^{(\alpha,\beta)}(x;m,\lambda,\mu)$.

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1. Introduction and preliminaries

Throughout in the present paper, we use the following standard notations:
$\mathbb{N} := \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 := \{0, 1, 2, 3, \ldots\} = \mathbb{N} \cup \{0\}$, $\mathbb{Z}_0^- := \{0, -1, -2, -3, \ldots\}$,
$\mathbb{Z}^- := \{-1, -2, -3, \ldots\} = \mathbb{Z}_0^- \backslash \{0\}$ and $\mathbb{Z} = (\mathbb{Z}_0^- \cup \mathbb{N})$.
Here, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of positive real numbers and $\mathbb{C}$ denotes the set of complex numbers.
The Pochhammer symbol (or the shifted factorial) $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ is defined, in terms of the familiar Gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C}\backslash\{0\}) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}) \end{cases} \tag{1.1}$$

it is being understood *conventionally* that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.
Some useful consequences of Lagrange's expansion [13, p.133];see also [5, p.146,problem 207]

include the following generalization [5, p.349,problem 216] of the familiar binomial expansion :

$$\sum_{n=0}^{\infty} \binom{\theta + (\beta+1)n}{n} t^n = \frac{(1+\zeta)^{\theta+1}}{(1-\beta\zeta)} \tag{1.2}$$

where $\binom{\theta+(\beta+1)n}{n}$ is a binomial coefficient and θ, β are complex numbers independent of n and ζ is a function of 't' defined implicitly by

$$\zeta = t(1+\zeta)^{1+\beta} \tag{1.3}$$

subject to the condition

$$\zeta(0) = 0 \tag{1.4}$$

Another generalization [5, p.348,problem 212] related with the equation (1.2),is given as:

$$\sum_{n=0}^{\infty} \frac{\theta}{\{\theta + (\beta+1)n\}} \binom{\theta + (\beta+1)n}{n} t^n$$

$$= \ 1 \ + \ \theta \sum_{n=1}^{\infty} \binom{\theta + (\beta+1)n - 1}{n-1} \frac{t^n}{n} \ = (1+\zeta)^{\theta} \tag{1.5}$$

where ζ is defined by the equations (1.3) and (1.4).
When $\beta = -1$,both results (1.2) and (1.5) reduce immediately to the binomial expansion.
Gould [2, p.90]; see also [7, p.169] gave the following identity:

$$\sum_{n=0}^{\infty} \frac{\theta(\sigma + \mu n)}{\theta + (\beta+1)n} \binom{\theta + (\beta+1)n}{n} t^n = (1+\zeta)^{\theta} \left(\sigma + \frac{\mu\theta\zeta}{1-\beta\zeta}\right) \tag{1.6}$$

where θ, β, σ, μ are complex parameters independent of n and ζ is given by the equations (1.3) and (1.4).
If we put $\theta = \{\alpha + (\beta+1)mr\}$ and $\sigma = \{\lambda + \mu\, mr\}$ in Gould's identity (1.6), we get the first modified form of Gould's identity:

$$\sum_{n=0}^{\infty} \frac{\{\alpha + (\beta+1)mr\}(\lambda + \mu n + \mu\, mr)}{\{\alpha + (\beta+1)mr + (\beta\ +1)n\}} \binom{\alpha + (\beta+1)mr + (\beta\ +1)n}{n} t^n$$

$$= (1+\zeta)^{\{\alpha+(\beta+1)mr\}} \left(\lambda + \mu\, mr + \frac{\mu\{\alpha + (\beta+1)mr\}\zeta}{(1-\beta\zeta)}\right) \tag{1.7}$$

with

$$\zeta = t(1+\zeta)^{\beta+1}\ ; \zeta(0) = 0$$

If we put $\theta = \{\alpha + (\beta+1)mr\}$ and $\sigma = \{\lambda + \mu\,(\beta+1)r\}$ in Gould's identity (1.6), we get the second modified form of Gould's identity:

$$\sum_{n=0}^{\infty} \frac{\{\alpha+(\beta+1)mr\}\{\lambda+\mu n+\mu\,(\beta+1)r\}}{\{\alpha+(\beta+1)mr+(\beta\ +1)n\}} \binom{\alpha+(\beta+1)mr+(\beta\ +1)n}{n} t^n$$

$$= (1+\zeta)^{\{\alpha+(\beta+1)mr\}} \left(\lambda + \mu\,(\beta+1)r + \frac{\mu\{\alpha+(\beta+1)mr\}\zeta}{(1-\beta\zeta)}\right) \tag{1.8}$$

with

$$\zeta = t(1+\zeta)^{\beta+1}\ ; \zeta(0) = 0$$

Gauss's Multiplication Theorem

For every positive integer m, we have

$$(b)_{mr} = m^{mr} \prod_{j=1}^{m} \left(\frac{b+j-1}{m}\right)_r \ ;\ r = 0, 1, 2, \ldots \tag{1.9}$$

Summation identity[12, p.101,Lemma(3),(2.1.6)]

$$\sum_{n=0}^{\infty} \sum_{r=0}^{[\frac{n}{m}]} B(r,n) = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} B(r, n+mr) \tag{1.10}$$

([x] denotes the greatest integer in x ; $m \in \mathbb{N}$),

provided that series involved are absolutely convergent.

The generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ [6, p.200,112(1)]are defined by

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} {}_1F_1\left[\ -n; 1+\alpha;\ x\right];\ n \in\ N_0 \tag{1.11}$$

Replacing α by $\alpha + n\beta$ in equation (1.11),we get

$${}_1F_1\left[\begin{matrix} -n & ; \\ 1+\alpha+n\beta; & \end{matrix}\ x\right] = \frac{n!}{(1+\alpha+n\beta)_n} L_n^{(\alpha+n\beta)}(x) \tag{1.12}$$

The Jacobi Polynomials of first kind $P_n^{(\alpha,\beta)}(x)$ [6, p.254 (132.1),p.255 (132.7)] are defined by the following equations:

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1\left[\begin{matrix} -n, 1+\alpha+\beta+n; \\ 1+\alpha \qquad ; \end{matrix}\ \frac{1-x}{2}\right] \tag{1.13}$$

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha+\beta)_{2n}}{n!(1+\alpha+\beta)_n} \left(\frac{x-1}{2}\right)^n {}_2F_1\left[\begin{matrix} -n,\ -\alpha-n\ ; \\ -\alpha-\beta-2n; \end{matrix}\ \frac{2}{1-x}\right] \tag{1.14}$$

where n is a non-negative integer.

Replacing α by $(\alpha+bn)$ and β by $\{\beta-(b+1)n\}$ in equation (1.13), we get

$$(1.15)\quad {}_2F_1\left[\begin{matrix}-n, 1+\alpha+\beta; & \frac{1-x}{2}\\ 1+\alpha+bn\ ; & \end{matrix}\right] = \frac{n!\,\Gamma(\alpha+bn+1)}{\Gamma\{\alpha+(b+1)n+1\}} P_n^{(\alpha+bn\,,\,\beta-(b+1)n)}(x)$$

Replacing α by $(\alpha-n)$ and β by $\{\beta-(b+1)n\}$ in equation (1.14),we get the following result

$$(1.16)\quad {}_2F_1\left[\begin{matrix}-n,\ -\alpha\ ; & \frac{2}{1-x}\\ bn-\alpha-\beta; & \end{matrix}\right] = \frac{\Gamma(1+\alpha+\beta-bn-n)n!}{\Gamma(1+\alpha+\beta-bn)}\left(\frac{2}{x-1}\right)^n P_n^{(\alpha-n,\beta-bn-n)}(x)$$

The generalized Rice Polynomials $H_n^{(\alpha,\beta)}[\nu,\sigma,x]$ of Khandekar [4, p.158,eq.(2.3)] are defined by

$$(1.17)\quad H_n^{(\alpha,\beta)}[\nu,\sigma,x] = \binom{\alpha+n}{n}\,{}_3F_2\left[\begin{matrix}-n,\ \alpha+\beta+n+1,\nu; & x\\ \alpha+1,\sigma\qquad ; & \end{matrix}\right]$$

$$(1.18)\quad H_n[\nu,\sigma,x] = H_n^{(0,0)}[\nu,\sigma,x]$$

$$(1.19)\quad P_n^{(\alpha,\beta)}(x) = H_n^{(\alpha,\beta)}\left[\nu,\nu,\frac{1-x}{2}\right]$$

Replacing α by $(\alpha+b\,n)$ and β by $\{\beta-(b+1)n\}$ in eq.(1.17), we get

$$(1.20)\quad {}_3F_2\left[\begin{matrix}-n,\ \nu,\ 1+\alpha+\beta\ ; & x\\ 1+\alpha+bn,\ \sigma; & \end{matrix}\right] = \frac{n!}{(1+\alpha+b\,n)_n}\ H_n^{(\alpha+bn,\beta-(b+1)n)}\,[\nu,\sigma,x]$$

<u>Some useful Pochhammer's relations</u>

$$(1.21)\quad \frac{(\lambda+\mu\,n)\left(\frac{\lambda+\mu\,n}{\mu(\beta+1-m)}+1\right)_r}{\left(\frac{\lambda+\mu\,n}{\mu(\beta+1-m)}\right)_r} = \lambda+\mu\,n+\mu(\beta+1)r-\mu\,mr$$

(1.22)

$$\left(\lambda+\frac{\mu\zeta\alpha}{(1-\beta\zeta)}\right)+\left(\mu\,mr+\frac{\mu\zeta\beta\,mr+\mu\zeta\,mr}{(1-\beta\zeta)}\right) = \left(\frac{\lambda(1-\beta\zeta)+\mu\,\zeta\alpha}{(1-\beta\zeta)}\right)\frac{\left(\frac{\lambda(1-\beta\zeta)+\mu\,\zeta\alpha}{\mu\,m(\zeta+1)}+1\right)_r}{\left(\frac{\lambda(1-\beta\zeta)+\mu\,\zeta\alpha}{\mu\,m(\zeta+1)}\right)_r}$$

$$(1.23)\quad \alpha+m(\beta+1)r = \alpha\,\frac{\left(\frac{\alpha}{m(\beta+1)}+1\right)_r}{\left(\frac{\alpha}{m(\beta+1)}\right)_r}$$

$$\left(\lambda+\frac{\mu\,\zeta\,\alpha}{(1-\beta\,\zeta)}\right)+\left\{\mu\,(\beta+1)r+\frac{\mu\,\zeta\,\beta\,m\,r}{(1-\beta\,\zeta)}+\frac{\mu\,\zeta\,m\,r}{(1-\beta\,\zeta)}\right\}$$
$$= \frac{\{\lambda(1-\beta\,\zeta)+\mu\,\zeta\,\alpha\}}{(1-\beta\,\zeta)}\frac{\left(\frac{\{\lambda(1-\beta\,\zeta)+\mu\,\zeta\,\alpha\}}{\mu\,(\beta+1)(1-\beta\,\zeta+m\,\zeta)}+1\right)_r}{\left(\frac{\{\lambda(1-\beta\,\zeta)+\mu\,\zeta\,\alpha\}}{\mu\,(\beta+1)(1-\beta\,\zeta+m\,\zeta)}\right)_r}\qquad \text{‘}(1.23a)$$

where

$$r = 0, 1, 2, 3, \ldots$$

Now we shall discuss some special cases of the implicit functions defined by equation (1.3) subject to the condition (1.4).Using Mathematica 9.0, we can find the roots of resulting cubic equation in ζ for different values of β in equation (1.3).

Case I:- When $\beta = 0$ in (1.3),then particular value of ζ(satisfying the condition (1.4)) is denoted by

$$\Theta = \frac{t}{1-t} \tag{1.24}$$

Case II:- When $\beta = 1$ in (1.3),we get

$$t\zeta^2 + (2t-1)\zeta + t = 0$$

then one of the values of ζ (satisfying the condition (1.4)) is given by

$$\Lambda = \frac{1 - 2t - \sqrt{(1-4t)}}{2t} \tag{1.25}$$

Case III:- When $\beta = -2$ in (1.3),we get

$$\zeta^2 + \zeta - t = 0$$

then the particular value of ζ(satisfying the condition (1.4)) is given by

$$\Xi = \frac{-1 + \sqrt{(1+4t)}}{2} \tag{1.26}$$

Case IV:- When $\beta = -3$ in (1.3), we get

$$\zeta^3 + 2\zeta^2 + \zeta - t = 0$$

then one of the roots(satisfying the condition (1.4)) of above equation is given by

$$\Upsilon = \frac{1}{3}\left[-2 + \frac{2^{\frac{1}{3}}}{\left\{2 + 27t + 3\sqrt{3}\sqrt{(4t+27t^2)}\right\}^{\frac{1}{3}}} + \frac{\left\{2 + 27t + 3\sqrt{3}\sqrt{(4t+27t^2)}\right\}^{\frac{1}{3}}}{2^{\frac{1}{3}}}\right] \tag{1.27}$$

Case V:- When $\beta = -\frac{1}{2}$ in (1.3), we get

$$\zeta^2 - t^2\zeta - t^2 = 0$$

then one of the roots(satisfying the condition (1.4)) of above equation is given by

$$U = \frac{t}{2}\{t + \sqrt{(t^2+4)}\} \tag{1.28}$$

Case VI:- When $\beta = \frac{-3}{2}$ in (1.3),we get

$$\zeta^3 + \zeta^2 - t^2 = 0$$

then one of the roots(satisfying the condition (1.4)) of above equation is given by

$$\Psi = -\frac{1}{3}\left[1 + \frac{(1+\iota\sqrt{3})}{2^{\frac{2}{3}}\left\{-2+27t^2+3\sqrt{3}\sqrt{(-4t^2+27t^4)}\right\}^{\frac{1}{3}}}\right.$$

$$\left. + \frac{(1-\iota\sqrt{3})\left\{-2+27t^2+3\sqrt{3}\sqrt{(-4t^2+27t^4)}\right\}^{\frac{1}{3}}}{2^{\frac{4}{3}}}\right] \tag{1.29}$$

where $\iota = \sqrt{(-1)}$.

Case VII:- When $\beta = -\frac{1}{3}$ in (1.3),we obtain

$$\zeta^3 - t^3\zeta^2 - 2t^3\zeta - t^3 = 0$$

then one of the values of ζ (satisfying the condition (1.4)) is denoted by

$$\Pi = \frac{t^3}{3} - \frac{2^{\frac{1}{3}}(-6t^3 - t^6)}{3\{27t^3 + 18t^6 + 2t^9 + 3\sqrt{3}\sqrt{(27t^6 + 4t^9)}\}^{\frac{1}{3}}} +$$

$$\frac{\{27t^3 + 18t^6 + 2t^9 + 3\sqrt{3}\sqrt{(27t^6 + 4t^9)}\}^{\frac{1}{3}}}{3.2^{\frac{1}{3}}} \tag{1.30}$$

Case VIII:- When $\beta = -\frac{2}{3}$ in (1.3),we obtain

$$\zeta^3 - t^3\zeta - t^3 = 0$$

then one of the values of ζ (satisfying the condition (1.4)) is denoted by

$$\Phi = \frac{\left(\frac{2}{3}\right)^{\frac{1}{3}} t^3}{\left\{9t^3 + \sqrt{3}\sqrt{(27t^6 - 4t^9)}\right\}^{\frac{1}{3}}} + \frac{\left\{9t^3 + \sqrt{3}\sqrt{(27t^6 - 4t^9)}\right\}^{\frac{1}{3}}}{2^{\frac{1}{3}}3^{\frac{2}{3}}} \tag{1.31}$$

2. Main generating relations

First Generating Relation:

If any values of variables and parameters leading to the results which do not make sense, are tacitly excluded, then

$$\sum_{n=0}^{\infty} \frac{(\lambda + \mu n)}{\{\alpha + (\beta+1)n\}} H_n^{(\alpha,\beta)}(x;m)\ t^n = (1+\zeta)^{\alpha}.$$

$$(2.1)\qquad .\left\{\frac{\lambda(1-\beta\zeta)+\mu\zeta\alpha}{\alpha(1-\beta\zeta)}\right\}{}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{\alpha}{m\,(\beta+1)},1+\frac{\lambda\,(1-\beta\zeta)+\mu\zeta\alpha}{m\,\mu\,(1+\zeta)},(a_p);\\ 1+\frac{\alpha}{m\,(\beta+1)},\frac{\lambda\,(1-\beta\zeta)+\mu\zeta\alpha}{m\,\mu\,(1+\zeta)},(b_q);\end{array} x(-\zeta)^m\right]$$

where

$$\zeta=t(1+\zeta)^{1+\beta};\zeta(0)=0.$$

provided that involved series on both sides are absolutely convergent.
Here Srivastava's generalized hypergeometric polynomials of one variable $H_n^{(\alpha,\beta)}(x;m)$ [12, p.360,eq.(7.3.3)];see also [11, pp.331-332] are given by

$$(2.2)\qquad H_n^{(\alpha,\beta)}(x;m)=\binom{\alpha+(\beta+1)n}{n}{}_{p+m}F_{q+m}\left[\begin{array}{cc}\Delta(m;-n),(a_p) & ;\\ \Delta(m;1+\alpha+\beta\,n),(b_q); & \end{array} x\right]$$

where α,β are complex parameters independent of 'n' and $\Delta(m;\lambda)$ abbreviates the array of m number of parameters given by

$$\frac{\lambda}{m},\frac{\lambda+1}{m},\ldots,\frac{\lambda+m-1}{m};m\in\mathbb{N}$$

Independent Demonstration:
Using the definition (2.2) of $H_n^{(\alpha,\beta)}(x;m)$ and then the power series form of ${}_{p+m}F_{q+m}[x]$ in left hand side of equation (2.1),we get

$$\Omega=\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{\{\alpha+(\beta+1)n\}}H_n^{(\alpha,\beta)}(x;m)\ t^n=\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)\ \Gamma\{\alpha+(\beta+1)n+1\}}{\{\alpha+(\beta+1)n\}\ \Gamma(n+1)\ \Gamma\{\alpha+\beta\ n+1\}}\ .$$

$$.\sum_{r=0}^{[\frac{n}{m}]}\frac{\prod_{j=1}^{m}\left(\frac{-n+j-1}{m}\right)_r\ [(a_p)]_r\ x^r\ t^n}{\prod_{j=1}^{m}\left(\frac{1+\alpha+n\beta+j-1}{m}\right)_r\ [(b_q)]_r\ r!}$$

Using Gauss's multiplication theorem (1.9) in above equation, we get

$$(2.3)\quad \Omega=\sum_{n=0}^{\infty}\sum_{r=0}^{[\frac{n}{m}]}\frac{(\lambda+\mu n)\ (-n)_{mr}\ (\alpha)_{n(\beta+1)}\ [(a_p)]_r\ x^r\ t^n}{\alpha\ (\alpha+1)_{n\beta}\ (1+\alpha+n\beta)_{mr}\ [(b_q)]_r\ r!\ n!}$$

Now applying summation identity (1.10) and then simplifying further, we get

$$\Omega=\sum_{r=0}^{\infty}\frac{(\alpha)_{m(\beta+1)r}\ [(a_p)]_r\ x^r\ (-t)^{mr}}{\alpha\ (\alpha+1)_{m(\beta+1)r}\ [(b_q)]_r\ r!}\sum_{n=0}^{\infty}\frac{\{\alpha+(\beta+1)mr\}(\lambda+\mu n+\mu\,mr)}{\{\alpha+(\beta+1)mr+(\beta\ +1)n\}}.$$

$$(2.4)\qquad .\binom{\alpha+(\beta+1)mr+(\beta\ +1)n}{n}t^n$$

Now using first modified Gould's identity (1.7),we get

$$\Omega=(1+\zeta)^{\alpha}\sum_{r=0}^{\infty}\frac{(\alpha)_{m(\beta+1)r}\ [(a_p)]_r\ x^r\ (-t)^{mr}}{\alpha\ (\alpha+1)_{m(\beta+1)r}\ [(b_q)]_r\ r!}(1+\zeta)^{m(\beta+1)r}.$$

$$(2.5)\qquad .\left[\left(\lambda+\frac{\mu\zeta\alpha}{(1-\beta\zeta)}\right)+\left(\mu\, mr+\frac{\mu\zeta(\beta+1)mr\}}{(1-\beta\zeta)}\right)\right]$$

Now using (1.22) and simplifying it further, we get

$$(2.6)\quad \Omega=(1+\zeta)^{\alpha}\left(\frac{\lambda(1-\beta\zeta)+\mu\zeta\alpha}{(1-\beta\zeta)}\right)\sum_{r=0}^{\infty}\frac{(\frac{\alpha}{m(\beta+1)})_r\ [(a_p)]_r\ x^r\ (-\zeta)^{mr}}{\alpha\ (\frac{\alpha}{m(\beta+1)}+1)_r\ [(b_q)]_r\ r!}.\frac{\left(\frac{\lambda(1-\beta\zeta)+\mu\zeta\alpha}{\mu\, m(1+\zeta)}+1\right)_r}{\left(\frac{\lambda(1-\beta\zeta)+\mu\zeta\alpha}{\mu\, m(1+\zeta)}\right)_r}$$

After solving it further,we get the result (2.1) in the form of a generalized hypergeometric function of one variable.

Second Generating Relation:

If any values of variables and parameters leading to the results which do not make sense, are tacitly excluded, then

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{\{\alpha+(\beta+1)n\}}\mathscr{B}_n^{(\alpha,\beta)}(x;m,\lambda,\mu)\ t^n=(1+\zeta)^{\alpha}\ .$$

$$(2.7)\qquad .\left\{\frac{\lambda(1-\beta\zeta)+\mu\zeta\alpha}{\alpha(1-\beta\zeta)}\right\}{}_{p+2}F_{q+2}\left[\begin{array}{l}\frac{\alpha}{m\,(\beta+1)},1+\frac{\lambda\,(1-\beta\zeta)+\mu\zeta\alpha}{\mu\,(1+\beta)(1-\beta\zeta+m\zeta)},(a_p);\\ 1+\frac{\alpha}{m\,(\beta+1)},\frac{\lambda\,(1-\beta\zeta)+\mu\zeta\alpha}{\mu\,(1+\beta)(1-\beta\zeta+m\zeta)},(b_q);\end{array}\ x(-\zeta)^m\right]$$

where

$$\zeta=t(1+\zeta)^{1+\beta};\zeta(0)=0.$$

provided that involved series on both sides are absolutely convergent.

Here we define new generalized hypergeometric polynomials$\mathscr{B}_n^{(\alpha,\beta)}(x;m,\lambda,\mu)$,known as "Pathan's generalized hypergeometric polynomials of one variable", given by

(2.8)

$$\mathscr{B}_n^{(\alpha,\beta)}(x;m,\lambda,\mu)=\binom{\alpha+(\beta+1)n}{n}{}_{p+m+1}F_{q+m+1}\left[\begin{array}{c}\Delta(m;-n),1+\frac{\lambda+\mu\,n}{\mu(\beta+1-m)},(a_p);\\ \Delta(m;1+\alpha+\beta\,n),\frac{\lambda+\mu\,n}{\mu(\beta+1-m)},(b_q);\end{array}\ x\right]$$

where α,β are complex parameters independent of 'n' and $\Delta(m;\lambda)$ abbreviates the array of m number of parameters given by

$$\frac{\lambda}{m},\frac{\lambda+1}{m},\ldots,\frac{\lambda+m-1}{m};m\in\mathbb{N}$$

Independent Demonstration:

Using the definition (2.8) of $\mathscr{B}_n^{(\alpha,\beta)}(x;m,\lambda,\mu)$ and then the power series form of ${}_{p+m+1}F_{q+m+1}[x]$ in left hand side of equation (2.7),using Gauss's multiplication theorem (1.9) and result (1.21),we get

$$\Omega^*=\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{\{\alpha+(\beta+1)n\}}\mathscr{B}_n^{(\alpha,\beta)}(x;m,\lambda,\mu)\ t^n=$$

$$(2.9)\qquad \sum_{n=0}^{\infty}\sum_{r=0}^{[\frac{n}{m}]}\frac{[\lambda+\mu n+\mu(\beta+1)r-\mu\, mr]\ (-n)_{mr}\ \Gamma\{\alpha+n(\beta+1)\}\ [(a_p)]_r\ x^r\ t^n}{\Gamma\{\alpha+n\beta+1+mr\}\ [(b_q)]_r\ r!\ n!}$$

Now applying summation identity (1.10) in above equation then simplifying further, we get

$$\Omega^*=\sum_{r=0}^{\infty}\frac{[(a_p)]_r\ x^r\ (-t)^{mr}}{[\alpha+m(\beta+1)r]\ [(b_q)]_r\ r!}\sum_{n=0}^{\infty}\frac{\{\alpha+(\beta+1)mr\}(\lambda+\mu n+\mu(\beta+1)r)}{\{\alpha+(\beta+1)mr+(\beta\ +1)n\}}.$$

$$(2.10)\qquad .\binom{\alpha+(\beta+1)mr+(\beta\ +1)n}{n}t^n$$

Now using second modified Gould's identity (1.8) and eq.(1.23),we get

$$\Omega^*=(1+\zeta)^{\alpha}\sum_{r=0}^{\infty}\frac{\left(\frac{\alpha}{m(\beta+1)}\right)_r[(a_p)]_r\ x^r\ (-\zeta)^{mr}}{\alpha\left(\frac{\alpha}{m(\beta+1)}+1\right)_r\ [(b_q)]_r\ r!}$$

$$(2.11)\qquad .\left[\left(\lambda+\frac{\mu\,\zeta\alpha}{(1-\beta\zeta)}\right)+\left\{\mu(\beta+1)r+\frac{\mu\zeta(\beta+1)\,mr}{(1-\beta\zeta)}\right\}\right]$$

Now using equation $(1.23a)$ in above equation and summing it up into hypergeometric form further, we get the desired result (2.7).

3. Known applications of first generating relation (**2.1**)

(i). Putting $\lambda=1,\mu=\frac{\beta+1}{\alpha}$ in equation (2.1) and after simplifying, we get

$$(3.1)\qquad \sum_{n=0}^{\infty}H_n^{(\alpha,\beta)}(x;m)t^n=\frac{(1+\zeta)^{\alpha+1}}{(1-\beta\zeta)}\ {}_pF_q\left[\begin{array}{l}(a_p);\\(b_q);\end{array}x(-\zeta)^m\right]$$

which is the result of Srivastava [9, p.975];see also [10, p.233,eq.(12)].Here ζ being given by equation (1.3) and (1.4) and $H_n^{(\alpha,\beta)}(x;m)$ is given by equation (2.2).

(ii). Putting $\lambda=1,\mu=\frac{\beta+1}{\alpha}$ and $m=1$ in equation (2.1) and using the definition of $H_n^{(\alpha,\beta)}(x;1)$ and after simplification, we get

$$\sum_{n=0}^{\infty}\binom{\alpha+(\beta+1)n}{n}{}_{p+1}F_{q+1}\left[\begin{array}{lc}-n,(a_p)&;\\1+\alpha+n\beta,(b_q);&\end{array}x\right]t^n$$

$$(3.2)\qquad =\frac{(1+\zeta)^{\alpha+1}}{(1-\beta\zeta)}{}_pF_q\left[\begin{array}{l}(a_p);\\(b_q);\end{array}x(-\zeta)\right]$$

which is the result of Srivastava [8, p.591,eq.(9)];see also [3, p.1186].ζ is given by equations (1.3) and (1.4).

(iii). Putting $\lambda=1,\mu=\frac{1}{2\alpha},\beta=\frac{-1}{2}$ and $m=1$ in equation (2.1),using the definition(2.2) of $H_n^{(\alpha,-\frac{1}{2})}(x;1)=f_n^{(\alpha)}(x)$ and replacing ζ by U, we get

$$\sum_{n=0}^{\infty} f_n^{(\alpha)}(x)t^n=\sum_{n=0}^{\infty}\binom{\alpha+\frac{n}{2}}{n}{}_{p+1}F_{q+1}\left[\begin{array}{c}-n,(a_p)\qquad ;\\ 1+\alpha-\frac{n}{2},\,(b_q)\;;\end{array} x\right]t^n$$

$$=\frac{[1+U]^{\alpha+1}}{(1+\frac{U}{2})}{}_pF_q\left[\begin{array}{c}(a_p);\\ (b_q);\end{array} -xU\right] \tag{3.3}$$

where

$$U=\frac{t}{2}\{t+\sqrt{(t^2+4)}\}$$

which is the known result of Brown [1, p.264,eq.(7)] and $f_n^{(\alpha)}(x)$ is Brown's generalized hypergeometric polynomial [12, p.358,eq.(7.2.4)].

4. New applications of first generating relation (2.1)

(i). Putting $\beta=0$ and $\zeta=\Theta=\frac{t}{1-t}$ from equation (1.24) in equation (2.1), we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha+n)}\binom{\alpha+n}{n}{}_{p+m}F_{q+m}\left[\begin{array}{c}\Delta(m;-n),(a_p)\quad ;\\ \Delta(m;1+\alpha),(b_q);\end{array} x\right]\;t^n=(1-t)^{-\alpha-1}.$$

$$.\left\{\frac{\lambda(1-t)+\mu\alpha\, t}{\alpha}\right\}\;{}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{\alpha}{m},1+\frac{\lambda\,(1-t)+\mu\alpha\, t}{m\,\mu},(a_p);\\ 1+\frac{\alpha}{m},\frac{\lambda\,(1-t)+\mu\alpha\, t}{m\,\mu},(b_q);\end{array} x\left(\frac{-t}{1-t}\right)^m\right] \tag{4.1}$$

(ii). Putting $\beta=1$ and $\zeta=\Lambda=\frac{1-2t-\sqrt{(1-4t)}}{2t}$from equation (1.25) in equation (2.1),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha+2n)}\binom{\alpha+2n}{n}{}_{p+m}F_{q+m}\left[\begin{array}{c}\Delta(m;-n),(a_p)\qquad ;\\ \Delta(m;1+\alpha+n),(b_q);\end{array} x\right]\;t^n=(1+\Lambda)^{\alpha}.$$

$$.\left\{\frac{\lambda(1-\Lambda)+\mu\,\alpha\,\Lambda}{\alpha(1-\Lambda)}\right\}\;{}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{\alpha}{2m},1+\frac{\lambda(1-\Lambda)+\mu\,\alpha\,\Lambda}{m\,\mu\,(1+\Lambda)},(a_p);\\ 1+\frac{\alpha}{2m},\frac{\lambda(1-\Lambda)+\mu\,\alpha\,\Lambda}{m\,\mu\,(1+\Lambda)},(b_q);\end{array} x(-\Lambda)^m\right] \tag{4.2}$$

(iii). Putting $\beta=-2$ and $\zeta=\Xi=\frac{-1+\sqrt{(1+4t)}}{2}$from equation (1.26) in equation (2.1),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha-n)}\binom{\alpha-n}{n}{}_{p+m}F_{q+m}\left[\begin{array}{c}\Delta(m;-n),(a_p)\qquad ;\\ \Delta(m;1+\alpha-2n),(b_q);\end{array} x\right]\;t^n=(1+\Xi)^{\alpha}.$$

$$.\left\{\frac{\lambda(1+2\Xi)+\mu\alpha\Xi}{\alpha(1+2\Xi)}\right\}\;{}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{-\alpha}{m},1+\frac{\lambda\,(1+2\Xi)+\mu\alpha\Xi}{m\,\mu\,(1+\Xi)},(a_p);\\ 1-\frac{\alpha}{m},\frac{\lambda\,(1+2\Xi)+\mu\alpha\Xi}{m\,\mu\,(1+\Xi)},(b_q);\end{array} x(-\Xi)^m\right] \tag{4.3}$$

(**iv**). Putting $\beta=-3$ and $\zeta=\Upsilon$ from equation (1.27) in equation (2.1),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha-2n)}\binom{\alpha-2n}{n}{}_{p+m}F_{q+m}\left[\begin{array}{cc}\Delta(m;-n),(a_p) & ;\\ \Delta(m;1+\alpha-3n),(b_q); & \end{array} x\right]\ t^n=(1+\Upsilon)^{\alpha}.$$

$$\cdot\left\{\frac{\lambda(1+3\Upsilon)+\mu\alpha\Upsilon}{\alpha(1+3\Upsilon)}\right\}\ {}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{-\alpha}{2m},1+\frac{\lambda(1+3\Upsilon)+\mu\alpha\Upsilon}{m\,\mu\,(1+\Upsilon)},(a_p);\\ 1-\frac{\alpha}{2\,m},\frac{\lambda(1+3\Upsilon)+\mu\alpha\Upsilon}{m\,\mu\,(1+\Upsilon)},(b_q);\end{array} x(-\Upsilon)^m\right] \tag{4.4}$$

(**v**). Putting $\beta=\frac{-1}{2}$ and $\zeta=U$ from equation (1.28) in equation (2.1),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha+\frac{1}{2}n)}\binom{\alpha+\frac{1}{2}n}{n}{}_{p+m}F_{q+m}\left[\begin{array}{cc}\Delta(m;-n),(a_p) & ;\\ \Delta(m;1+\alpha-\frac{1}{2}n),(b_q); & \end{array} x\right]\ t^n=(1+U)^{\alpha}.$$

$$\cdot\left\{\frac{\lambda(1+\frac{U}{2})+\alpha\mu\,U}{\alpha(1+\frac{U}{2})}\right\}\ {}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{2\alpha}{m},1+\frac{\lambda(1+\frac{U}{2})+\alpha\mu\,U}{m\,\mu\,(1+U)},(a_p);\\ 1+\frac{2\alpha}{m},\frac{\lambda(1+\frac{U}{2})+\alpha\mu\,U}{m\,\mu\,(1+U)},(b_q);\end{array} x(-U)^m\right] \tag{4.5}$$

(**vi**). Putting $\beta=\frac{-3}{2}$ and $\zeta=\Psi$ from equation (1.29) in equation (2.1),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha-\frac{1}{2}n)}\binom{\alpha-\frac{1}{2}n}{n}{}_{p+m}F_{q+m}\left[\begin{array}{cc}\Delta(m;-n),(a_p) & ;\\ \Delta(m;1+\alpha-\frac{3}{2}n),(b_q); & \end{array} x\right]\ t^n=(1+\Psi)^{\alpha}.$$

$$\cdot\left\{\frac{\lambda(1+\frac{3}{2}\,\Psi)+\alpha\mu\Psi}{\alpha(1+\frac{3}{2}\,\Psi)}\right\}\ {}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{-2\alpha}{m},1+\frac{\lambda(1+\frac{3}{2}\,\Psi)+\alpha\mu\Psi}{m\mu(1+\Psi)},(a_p);\\ 1-\frac{2\alpha}{m},\frac{\lambda(1+\frac{3}{2}\,\Psi)+\alpha\mu\Psi}{m\mu(1+\Psi)},(b_q);\end{array} x(-\Psi)^m\right] \tag{4.6}$$

(**vii**). Putting $\beta=\frac{-1}{3}$ and $\zeta=\Pi$ from equation (1.30) in equation (2.1),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha+\frac{2}{3}n)}\binom{\alpha+\frac{2}{3}n}{n}{}_{p+m}F_{q+m}\left[\begin{array}{cc}\Delta(m;-n),(a_p) & ;\\ \Delta(m;1+\alpha-\frac{1}{3}n),(b_q); & \end{array} x\right]\ t^n=(1+\Pi)^{\alpha}.$$

$$\cdot\left\{\frac{\lambda(1+\frac{1}{3}\,\Pi)+\alpha\mu\Pi}{\alpha(1+\frac{1}{3}\,\Pi)}\right\}\ {}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{3\alpha}{2m},1+\frac{\lambda(1+\frac{1}{3}\,\Pi)+\alpha\mu\Pi}{m\mu(1+\Pi)},(a_p);\\ 1+\frac{3\alpha}{2\,m},\frac{\lambda(1+\frac{1}{3}\,\Pi)+\alpha\mu\Pi}{m\mu(1+\Pi)},(b_q);\end{array} x(-\Pi)^m\right] \tag{4.7}$$

(**viii**). Putting $\beta=\frac{-2}{3}$ and $\zeta=\Phi$ from equation (1.31) in equation (2.1),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha+\frac{1}{3}n)}\binom{\alpha+\frac{1}{3}n}{n}{}_{p+m}F_{q+m}\left[\begin{array}{cc}\Delta(m;-n),(a_p) & ;\\ \Delta(m;1+\alpha-\frac{2}{3}n),(b_q); & \end{array} x\right]\ t^n=(1+\Phi)^{\alpha}.$$

$$\cdot\left\{\frac{\lambda(1+\frac{2}{3}\,\Phi)+\mu\alpha\Phi}{\alpha(1+\frac{2}{3}\,\Phi)}\right\}\ {}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{3\alpha}{m},1+\frac{\lambda(1+\frac{2}{3}\,\Phi)+\mu\alpha\Phi}{m\,\mu\,(1+\Phi)},(a_p);\\ 1+\frac{3\alpha}{m},\frac{\lambda(1+\frac{2}{3}\,\Phi)+\mu\alpha\Phi}{m\,\mu\,(1+\Phi)},(b_q);\end{array} x(-\Phi)^m\right] \tag{4.8}$$

5. New applications of second generating relation (**2.7**)

(i). Putting $\lambda=1,\mu=\frac{\beta+1}{\alpha}$ in equation (2.7) and after simplifying, we get

$$\sum_{n=0}^{\infty}\binom{\alpha+(\beta+1)n}{n}{}_{p+m+1}F_{q+m+1}\left[\begin{array}{c}\Delta(m;-n),\ 1+\ \frac{\alpha+(\beta+1)\,n}{(\beta+1)(\beta+1-m)},\ (a_p);\\ \Delta(m;1+\alpha+\beta\,n),\ \frac{\alpha+(\beta+1)\,n}{(\beta+1)(\beta+1-m)},\ (b_q);\end{array} x\right]\ t^n$$

$$=\frac{(1+\zeta)^{(\alpha+1)}}{(1-\beta\zeta)}{}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{\alpha}{m(\beta+1)},1+\frac{\alpha(1+\zeta)}{(\beta+1)^2(1-\beta\zeta+m\zeta)},(a_p);\\ 1+\frac{\alpha}{m(\beta+1)},\frac{\alpha(1+\zeta)}{(\beta+1)^2(1-\beta\zeta+m\zeta)},(b_q);\end{array} x(-\zeta)^m\right] \tag{5.1}$$

where ζ is given by equations (1.3) and (1.4).

(ii). Putting $\lambda=1,\mu=\frac{\beta+1}{\alpha},\ m=1$ in equation (2.7) and using the definition of $\mathscr{B}_n^{(\alpha,\beta)}(x;1,1,\frac{\beta+1}{\alpha})$, we get

$$\sum_{n=0}^{\infty}\binom{\alpha+(\beta+1)n}{n}{}_{p+2}F_{q+2}\left[\begin{array}{c}-n,\ 1+\ \frac{\alpha+(\beta+1)\,n}{(\beta+1)\beta},\ (a_p);\\ 1+\alpha+\beta\,n,\ \frac{\alpha+(\beta+1)\,n}{(\beta+1)\beta},\ (b_q);\end{array} x\right]\ t^n$$

$$=\frac{(1+\zeta)^{(\alpha+1)}}{(1-\beta\zeta)}{}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{\alpha}{(\beta+1)},1+\frac{\alpha(1+\zeta)}{(\beta+1)^2(1-\beta\zeta+\zeta)},(a_p);\\ 1+\frac{\alpha}{(\beta+1)},\frac{\alpha(1+\zeta)}{(\beta+1)^2(1-\beta\zeta+\zeta)},(b_q);\end{array} x(-\zeta)\right] \tag{5.2}$$

where ζ is given by equations (1.3) and (1.4).

(iii). Putting $\lambda=1,\mu=\frac{1}{2\alpha},\beta=\frac{-1}{2},\ m=1$ in equation (2.7),using the definition of $\mathscr{B}_n^{(\alpha,-\frac{1}{2})}(x;1,1,\frac{1}{2\alpha})$ and replacing ζ by U from equation (1.28), we get

$$\sum_{n=0}^{\infty}\binom{\alpha+\frac{n}{2}}{n}{}_{p+2}F_{q+2}\left[\begin{array}{c}-n,1-4\alpha-2n,(a_p);\\ 1+\alpha-\frac{n}{2},-4\alpha-2n,\ (b_q)\ ;\end{array} x\right]t^n$$

$$=\frac{(1+U)^{\alpha+1}}{(1+\frac{U}{2})}{}_{p+2}F_{q+2}\left[\begin{array}{c}2\alpha,1+\frac{8\alpha(1+U)}{(2+3U)},(a_p);\\ 1+2\alpha,\frac{8\alpha(1+U)}{(2+3U)},(b_q);\end{array} x(-U)\right] \tag{5.3}$$

(iv). Putting $\beta=0$ and $\zeta=\Theta=\frac{t}{1-t}$ from equation (1.24) in equation (2.7), we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha+n)}\binom{\alpha+n}{n}{}_{p+m+1}F_{q+m+1}\left[\begin{array}{c}\Delta(m;-n),1+\frac{\lambda+\mu\,n}{\mu(1-m)},(a_p);\\ \Delta(m;1+\alpha),\ \frac{\lambda+\mu\,n}{\mu(1-m)},\ (b_q)\ ;\end{array} x\right]\ t^n=(1-t)^{-\alpha-1}.$$

$$.\left\{\frac{\lambda(1-t)+\mu\alpha\,t}{\alpha}\right\}\ {}_{p+2}F_{q+2}\left[\begin{array}{c}\frac{\alpha}{m},1+\frac{\{\lambda\,(1-t)+\mu\alpha\,t\}}{\mu\,(1-t+mt)},(a_p);\\ 1+\frac{\alpha}{m},\frac{\{\lambda\,(1-t)+\mu\alpha\,t\}}{\mu\,(1-t+mt)},(b_q);\end{array} x\left(\frac{t}{t-1}\right)^m\right] \tag{5.4}$$

(v). Putting $\beta=1$ and $\zeta=\Lambda=\frac{1-2t-\sqrt{(1-4t)}}{2t}$from equation (1.25) in equation (2.7),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha+2n)}\binom{\alpha+2n}{n}{}_{p+m+1}F_{q+m+1}\left[\begin{array}{c}\Delta(m;-n),1+\frac{\lambda+\mu\,n}{\mu(2-m)},(a_p);\\ \Delta(m;1+\alpha+\ n),\frac{\lambda+\mu\,n}{\mu(2-m)},(b_q);\end{array} x\right]\ t^n$$

$$(5.5)\quad = (1+\Lambda)^{\alpha}\left\{\frac{\lambda(1-\Lambda)+\mu\,\alpha\,\Lambda}{\alpha(1-\Lambda)}\right\}\ {}_{p+2}F_{q+2}\left[\begin{array}{l} \frac{\alpha}{2m}, 1+\frac{\{\lambda(1-\Lambda)+\mu\,\alpha\,\Lambda\}}{2\,\mu\,(1-\Lambda+m\Lambda)}, (a_p); \\ 1+\frac{\alpha}{2m}, \frac{\{\lambda(1-\Lambda)+\mu\,\alpha\,\Lambda\}}{2\,\mu\,(1-\Lambda+m\Lambda)}, (b_q); \end{array} x(-\Lambda)^m\right]$$

(vi). Putting $\beta=-2$ and $\zeta=\Xi=\frac{-1+\sqrt{(1+4t)}}{2}$from equation (1.26) in equation (2.7),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha-n)}\binom{\alpha-n}{n}{}_{p+m+1}F_{q+m+1}\left[\begin{array}{c} \Delta(m;-n), 1-\frac{(\lambda+\mu\,n)}{\mu(1+m)}, (a_p)\ ; \\ \Delta(m;1+\alpha-2\,n), -\frac{(\lambda+\mu\,n)}{\mu(1+m)}, (b_q); \end{array} x\right]\ t^n$$

$$(5.6)\quad = (1+\Xi)^{\alpha}\left\{\frac{\lambda(1+2\Xi)+\mu\alpha\Xi}{\alpha(1+2\Xi)}\right\}\ {}_{p+2}F_{q+2}\left[\begin{array}{l} \frac{-\alpha}{m}, 1-\frac{\{\lambda\,(1+2\Xi)+\mu\alpha\Xi\}}{\mu\,(1+2\Xi+m\Xi)}, (a_p); \\ 1-\frac{\alpha}{m}, -\frac{\{\lambda\,(1+2\Xi)+\mu\alpha\Xi\}}{\mu\,(1+2\Xi+m\Xi)}, (b_q); \end{array} x(-\Xi)^m\right]$$

(vii). Putting $\beta=-3$ and $\zeta=\Upsilon$ from equation (1.27) in equation (2.7),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha-2n)}\binom{\alpha-2n}{n}{}_{p+m+1}F_{q+m+1}\left[\begin{array}{c} \Delta(m;-n),\ 1-\frac{(\lambda+\mu\,n)}{\mu(2+m)},\ (a_p); \\ \Delta(m;1+\alpha-3\,n), -\frac{(\lambda+\mu\,n)}{\mu(2+m)}, (b_q); \end{array} x\right]\ t^n$$

$$(5.7)\quad = (1+\Upsilon)^{\alpha}\left\{\frac{\lambda(1+3\Upsilon)+\mu\alpha\Upsilon}{\alpha(1+3\Upsilon)}\right\}\ {}_{p+2}F_{q+2}\left[\begin{array}{l} \frac{-\alpha}{2m}, 1-\frac{\{\,\lambda(1+3\Upsilon)+\mu\alpha\Upsilon\}}{2\,\mu\,(1+3\Upsilon+m\Upsilon)}, (a_p); \\ 1-\frac{\alpha}{2\,m}, -\frac{\{\lambda(1+3\Upsilon)+\mu\alpha\Upsilon\}}{2\,\mu\,(1+3\Upsilon+m\Upsilon)}, (b_q); \end{array} x(-\Upsilon)^m\right]$$

(viii). Putting $\beta=\frac{-1}{2}$ and $\zeta=U$ from equation (1.28) in equation (2.7),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha+\frac{1}{2}n)}\binom{\alpha+\frac{1}{2}n}{n}{}_{p+m+1}F_{q+m+1}\left[\begin{array}{c} \Delta(m;-n), 1+\frac{(\lambda+\mu\,n)}{\mu(\frac{1}{2}-m)}, (a_p); \\ \Delta(m;1+\alpha-\frac{1}{2}n), \frac{(\lambda+\mu\,n)}{\mu(\frac{1}{2}-m)}, (b_q); \end{array} x\right]\ t^n$$

$$(5.8)\quad = (1+U)^{\alpha}\left\{\frac{\lambda(1+\frac{U}{2})+\alpha\mu\,U}{\alpha(1+\frac{U}{2})}\right\}\ {}_{p+2}F_{q+2}\left[\begin{array}{l} \frac{2\alpha}{m}, 1+\frac{2\{\lambda(1+\frac{U}{2})+\alpha\mu\,U\,\}}{\mu\,(1+\frac{U}{2}+mU)}, (a_p); \\ 1+\frac{2\alpha}{m}, \frac{2\{\lambda(1+\frac{U}{2})+\alpha\mu\,U\,\}}{\mu\,(1+\frac{U}{2}+mU)}, (b_q); \end{array} x(-U)^m\right]$$

(ix). Putting $\beta=\frac{-3}{2}$ and $\zeta=\Psi$ from equation (1.29) in equation (2.7),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha-\frac{1}{2}n)}\binom{\alpha-\frac{1}{2}n}{n}{}_{p+m+1}F_{q+m+1}\left[\begin{array}{c} \Delta(m;-n), 1-\frac{(\lambda+\mu\,n)}{\mu(\frac{1}{2}+m)}, (a_p); \\ \Delta(m;1+\alpha-\frac{3}{2}\,n), -\frac{(\lambda+\mu\,n)}{\mu(\frac{1}{2}+m)}, (b_q); \end{array} x\right]\ t^n$$

$$(5.9)\quad = (1+\Psi)^{\alpha}\left\{\frac{\lambda(1+\frac{3}{2}\,\Psi)+\alpha\mu\Psi}{\alpha(1+\frac{3}{2}\,\Psi)}\right\}\ {}_{p+2}F_{q+2}\left[\begin{array}{l} \frac{-2\alpha}{m}, 1-2\frac{\{\lambda(1+\frac{3}{2}\,\Psi)+\alpha\mu\Psi\}}{\mu(1+\frac{3}{2}\,\Psi+m\,\Psi)}, (a_p); \\ 1-\frac{2\alpha}{m}, -\frac{2\{\lambda(1+\frac{3}{2}\,\Psi)+\alpha\mu\Psi\}}{\mu(1+\frac{3}{2}\,\Psi+m\,\Psi)}, (b_q); \end{array} x(-\Psi)^m\right]$$

(x). Putting $\beta=\frac{-1}{3}$ and $\zeta=\Pi$ from equation (1.30) in equation (2.7),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha+\frac{2}{3}n)}\binom{\alpha+\frac{2}{3}n}{n}{}_{p+m+1}F_{q+m+1}\left[\begin{array}{c} \Delta(m;-n), 1+\frac{(\lambda+\mu\,n)}{\mu(\frac{2}{3}-m)}, (a_p); \\ \Delta(m;1+\alpha-\frac{1}{3}\,n), \frac{(\lambda+\mu\,n)}{\mu(\frac{2}{3}-m)}, (b_q); \end{array} x\right]\ t^n$$

$$(5.10)\ = (1+\Pi)^{\alpha}\left\{\frac{\lambda(1+\frac{1}{3}\,\Pi)+\alpha\mu\Pi}{\alpha(1+\frac{1}{3}\,\Pi)}\right\}\ {}_{p+2}F_{q+2}\left[\begin{array}{c} \frac{3\alpha}{2m},1+\frac{3\{\lambda(1+\frac{1}{3}\,\Pi)+\alpha\mu\Pi\}}{2\mu(1+\frac{1}{3}\,\Pi+m\Pi)},(a_p);\\ 1+\frac{3\alpha}{2\,m},\frac{3\{\lambda(1+\frac{1}{3}\,\Pi)+\alpha\mu\Pi\}}{2\mu(1+\frac{1}{3}\,\Pi+m\Pi)},(b_q);\end{array} x(-\Pi)^m\right]$$

(xi). Putting $\beta=\frac{-2}{3}$ and $\zeta=\Phi$ from equation (1.31) in equation (2.7),we get

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{(\alpha+\frac{1}{3}n)}\binom{\alpha+\frac{1}{3}n}{n}{}_{p+m+1}F_{q+m+1}\left[\begin{array}{c}\Delta(m;-n),1+\frac{(\lambda+\mu\,n)}{\mu(\frac{1}{3}-m)},(a_p);\\ \Delta(m;1+\alpha-\frac{2}{3}\,n),\frac{(\lambda+\mu\,n)}{\mu(\frac{1}{3}-m)},(b_q);\end{array} x\right]\ t^n$$

$$(5.11)\ = (1+\Phi)^{\alpha}\left\{\frac{\lambda(1+\frac{2}{3}\,\Phi)+\mu\alpha\Phi}{\alpha(1+\frac{2}{3}\,\Phi)}\right\}\ {}_{p+2}F_{q+2}\left[\begin{array}{c} \frac{3\alpha}{m},1+\frac{3\{\lambda(1+\frac{2}{3}\,\Phi)+\mu\alpha\Phi\}}{\mu\,(1+\frac{2}{3}\,\Phi+m\Phi)},(a_p);\\ 1+\frac{3\alpha}{m},\frac{3\{\lambda(1+\frac{2}{3}\,\Phi)+\mu\alpha\Phi\}}{\mu\,(1+\frac{2}{3}\,\Phi+m\Phi)},(b_q);\end{array} x(-\Phi)^m\right]$$

Making suitable adjustments of parameters and variables in all generating relations of sections 4 and5, we can also obtain a number of new generating relations involving restricted generalized Laguerre polynomials, restricted Jacobi polynomials, restricted generalized Rice polynomials of Khandekar and other orthogonal polynomials.

6. Further generalizations of generating relations (2.1) and (2.7)

Generalization of (2.1):

Let

$$(6.1)\quad S_n^{(\alpha,\beta)}(x;m)=\sum_{r=0}^{\left[\frac{n}{m}\right]}\binom{\alpha+(\beta+1)n}{n-mr}\gamma_r\,x^r$$

where α,β are complex parameters independent of 'n'; m is an arbitrary positive integer and $\{\gamma_r\}$ is a bounded sequence of arbitrary real and complex numbers such that $\gamma_r\neq 0$.Then

$$\sum_{n=0}^{\infty}\frac{\lambda+\mu\,n}{\{\alpha+(\beta+1)n\}}S_n^{(\alpha,\beta)}(x;m)t^n$$

$$(6.2)\qquad =\frac{(1+\zeta)^{\alpha}}{(1-\beta\zeta)}\sum_{n=0}^{\infty}\left[\frac{\{\lambda(1-\beta\zeta)+\mu\zeta\alpha+\mu\,m(1+\zeta)n\}}{\{\alpha+(\beta+1)m\,n\}}\right]\gamma_n\,x^n\,\zeta^{mn}$$

where ζ is given by

$$(6.3)\qquad \zeta=t(1+\zeta)^{(\beta+1)};\zeta(0)=0$$

provided that each of the series involved is absolutely convergent.

Independent Demonstration:

Using the definition (6.1) of $S_n^{(\alpha,\beta)}(x;m)$ in left hand side of equation (6.2),we get

$$\Omega^{**}=\sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{\{\alpha+(\beta+1)n\}}S_n^{(\alpha,\beta)}(x;m)\ t^n$$

$$(6.4) \qquad = \sum_{n=0}^{\infty}\sum_{r=0}^{[\frac{n}{m}]} \frac{(\lambda+\mu n)\ \Gamma\{\alpha+(\beta+1)n+1\}}{\{\alpha+(\beta+1)n\}\ \Gamma(n-mr+1)\ \Gamma\{\alpha+\beta\ n+mr+1\}}\ \gamma_r\, x^r\, t^n$$

Applying summation identity (1.10) and then simplifying further, we get

$$\Omega^{**} = \sum_{r=0}^{\infty} \frac{\gamma_r\ x^r\ (t)^{mr}}{\{\alpha+m(\beta+1)r\}}\ \left\{\sum_{n=0}^{\infty} \frac{\{\alpha+(\beta+1)mr\}(\lambda+\mu n+\mu\, mr)}{\{\alpha+(\beta+1)mr+(\beta\ +1)n\}}\right.$$

$$(6.5) \qquad \left.\binom{\alpha+(\beta+1)mr+(\beta\ +1)n}{n} t^n\right\}$$

Now using first modified Gould's identity (1.7) with condition (6.3),we get

$$(6.6) \qquad \Omega^{**} = (1+\zeta)^{\alpha}\sum_{r=0}^{\infty}\left[\left\{\lambda+\frac{\mu\zeta\alpha}{(1-\beta\zeta)}\right\}+\left\{\mu\, mr+\frac{\mu\zeta(\beta+1)mr\}}{(1-\beta\zeta)}\right\}\right]\frac{\gamma_r\ x^r\ (\zeta)^{mr}}{\{\alpha+m(\beta+1)r\}}$$

Changing the summation index from r to n and after solving it further,we get the general result (6.2) corresponding to our first generating relation (2.1) subject to the conditions (6.3).

Generalization of (2.7):

Let

(6.7)

$$T_n^{(\alpha,\beta)}(x;m,\lambda,\mu) = \frac{1}{(\lambda+\mu\, n)}\sum_{r=0}^{[\frac{n}{m}]}\left\{\binom{\alpha+(\beta+1)n}{n-mr}[\lambda+\mu\, n+\mu(\beta+1-m)r]\right\}\gamma_r\, x^r$$

where α, β,λ,μ are complex parameters independent of 'n'; m is an arbitrary positive integer and $\{\gamma_r\}$ is a bounded sequence of arbitrary real and complex numbers such that $\gamma_r \neq 0$.Then

$$\sum_{n=0}^{\infty}\frac{(\lambda+\mu\, n)}{\{\alpha+(\beta+1)n\}}T_n^{(\alpha,\beta)}(x;m,\lambda,\mu)t^n$$

$$(6.8) \qquad = \frac{(1+\zeta)^{\alpha}}{(1-\beta\zeta)}\sum_{n=0}^{\infty}\left[\frac{\{\lambda(1-\beta\zeta)+\mu\zeta\alpha+\mu\,(1+\beta)(1-\beta\zeta+\ m\zeta)n\}}{\{\alpha+(\beta+1)m\, n\}}\right]\gamma_n\, x^n\, \zeta^{m\, n}$$

where ζ is given by

$$\zeta = t(1+\zeta)^{(\beta+1)}; \zeta(0) = 0$$

provided that each of the series involved is absolutely convergent.

Independent Demonstration:

Using the definition (6.7) of $T_n^{(\alpha,\beta)}(x;m,\lambda,\mu)$ in left hand side of equation (6.8),we get

$$\Omega^{***} = \sum_{n=0}^{\infty}\frac{(\lambda+\mu n)}{\{\alpha+(\beta+1)n\}}T_n^{(\alpha,\beta)}(x;m,\lambda,\mu)\ t^n$$

$$\text{(6.9)} \qquad = \sum_{n=0}^{\infty} \sum_{r=0}^{[\frac{n}{m}]} \left[\binom{\alpha+(\beta+1)n}{n-mr} \frac{\{\lambda+\mu n+\mu(\beta+1)r-\mu\, mr\}}{\{\alpha+(\beta+1)n\}} \right] \gamma_r \, x^r \, t^n$$

Applying summation identity (1.10) and then simplifying further, we get

$$\Omega^{***} = \sum_{r=0}^{\infty} \frac{\gamma_r \; x^r \; (t)^{mr}}{\{\alpha+m(\beta+1)r\}} \; \left\{ \sum_{n=0}^{\infty} \frac{\{\alpha+(\beta+1)mr\}\{\lambda+\mu n+\mu\,(\beta+1)r\}}{\{\alpha+(\beta+1)mr+(\beta\;+1)n\}} \right.$$

$$\text{(6.10)} \qquad \left. \binom{\alpha+(\beta+1)mr+(\beta\;+1)n}{n} t^n \right\}$$

Now using second modified Gould's identity (1.8) with condition (6.3),we get

$$\Omega^{***} = (1+\zeta)^{\alpha}$$

$$\text{(6.11)} \qquad \sum_{r=0}^{\infty} \left[\left\{ \lambda + \frac{\mu\zeta\alpha}{(1-\beta\zeta)} \right\} + \left\{ \mu\,(\beta+1)r + \frac{\mu\zeta(\beta+1)mr\}}{(1-\beta\zeta)} \right\} \right] \frac{\gamma_r \; x^r \; (\zeta)^{mr}}{\{\alpha+m(\beta+1)r\}}$$

Changing the summation index from r to n and after solving it further,we get the general result (6.8) corresponding to our second generating relation (2.7) subject to the conditions (6.3).

In the definitions of generalized polynomials given by $S_n^{(\alpha,\beta)}(x;m)$ and $T_n^{(\alpha,\beta)}(x;m,\lambda,\mu)$, putting

$$\gamma_r = \frac{(-1)^{m\,r}(a_1)_r \; \ldots (a_p)_r}{(b_1)_r \; \ldots (b_q)_r \, r!},$$

we obtain Srivastava's generalized hypergeometric polynomials of one variable $H_n^{(\alpha,\beta)}(x;m)$ and Pathan's generalized hypergeometric polynomials of one variable $\mathscr{B}_n^{(\alpha,\beta)}(x;m,\lambda,\mu)$ respectively.

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