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SOME PROPERTIES OF ANALYTIC FUNCTIONS OBTAINED BY USING A FRACTIONAL OPERATOR

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Abstract. In this paper, geometric properties of analytic functions are obtained, by using a fractional operator introduced in [6].The results are obtained in a similar manner to those given in [6]. Are also derived certain corollaries of the main results.
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1. Introduction

Let $\mathcal{A}$ be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, of the following form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{k+1} z^{k+1}, z \in U. \tag{1.1}$$

St. Ruscheweyh [4] defined the operator $\mathcal{R}^\lambda : \mathcal{A} \to \mathcal{A}$ for $\lambda \geq -1$ by

$$\mathcal{R}^\lambda f(z) = \frac{z}{(1-z)^{1+\lambda}} * f(z), z \in U.$$

For $\lambda \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$ this operator is defined by

$$\mathcal{R}^\lambda f(z) = \frac{z(z^{\lambda-1} f(z))^\lambda}{\lambda!}, z \in U.$$

G. Ş. Sălăgean [5] defined the operator $\mathcal{D}^n$ of order n, $n \in \mathbb{N}_0$, for $f \in \mathcal{A}$, by

$$\mathcal{D}^0 f(z) = f(z)$$

$$\mathcal{D}^1 f(z) = \mathcal{D} f(z) = z f'(z)$$

$$\mathcal{D}^n f(z) = \mathcal{D}(\mathcal{D}^{n-1} f(z)), n \in \mathbb{N} = \{1, 2, \cdots\}.$$

S. Owa [2] defined the operator $D_z^{-\mu}$ of order $\mu, \mu > 0$, for the function $f \in \mathcal{A}$ by

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int\limits_0^z \frac{f(t)}{(z-t)^{1-\mu}} dt, z \in U,$$

where the multiplicity of $(z-t)^{\mu-1}$ is removed by imposing the condition $\log(z-t)$ to be real when $z-t>0$, and $\Gamma(\mu)$ is the familiar Gamma function.

Also, is defined the operator D_z^{λ} of order $\lambda, \lambda \geq 0$, for the function $f \in \mathcal{A}$ by

$$D_z^{\lambda} f(z) = \begin{cases} \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int\limits_0^z \frac{f(t)}{(z-t)^{\lambda}} dt, & 0 \leq \lambda < 1 \\ \frac{d^n}{dz^n} D_z^{\lambda-n} f(z), & n \leq \lambda < n+1 \end{cases}, n \in \mathbb{N}_0,$$

where the multiplicity of $(z-t)^{-\lambda}$ is removed by using the same condition as in the case of the above operator.

Using the above fractional integral and fractional derivative operators, S. Owa and H. M. Srivastava [3] defined the operator $\Omega_z^{\lambda} : \mathcal{A} \to \mathcal{A}, -\infty < \lambda < 2$, by

$$\Omega_z^{\lambda} f(z) = \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z), z \in U,$$

where $D_z^{\lambda} f(z)$ is the fractional integral operator of order$\lambda, -\infty < \lambda < 0$, and a fractional derivative operator of order $\lambda, 0 \leq \lambda < 2$. Ω_z^{λ} is called fractional differintegral operator.

P. Sharma, R. K. Raina, G. Ş. Sălăgean [6] defined the fractional operator $\mathbb{D}_{\lambda}^{\nu,n} : \mathcal{A} \to \mathcal{A}$ for $\nu > -1, n \in \mathbb{N}_0, -\infty < \lambda < 2$ by

$$\mathbb{D}_{\lambda}^{\nu,n} f(z) = \mathcal{R}^{\nu} \mathcal{D}^n \Omega_z^{\lambda} f(z).$$

Hence, this operator is the composition of fractional differintegral operator, the Sălăgean operator and the Ruscheweyh operator:

The series expression of $\mathbb{D}_{\lambda}^{\nu,n} f(z)$ for $f \in \mathcal{A}$ of the form (1.1) is given by

$$\mathbb{D}_{\lambda}^{\nu,n} f(z) = z + \sum_{k=1}^{\infty} \frac{(\nu+1)_k}{(2-\lambda)_k} (k+1)^{n+1} a_{k+1} z^{k+1},$$

where $\nu > -1, n \in \mathbb{N}_0, -\infty < \lambda < 2, z \in U$, and the symbol $(x)_k$ is the Pochhammer symbol, defined by

$$(x)_k = \begin{cases} 1, k = 0 \\ x(x+1)...(x+k-1), k \in \mathbb{N} \end{cases} = \frac{\Gamma(x+k)}{\Gamma(x)}, x \in \mathbb{C} \setminus \mathbb{Z}_0^-.$$

Remark 1. [6] The fractional operator $\mathbb{D}_0^{\nu,0}$ is the Ruscheweyh operator $\mathcal{R}^{\nu}$ of order $\nu, \nu > -1$, and $\mathbb{D}_{\lambda}^{0,0}$ is the fractional differintegral operator Ω_z^{λ} of order $\lambda, -\infty < \lambda < 2$, while $\mathbb{D}_0^{0,n} = \mathcal{D}^n$ and $\mathbb{D}_{\lambda}^{1-\lambda,n} = \mathcal{D}^{n+1}$ are the Sălăgean operators, of order n and n+1, $n \in \mathbb{N}_0$.

Remark 2. [6] The operator $\mathbb{D}_\lambda^{\nu,n}$ satisfies the following identity:

$$\mathbb{D}_\lambda^{\nu+1,n}f(z) = \frac{\nu}{\nu+1}\mathbb{D}_\lambda^{\nu,n}f(z) + \frac{1}{\nu+1}z(\mathbb{D}_\lambda^{\nu,n}f(z))', \tag{1.2}$$

where $\nu > -1, n \in \mathbb{N}_0, -\infty < \lambda < 2$.

Remark 3. [7] The operator $\mathbb{D}_\lambda^{\nu,n}$ satisfies the following identities:

$$\mathbb{D}_\lambda^{\nu,n+1}f(z) = z(\mathbb{D}_\lambda^{\nu,n}f(z))', \tag{1.3}$$

where $\nu > -1, n \in \mathbb{N}_0, -\infty < \lambda < 2$,

and

$$\mathbb{D}_{\lambda+1}^{\nu,n}f(z) = -\frac{\lambda}{1-\lambda}\mathbb{D}_\lambda^{\nu,n}f(z) + \frac{1}{1-\lambda}z(\mathbb{D}_\lambda^{\nu,n}f(z))', \tag{1.4}$$

where $\nu > -1, n \in \mathbb{N}_0, -\infty < \lambda < 1$.

Definition 1. [1, p. 4] Let f, F analytic functions in the open unit disk. The function f is said to be subordinate to F, written $f \prec F$, or $f(z) \prec F(z)$, if there exists an analytic function in the open unit disk w, with $w(0) = 0$ and $\left|w(z)\right| < 1, z \in U$, such that $f(z) = F\left[w(z)\right], z \in U$.

Definition 2. [1, p. 15] Let Ω and Δ be any sets in $\mathbb{C}$, let p be analytic in the unit disk U with $p(0) = a$ and let $\psi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$. If

$$\{\psi(p(z), zp'(z), z^2p''(z); z)|z \in U\} \subset \Omega \implies p(U) \subset \Delta,$$

then ψ is called admissible function.

Definition 3. [1, Case 1, p.33] Let k be a positive integer, $a \in \mathbb{C}$ with $|a| < M, M > 0$. The class of admissible functions $\Psi_k[M, a]$, consists of those functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition:

$$\left|\psi(r, s, t; z)\right| \geq M, z \in U,$$

where

$$r = Me^{i\theta},$$

$$s = m\frac{M\left|M - \overline{a}e^{i\theta}\right|^2}{M^2 - |a|^2}e^{i\theta},$$

$$\Re\frac{t}{s} + 1 \geq m\frac{|M - \overline{a}e^{i\theta}|^2}{M^2 - |a|^2},$$

$\theta \in \mathbb{R}$, and $m \geq k$.

Definition 4. [1, Case 2, p. 34] Let k be a positive integer, $a \in \mathbb{C}$ with $\Re a > 0$. The class of admissible functions $\Psi_k[a]$, consists of those functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition:

$$\Re\psi(\rho i, \sigma, \mu + i\nu; z) \le 0, z \in U,$$

where $\rho, \sigma, \mu, \nu \in \mathbb{R}$,

$$\sigma \le -\frac{k}{2}\frac{\left|a - i\rho\right|^2}{\Re a}, \sigma + \mu \le 0.$$

We shall use the followings to prove our results.

Let $\mathcal{H}[a,k]$ be the class of analytic functions of the form $f(z) = a + a_k z^k + a_{k+1}z^{k+1} + \cdots$.

Lemma 1.1. [1, Theorem 2.3 h, (ii), p. 34] *Let $p \in \mathcal{H}[a,k]$. If $\psi \in \Psi_k[M,a]$, then*

$$\left|\psi(p(z), zp'(z), z^2p"(z); z)\right| < M \implies \left|p(z)\right| < M.$$

Lemma 1.2. [1, Theorem 2.3 i, (ii), p. 35] *Let $p \in \mathcal{H}[a,k]$. If $\psi \in \Psi_k[a]$, then*

$$\Re\psi\big(p(z), zp'(z), z^2p"(z); z\big) > 0 \implies \Re p(z) > 0.$$

2. Main results

Theorem 2.1. *Let $\nu > -1, n \in \mathbb{N}_0, -\infty < \lambda < 2, M > 1, m \ge 1, \theta \in \mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\left|\phi\left(Me^{i\theta}, \frac{mMe^{i\theta}}{\nu + Me^{i\theta}} + Me^{i\theta}\right)\right| \ge M.$$

Also, let $f \in \mathcal{A}$ with $\mathbb{D}_\lambda^{\nu,n} f(z) \ne 0$ and $\mathbb{D}_\lambda^{\nu+1,n} f(z) \ne 0$ for $z \in U \setminus \{0\}$.

The inequality

$$\left|\phi\left(\frac{z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n} f(z)}, \frac{z\big(\mathbb{D}_\lambda^{\nu+1,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu+1,n} f(z)}\right)\right| < M, z \in U, \tag{2.1}$$

implies

$$\left|\frac{z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n} f(z)}\right| < M, z \in U. \tag{2.2}$$

Proof. Let

$$p(z) = \frac{z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n} f(z)}, \tag{2.3}$$

then $p \in \mathcal{H}[1,1]$. From (1.2), we have

$$\mathbb{D}_\lambda^{\nu+1,n} f(z) = \frac{\nu}{\nu+1}\mathbb{D}_\lambda^{\nu,n} f(z) + \frac{1}{\nu+1} z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)',$$

which on using (2.3) gives

$$\mathbb{D}_\lambda^{\nu+1,n} f(z) = \frac{\nu + p(z)}{\nu + 1} \mathbb{D}_\lambda^{\nu,n} f(z). \tag{2.4}$$

Differentiating we obtain

$$\left(\mathbb{D}_\lambda^{\nu+1,n} f(z)\right)' = \frac{p'(z)}{\nu+1} \mathbb{D}_\lambda^{\nu,n} f(z) + \frac{\nu + p(z)}{\nu + 1} \left(\mathbb{D}_\lambda^{\nu,n} f(z)\right)'. \tag{2.5}$$

Using (2.3), (2.4) and (2.5) we get

$$\frac{z\left(\mathbb{D}_\lambda^{\nu+1,n} f(z)\right)'}{\mathbb{D}_\lambda^{\nu+1,n} f(z)} = \frac{zp'(z)}{\nu + p(z)} + p(z).$$

If we put

$$p(z) = r, zp'(z) = s,$$

and use the transformation

$$\begin{cases} b_1 = r \\ c_1 = \frac{s}{\nu + r} + r \end{cases},$$

we get

$$\phi(b_1, c_1) = \psi(r, s),$$

where $\psi(r, s) = \psi(r, s, 0; z)$.

Let $M > 1, m \geq 1$ and $\theta \in \mathbb{R}$. If we put $r = Me^{i\theta}$ and $s = mMe^{i\theta}$ in the above transformation, then we get $\psi \in \Psi[M, 1]$. Applying Lemma 1.1, we have:

$$\left|\psi(p(z), zp'(z))\right| = \left|\phi\left(\frac{z\left(\mathbb{D}_\lambda^{\nu,n} f(z)\right)'}{\mathbb{D}_\lambda^{\nu,n} f(z)}, \frac{z\left(\mathbb{D}_\lambda^{\nu+1,n} f(z)\right)'}{\mathbb{D}_\lambda^{\nu+1,n} f(z)}\right)\right| < M \implies \left|p(z)\right| = \left|\frac{z\left(\mathbb{D}_\lambda^{\nu,n} f(z)\right)'}{\mathbb{D}_\lambda^{\nu,n} f(z)}\right| < M.$$

□

Remark 4. Making use of (1.3), the inequalities (2.1) and (2.2) from Theorem 2.1 become:

$$\left|\phi\left(\frac{\mathbb{D}_\lambda^{\nu,n+1} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)}, \frac{\mathbb{D}_\lambda^{\nu+1,n+1} f(z)}{\mathbb{D}_\lambda^{\nu+1,n} f(z)}\right)\right| < M$$

and

$$\left|\frac{\mathbb{D}_\lambda^{\nu,n+1} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)}\right| < M.$$

Taking, respectively, $\nu = n = \lambda = 0$ and $\nu = \lambda = 0$, $n = 1$ in Theorem 2.1 we obtain the following corollaries.

Corollary 2.1.1. *Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\left|\phi(Me^{i\theta}, m + Me^{i\theta})\right| \geq M,$$

where $M > 1, m \geq 1, \theta \in \mathbb{R}$, and let $f \in \mathcal{A}$, with $f(z) \neq 0$ and $\mathcal{R}^1 f(z) \neq 0$ for $z \in U \setminus \{0\}$.

The inequality

$$\left|\phi\left(\frac{zf'(z)}{f(z)}, \frac{z\big(\mathcal{R}^1 f(z)\big)'}{\mathcal{R}^1 f(z)}\right)\right| < M, z \in U,$$

implies

$$\left|\frac{zf'(z)}{f(z)}\right| < M, z \in U.$$

Corollary 2.1.2. *Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\left|\phi(Me^{i\theta}, m + Me^{i\theta})\right| \geq M,$$

where $M > 1, m \geq 1, \theta \in \mathbb{R}$, and let $f \in \mathcal{A}$, with $\mathcal{D}^1 f(z) \neq 0$ and $\mathcal{R}^1\mathcal{D}^1 f(z) \neq 0$ for $z \in U \setminus \{0\}$.

The inequality

$$\left|\phi\left(\frac{z\big(\mathcal{D}^1 f(z)\big)'}{\mathcal{D}^1 f(z)}, \frac{z\big(\mathcal{R}^1\mathcal{D}^1 f(z)\big)'}{\mathcal{R}^1\mathcal{D}^1 f(z)}\right)\right| < M, z \in U,$$

implies

$$\left|\frac{z\big(\mathcal{D}^1 f(z)\big)'}{\mathcal{D}^1 f(z)}\right| < M, z \in U.$$

Theorem 2.2. *Let $\nu > -1, n \in \mathbb{N}_0, -\infty < \lambda < 2, \rho, \sigma \in \mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\Re\phi\left(\rho i, \frac{\sigma}{\nu + \rho i} + \rho i\right) \leq 0,$$

where $\sigma \leq -\frac{1}{2}(1+\rho^2)$. Also, let $f \in \mathcal{A}$ with $\mathbb{D}_\lambda^{\nu,n} f(z) \neq 0$ and $\mathbb{D}_\lambda^{\nu+1,n} f(z) \neq 0$ for $z \in U \setminus \{0\}$.

The inequality

$$\Re\phi\left(\frac{z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n} f(z)}, \frac{z\big(\mathbb{D}_\lambda^{\nu+1,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu+1,n} f(z)}\right) > 0, z \in U, \tag{2.6}$$

implies

$$\Re\frac{z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n} f(z)} > 0, z \in U. \tag{2.7}$$

Proof. Proceeding like in the proof of Theorem 2.1, and using the transformation

$$\begin{cases} b_1 = r \\ c_1 = \frac{s}{\nu+r} + r \end{cases},$$

we get

$$\phi(b_1, c_1) = \psi(r, s),$$

where $\psi(r, s) = \psi(r, s, 0; z)$.

Putting $r = \rho i, s = \sigma$ in the above transformation, we get $\psi \in \Psi_1[1]$ and

$$b_1 = \rho i,$$

$$c_1 = \frac{\sigma}{\nu + \rho i} + \rho i,$$

where $\sigma \leq -\frac{1}{2}(1 + \rho^2)$.

Applying Lemma 1.2, we obtain

$$\Re\psi\big(p(z), zp'(z)\big) = \Re\phi\left(\frac{z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n} f(z)}, \frac{z\big(\mathbb{D}_\lambda^{\nu+1,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu+1,n} f(z)}\right) > 0 \implies \Re p(z) = \Re\frac{z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n} f(z)} > 0.$$

□

Remark 5. The inequalities (2.6) and (2.7) from Theorem 2.2 can be expressed:

$$\Re\phi\left(\frac{\mathbb{D}_\lambda^{\nu,n+1} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)}, \frac{\mathbb{D}_\lambda^{\nu+1,n+1} f(z)}{\mathbb{D}_\lambda^{\nu+1,n} f(z)}\right) > 0$$

and

$$\Re\frac{\mathbb{D}_\lambda^{\nu,n+1} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)} > 0.$$

Taking, respectively, $\nu = n = \lambda = 0$ and $\nu = \lambda = 0, n = 1$ in Theorem 2.2, we obtain the following corollaries.

Corollary 2.2.1. *Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\Re\phi\left(\rho i, \frac{\sigma}{\rho i} + \rho i\right) \leq 0,$$

where $\rho, \sigma \in \mathbb{R}, \sigma \leq -\frac{1}{2}(1+\rho^2)$ and let $f \in \mathcal{A}$ with $f(z) \neq 0$ and $\mathcal{R}^1 f(z) \neq 0$ for $z \in U \setminus \{0\}$.

The inequality

$$\Re\phi\left(\frac{zf'(z)}{f(z)}, \frac{z\big(\mathcal{R}^1 f(z)\big)'}{\mathcal{R}^1 f(z)}\right) > 0, z \in U$$

implies

$$\Re\frac{zf'(z)}{f(z)} > 0, z \in U.$$

Remark 6. Corollary 2.2.1 is a starlikeness criterion.

Corollary 2.2.2. *Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\Re\phi\left(\rho i, \frac{\sigma}{\rho i} + \rho i\right) \leq 0,$$

where $\rho, \sigma \in \mathbb{R}, \sigma \leq -\frac{1}{2}(1 + \rho^2)$ and let $f \in \mathcal{A}$ with $\mathcal{D}^1 f(z) \neq 0$ and $\mathcal{R}^1\mathcal{D}^1 f(z) \neq 0$ for $z \in U \setminus \{0\}$.

The inequality

$$\Re\phi\left(\frac{z\big(\mathcal{D}^1 f(z)\big)'}{\mathcal{D}^1 f(z)}, \frac{z\big(\mathcal{R}^1\mathcal{D}^1 f(z)\big)'}{\mathcal{R}^1\mathcal{D}^1 f(z)}\right) > 0, z \in U$$

implies

$$\Re\frac{z\big(\mathcal{D}^1 f(z)\big)'}{\mathcal{D}^1 f(z)} > 0, z \in U.$$

Remark 7. Corollary 2.2.2 is a convexity criterion.

Taking $\psi(p(z), zp'(z)) = p(z) + \delta\frac{zp'(z)}{p(z)}$ in the proof of Theorem 2.1 and Theorem 2.2, we obtain the following corollaries.

Corollary 2.2.3. *Let* $\nu > -1, n \in \mathbb{N}_0, -\infty < \lambda < 2, M > 1, \delta \in \mathbb{R}$ *and let* $f \in \mathcal{A}$ *with* $\mathbb{D}_\lambda^{\nu,n} f(z) \neq 0$ *and* $\mathbb{D}_\lambda^{\nu+1,n} f(z) \neq 0$ *for* $z \in U \setminus \{0\}$.

The inequality

$$\left|(1-\delta)\frac{\mathbb{D}_\lambda^{\nu,n+1} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)} + \delta\frac{\mathbb{D}_\lambda^{\nu,n+2} f(z)}{\mathbb{D}_\lambda^{\nu,n+1} f(z)}\right| < M, z \in U,$$

implies

$$\left|\frac{\mathbb{D}_\lambda^{\nu,n+1} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)}\right| < M, z \in U.$$

Corollary 2.2.4. *Let* $\nu > -1, n \in \mathbb{N}_0, -\infty < \lambda < 2, \delta \in \mathbb{R}$ *and let* $f \in \mathcal{A}$ *with* $\mathbb{D}_\lambda^{\nu,n} f(z) \neq 0$ *and* $\mathbb{D}_\lambda^{\nu+1,n} f(z) \neq 0$ *for* $z \in U \setminus \{0\}$

The inequality

$$\Re\left[(1-\delta)\frac{\mathbb{D}_\lambda^{\nu,n+1} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)} + \delta\frac{\mathbb{D}_\lambda^{\nu,n+2} f(z)}{\mathbb{D}_\lambda^{\nu,n+1} f(z)}\right] > 0, z \in U,$$

implies

$$\Re\frac{\mathbb{D}_\lambda^{\nu,n+1} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)} > 0, z \in U.$$

Theorem 2.3. *Let* $\nu > -1, n \in \mathbb{N}_0, -\infty < \lambda < 2, M > 1, m \geq 1, \theta \in \mathbb{R}$ *and let* $\phi : \mathbb{C}^2 \to \mathbb{C}$ *be an admissible function that satisfies the condition*

$$\left|\phi\left(Me^{i\theta}, m + Me^{i\theta}\right)\right| \geq M.$$

Also, let $f \in \mathcal{A}$ *with* $\mathbb{D}_\lambda^{\nu,n} f(z) \neq 0$ *and* $\mathbb{D}_\lambda^{\nu,n+1} f(z) \neq 0$ *for* $z \in U \setminus \{0\}$.

The inequality

$$\left|\phi\left(\frac{z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n} f(z)}, \frac{z\big(\mathbb{D}_\lambda^{\nu,n+1} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n+1} f(z)}\right)\right| < M, z \in U, \tag{2.8}$$

implies

$$\left|\frac{z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n} f(z)}\right| < M, z \in U. \tag{2.9}$$

Proof. Let

$$p(z) = \frac{z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n} f(z)}, \tag{2.10}$$

then $p \in \mathcal{H}[1,1]$. From (1.3), we have

$$\mathbb{D}_\lambda^{\nu,n+1} f(z) = z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)',$$

which on using (2.10) gives

$$\mathbb{D}_\lambda^{\nu,n+1} f(z) = p(z)\mathbb{D}_\lambda^{\nu,n} f(z). \tag{2.11}$$

Differentiating we obtain

$$\big(\mathbb{D}_\lambda^{\nu,n+1} f(z)\big)' = p'(z)\mathbb{D}_\lambda^{\nu,n} f(z) + p(z)\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'. \tag{2.12}$$

Using (2.10), (2.11) and (2.12) we get

$$\frac{z\big(\mathbb{D}_\lambda^{\nu,n+1} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n+1} f(z)} = \frac{zp'(z)}{p(z)} + p(z).$$

If we put

$$p(z) = r, zp'(z) = s,$$

and use the transformation

$$\begin{cases} b_1 = r \\ c_1 = \frac{s}{r} + r \end{cases},$$

we get

$$\phi(b_1, c_1) = \psi(r, s),$$

where $\psi(r,s) = \psi(r,s,0;z)$.

Let $M > 1, m \geq 1$ and $\theta \in \mathbb{R}$. If we put $r = Me^{i\theta}, s = mMe^{i\theta}$ in the above transformation, then we get $\psi \in \Psi[M,1]$. Applying Lemma 1.1, we have

$$\big|\psi(p(z), zp'(z))\big| = \left|\phi\left(\frac{z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n} f(z)}, \frac{z\big(\mathbb{D}_\lambda^{\nu,n+1} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n+1} f(z)}\right)\right| < M \implies \big|p(z)\big| = \left|\frac{z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n} f(z)}\right| < M.$$

□

Remark 8. The inequalities (2.8) and (2.9) from Theorem 2.3 can be expressed:

$$\left|\phi\left(\frac{\mathbb{D}_\lambda^{\nu,n+1} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)}, \frac{\mathbb{D}_\lambda^{\nu,n+2} f(z)}{\mathbb{D}_\lambda^{\nu,n+1} f(z)}\right)\right| < M$$

and

$$\left|\frac{\mathbb{D}_\lambda^{\nu,n+1} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)}\right| < M.$$

Theorem 2.4. *Let $\nu > -1, n \in \mathbb{N}_0, -\infty < \lambda < 2, \rho, \sigma \in \mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\Re\phi\left(\rho i, \frac{\sigma}{\rho i} + \rho i\right) \leq 0,$$

where $\sigma \leq -\frac{1}{2}(1+\rho^2)$. Also, let $f \in \mathcal{A}$ with $\mathbb{D}_\lambda^{\nu,n} f(z) \neq 0$ and $\mathbb{D}_\lambda^{\nu,n+1} f(z) \neq 0$ for $z \in U \setminus \{0\}$.

The inequality

$$\Re\phi\left(\frac{z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n} f(z)}, \frac{z\big(\mathbb{D}_\lambda^{\nu,n+1} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n+1} f(z)}\right) > 0, z \in U, \tag{2.13}$$

implies

$$\Re\frac{z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n} f(z)} > 0, z \in U. \tag{2.14}$$

Proof. Proceeding like in the proof of Theorem 2.3, and using the transformation

$$\begin{cases} b_1 = r \\ c_1 = \frac{s}{r} + r \end{cases},$$

we get

$$\phi(b_1, c_1) = \psi(r, s),$$

where $\psi(r, s) = \psi(r, s, 0; z)$.

Putting $r = \rho i, s = \sigma$ in the above transformation, we get $\psi \in \Psi_1[1]$ and

$$b_1 = \rho i,$$

$$c_1 = \frac{\sigma}{\rho i} + \rho i,$$

where $\sigma \leq -\frac{1}{2}(1+\rho^2)$.

Applying Lemma 1.2, we obtain

$$\Re\psi\big(p(z), zp'(z)\big) = \Re\phi\left(\frac{z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n} f(z)}, \frac{z\big(\mathbb{D}_\lambda^{\nu,n+1} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n+1} f(z)}\right) > 0 \implies \Re p(z) = \Re\frac{z\big(\mathbb{D}_\lambda^{\nu,n} f(z)\big)'}{\mathbb{D}_\lambda^{\nu,n} f(z)} > 0.$$

□

Remark 9. The inequalities (2.13) and (2.14) from Theorem 2.4 can be expressed:

$$\Re\phi\left(\frac{\mathbb{D}_\lambda^{\nu,n+1} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)}, \frac{\mathbb{D}_\lambda^{\nu,n+2} f(z)}{\mathbb{D}_\lambda^{\nu,n+1} f(z)}\right) > 0$$

and

$$\Re\frac{\mathbb{D}_\lambda^{\nu,n+1} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)} > 0.$$

Theorem 2.5. *Let $\nu > -1, n \in \mathbb{N}_0, -\infty < \lambda < 1, M > 1, m \geq 1, \theta \in \mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\left|\phi\left(Me^{i\theta}, \frac{mMe^{i\theta}}{-\lambda + Me^{i\theta}} + Me^{i\theta}\right)\right| \geq M.$$

Also, let $f \in \mathcal{A}$ with $\mathbb{D}_{\lambda}^{\nu,n} f(z) \neq 0$ and $\mathbb{D}_{\lambda+1}^{\nu,n} f(z) \neq 0$ for $z \in U \setminus \{0\}$.

The inequality

$$\left|\phi\left(\frac{z\left(\mathbb{D}_{\lambda}^{\nu,n} f(z)\right)'}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}, \frac{z\left(\mathbb{D}_{\lambda+1}^{\nu,n} f(z)\right)'}{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}\right)\right| < M, z \in U, \tag{2.15}$$

implies

$$\left|\frac{z\left(\mathbb{D}_{\lambda}^{\nu,n} f(z)\right)'}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}\right| < M, z \in U. \tag{2.16}$$

Proof. Let

$$p(z) = \frac{z\left(\mathbb{D}_{\lambda}^{\nu,n} f(z)\right)'}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}, \tag{2.17}$$

then $p \in \mathcal{H}[1, 1]$. From (1.4), we have

$$\mathbb{D}_{\lambda+1}^{\nu,n} f(z) = -\frac{\lambda}{1-\lambda}\mathbb{D}_{\lambda}^{\nu,n} f(z) + \frac{1}{1-\lambda} z\left(\mathbb{D}_{\lambda}^{\nu,n} f(z)\right)',$$

which on using (2.17) gives

$$\mathbb{D}_{\lambda+1}^{\nu,n} f(z) = \frac{-\lambda + p(z)}{1-\lambda}\mathbb{D}_{\lambda}^{\nu,n} f(z). \tag{2.18}$$

Differentiating we obtain

$$\left(\mathbb{D}_{\lambda+1}^{\nu,n} f(z)\right)' = \frac{p'(z)}{1-\lambda}\mathbb{D}_{\lambda}^{\nu,n} f(z) + \frac{-\lambda + p(z)}{1-\lambda}\left(\mathbb{D}_{\lambda}^{\nu,n} f(z)\right)'. \tag{2.19}$$

Using (2.17), (2.18) and (2.19) we get

$$\frac{z\left(\mathbb{D}_{\lambda+1}^{\nu,n} f(z)\right)'}{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)} = \frac{zp'(z)}{-\lambda + p(z)} + p(z).$$

If we put

$$p(z) = r, zp'(z) = s,$$

and use the transformation

$$\begin{cases} b_1 = r \\ c_1 = \frac{s}{-\lambda + r} + r \end{cases},$$

we get

$$\phi(b_1, c_1) = \psi(r, s),$$

where $\psi(r, s) = \psi(r, s, 0; z)$.

Let $M > 1, m \geq 1$ and $\theta \in \mathbb{R}$. If we put $r = Me^{i\theta}, s = mMe^{i\theta}$ in the above transformation, then we get $\psi \in \Psi[M, 1]$. Applying Lemma 1.1, we have

$$\left|\psi(p(z), zp'(z))\right| = \left|\phi\left(\frac{z\left(\mathbb{D}_{\lambda}^{\nu,n} f(z)\right)'}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}, \frac{z\left(\mathbb{D}_{\lambda+1}^{\nu,n} f(z)\right)'}{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}\right)\right| < M \implies \left|p(z)\right| = \left|\frac{z\left(\mathbb{D}_{\lambda}^{\nu,n} f(z)\right)'}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}\right| < M.$$

□

Remark 10. The inequalities (2.15) and (2.16) from Theorem 2.5 can be expressed:

$$\left|\phi\left(\frac{\mathbb{D}_{\lambda}^{\nu,n+1} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}, \frac{\mathbb{D}_{\lambda+1}^{\nu,n+1} f(z)}{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}\right)\right| < M$$

and

$$\left|\frac{\mathbb{D}_{\lambda}^{\nu,n+1} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}\right| < M.$$

Theorem 2.6. *Let $\nu > -1, n \in \mathbb{N}_0, -\infty < \lambda < 1, \rho, \sigma \in \mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\Re\phi\left(\rho i, \frac{\sigma}{-\lambda + \rho i} + \rho i\right) \leq 0,$$

where $\sigma \leq -\frac{1}{2}(1 + \rho^2)$. Also, let $f \in \mathcal{A}$ with $\mathbb{D}_{\lambda}^{\nu,n} f(z) \neq 0$ and $\mathbb{D}_{\lambda+1}^{\nu,n} f(z) \neq 0$ for $z \in U \setminus \{0\}$.

The inequality

$$\Re\phi\left(\frac{z\left(\mathbb{D}_{\lambda}^{\nu,n} f(z)\right)'}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}, \frac{z\left(\mathbb{D}_{\lambda+1}^{\nu,n} f(z)\right)'}{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}\right) > 0, z \in U, \tag{2.20}$$

implies

$$\Re\frac{z\left(\mathbb{D}_{\lambda}^{\nu,n} f(z)\right)'}{\mathbb{D}_{\lambda}^{\nu,n} f(z)} > 0, z \in U. \tag{2.21}$$

Proof. Proceeding like in the proof of Theorem 2.5, and using the transformation

$$\begin{cases} b_1 = r \\ c_1 = \frac{s}{-\lambda + r} + r \end{cases},$$

we get

$$\phi(b_1, c_1) = \psi(r, s),$$

where $\psi(r, s) = \psi(r, s, 0; z)$.

Putting $r = \rho i, s = \sigma$ in the above transformation, we get $\psi \in \Psi_1[1]$ and

$$b_1 = \rho i,$$

$$c_1 = \frac{\sigma}{-\lambda + \rho i} + \rho i,$$

where $\sigma \leq -\frac{1}{2}(1 + \rho^2)$.

Applying Lemma 1.2, we obtain

$$\Re\psi\big(p(z), zp'(z)\big) = \Re\phi\left(\frac{z\big(\mathbb{D}_{\lambda}^{\nu,n}f(z)\big)'}{\mathbb{D}_{\lambda}^{\nu,n}f(z)}, \frac{z\big(\mathbb{D}_{\lambda+1}^{\nu,n}f(z)\big)'}{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}\right) > 0 \implies \Re p(z) = \Re\frac{z\big(\mathbb{D}_{\lambda}^{\nu,n}f(z)\big)'}{\mathbb{D}_{\lambda}^{\nu,n}f(z)} > 0.$$

□

Remark 11. The inequalities (2.20) and (2.21) from Theorem 2.6 can be expressed:

$$\Re\phi\left(\frac{\mathbb{D}_{\lambda}^{\nu,n+1}f(z)}{\mathbb{D}_{\lambda}^{\nu,n}f(z)}, \frac{\mathbb{D}_{\lambda+1}^{\nu,n+1}f(z)}{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}\right) > 0$$

and

$$\Re\frac{\mathbb{D}_{\lambda}^{\nu,n+1}f(z)}{\mathbb{D}_{\lambda}^{\nu,n}f(z)} > 0.$$

Theorem 2.7. *Let* $\nu > -1, n \in \mathbb{N}_0, -\infty < \lambda < 2, M > 0, m \geq 1, \theta \in \mathbb{R}$ *and let* $\phi : \mathbb{C}^3 \to \mathbb{C}$ *be an admissible function that satisfies the condition*

$$\left|\phi\left(Me^{i\theta}, mMe^{i\theta}, mMe^{i\theta} + L\right)\right| \geq M,$$

where $\Re(Le^{-i\theta}) \geq (m-1)mM$*. Also, let* $f \in \mathcal{A}$*.*

The inequality

$$\left|\phi\left(\mathbb{D}_{\lambda}^{\nu,n}f(z), \mathbb{D}_{\lambda}^{\nu,n+1}f(z), \mathbb{D}_{\lambda}^{\nu,n+2}f(z)\right)\right| < M, z \in U,$$

implies

$$\left|\mathbb{D}_{\lambda}^{\nu,n}f(z)\right| < M, z \in U.$$

Proof. Let

$$p(z) = \mathbb{D}_{\lambda}^{\nu,n}f(z), \tag{2.22}$$

then $p \in \mathcal{A}$. From (1.3), we have

$$\mathbb{D}_{\lambda}^{\nu,n+1}f(z) = z(\mathbb{D}_{\lambda}^{\nu,n}f(z))',$$

which on using (2.22) gives

$$\mathbb{D}_{\lambda}^{\nu,n+1}f(z) = zp'(z). \tag{2.23}$$

Replacing n by $n+1$ in the relation (1.3), we obtain

$$\mathbb{D}_{\lambda}^{\nu,n+2}f(z) = z(\mathbb{D}_{\lambda}^{\nu,n+1}f(z))',$$

and using (2.23), we have

$$\mathbb{D}_{\lambda}^{\nu,n+2}f(z) = zp'(z) + z^2p''(z).$$

If we put

$$p(z) = r, zp'(z) = s, z^2p''(z) = t,$$

and use the transformation

$$\begin{cases} b_1 = r \\ c_1 = s \\ d_1 = s + t \end{cases},$$

we obtain

$$\phi(b_1, c_1, d_1) = \psi(r, s, t).$$

Let $M > 0, m \geq 1, \Re(Le^{-i\theta}) \geq (m-1)mM$ and $\theta \in \mathbb{R}$. If we put $r = Me^{i\theta}, s = mMe^{i\theta}$ and $t = L$ in the above transformation, we get $\psi \in \Psi_1[M, 0]$. Applying Lemma 1.1, we have

$$\left|\psi(p(z), zp'(z), z^2p''(z))\right| = \left|\phi\left(\mathbb{D}_\lambda^{\nu,n} f(z), \mathbb{D}_\lambda^{\nu,n+1} f(z), \mathbb{D}_\lambda^{\nu,n+2} f(z)\right)\right| < M \implies$$

$$\left|p(z)\right| = \left|\mathbb{D}_\lambda^{\nu,n} f(z)\right| < M.$$

□

Theorem 2.8. *Let $\nu > -1, n \in \mathbb{N}_0, -\infty < \lambda < 1, M > 0, m \geq 1, \theta \in \mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\left|\phi\left(Me^{i\theta}, \frac{-\lambda + m}{1 - \lambda} Me^{i\theta}\right)\right| \geq M.$$

Also, let $f \in \mathcal{A}$.

The inequality

$$\left|\phi\left(\mathbb{D}_\lambda^{\nu,n} f(z), \mathbb{D}_{\lambda+1}^{\nu,n} f(z)\right)\right| < M, z \in U,$$

implies

$$\left|\mathbb{D}_\lambda^{\nu,n} f(z)\right| < M, z \in U.$$

Proof. Let

$$p(z) = \mathbb{D}_\lambda^{\nu,n} f(z),$$

then $p \in \mathcal{A}$. From (1.4), we have

$$\mathbb{D}_{\lambda+1}^{\nu,n} f(z) = -\frac{\lambda}{1 - \lambda} p(z) + \frac{1}{1 - \lambda} zp'(z).$$

If we put

$$p(z) = r, zp'(z) = s,$$

and use the transformation

$$\begin{cases} b_1 = r \\ c_1 = -\frac{\lambda}{1-\lambda} r + \frac{1}{1-\lambda} s \end{cases},$$

we obtain

$$\phi(b_1, c_1) = \psi(r, s),$$

where $\psi(r,s) = \psi(r,s,0;z)$.

Let $M > 0, m \geq 1$ and $\theta \in \mathbb{R}$. If we put $r = Me^{i\theta}, s = mMe^{i\theta}$ in the above transformation, we get $\psi \in \Psi_1[M,0]$. Applying Lemma 1.1, we have

$$\left|\psi(p(z), zp'(z))\right| = \left|\phi\Big(\mathbb{D}_{\lambda}^{\nu,n} f(z), \mathbb{D}_{\lambda+1}^{\nu,n} f(z)\Big)\right| < M \implies \left|p(z)\right| = \left|\mathbb{D}_{\lambda}^{\nu,n} f(z)\right| < M.$$

□

Theorem 2.9. *Let $\nu > -1, n \in \mathbb{N}_0, -\infty < \lambda < 1, M > 1, m \geq 1, \theta \in \mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\left|\phi\left(Me^{i\theta}, Me^{i\theta}\left(1 + \frac{m}{(\nu+1)Me^{i\theta} - \nu - \lambda}\right)\right)\right| \geq M.$$

Also, let $f \in \mathcal{A}$ with $\mathbb{D}_{\lambda}^{\nu,n} f(z) \neq 0$ and $\mathbb{D}_{\lambda+1}^{\nu,n} f(z) \neq 0$ for $z \in U \setminus \{0\}$.

The inequality

$$\left|\phi\left(\frac{\mathbb{D}_{\lambda}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}, \frac{\mathbb{D}_{\lambda+1}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}\right)\right| < M, z \in U,$$

implies

$$\left|\frac{\mathbb{D}_{\lambda}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}\right| < M, z \in U.$$

Proof. Let

$$p(z) = \frac{\mathbb{D}_{\lambda}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}, \tag{2.24}$$

then $p \in \mathcal{H}[1,1]$. From (1.4) and (2.24), we obtain

$$\mathbb{D}_{\lambda+1}^{\nu+1,n} f(z) = -\frac{\lambda}{1-\lambda} p(z)\mathbb{D}_{\lambda}^{\nu,n} f(z) + \frac{1}{1-\lambda} zp'(z)\mathbb{D}_{\lambda}^{\nu,n} f(z) + \frac{1}{1-\lambda} zp(z)\big(\mathbb{D}_{\lambda}^{\nu,n} f(z)\big)'. \tag{2.25}$$

We have

$$p(z) = \frac{\nu}{\nu+1} + \frac{1}{\nu+1}\frac{z(\mathbb{D}_{\lambda}^{\nu,n} f(z))'}{\mathbb{D}_{\lambda}^{\nu,n} f(z)},$$

which gives

$$\frac{z(\mathbb{D}_{\lambda}^{\nu,n} f(z))'}{\mathbb{D}_{\lambda}^{\nu,n} f(z)} = (\nu+1)p(z) - \nu. \tag{2.26}$$

Now, using (1.4) and (2.25), we get

$$\frac{\mathbb{D}_{\lambda+1}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)} = p(z) + \frac{zp'(z)}{-\lambda + \frac{z\left(\mathbb{D}_{\lambda}^{\nu,n} f(z)\right)'}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}},$$

which on using (2.26) gives

$$\frac{\mathbb{D}_{\lambda+1}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)} = p(z) + \frac{zp'(z)}{(\nu+1)p(z) - \nu - \lambda}.$$

If we put

$$p(z) = r, zp'(z) = s,$$

and use the transformation

$$\begin{cases} b_1 = r \\ c_1 = r + \frac{s}{(\nu+1)r-\nu-\lambda} \end{cases},$$

we get

$$\phi(b_1, c_1) = \psi(r, s),$$

where $\psi(r, s) = \psi(r, s, 0; z)$.

Let $M > 1, m \geq 1$ and $\theta \in \mathbb{R}$. If we put $r = Me^{i\theta}, s = mMe^{i\theta}$ in the above transformation, we get $\psi \in \Psi[M, 1]$. Applying Lemma 1.1, we have

$$\left|\psi(p(z), zp'(z))\right| = \left|\phi\left(\frac{\mathbb{D}_\lambda^{\nu+1,n} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)}, \frac{\mathbb{D}_{\lambda+1}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}\right)\right| < M \implies \left|p(z)\right| = \left|\frac{\mathbb{D}_\lambda^{\nu+1,n} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)}\right| < M.$$

□

Theorem 2.10. *Let $\nu > -1, n \in \mathbb{N}_0, -\infty < \lambda < 1, \rho, \sigma \in \mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\Re\phi\left(\rho i, \rho i + \frac{\sigma}{(\nu+1)\rho i - \nu - \lambda}\right) \leq 0,$$

where $\sigma \leq -\frac{1}{2}(1+\rho^2)$. Also, let $f \in \mathcal{A}$ with $\mathbb{D}_\lambda^{\nu,n} f(z) \neq 0$ and $\mathbb{D}_{\lambda+1}^{\nu,n} f(z) \neq 0$ for $z \in U \setminus \{0\}$.

The inequality

$$\Re\phi\left(\frac{\mathbb{D}_\lambda^{\nu+1,n} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)}, \frac{\mathbb{D}_{\lambda+1}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}\right) > 0, z \in U,$$

implies

$$\Re\frac{\mathbb{D}_\lambda^{\nu+1,n} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)} > 0, z \in U.$$

Proof. Proceeding like in the proof of Theorem 2.9, and using the transformation

$$\begin{cases} b_1 = r \\ c_1 = r + \frac{s}{(\nu+1)r-\nu-\lambda} \end{cases},$$

we get

$$\phi(b_1, c_1) = \psi(r, s),$$

where $\psi(r, s) = \psi(r, s, 0; z)$.

Putting $r = \rho i, s = \sigma$ in the above transformation, we get $\psi \in \Psi_1[1]$ and

$$b_1 = \rho i,$$

$$c_1 = \rho i + \frac{\sigma}{(\nu+1)\rho i - \nu - \lambda},$$

where $\sigma \le -\frac{1}{2}(1+\rho^2)$.

Applying Lemma 1.2, we have

$$\Re\psi\big(p(z), zp'(z)\big) = \Re\phi\left(\frac{\mathbb{D}_{\lambda}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}, \frac{\mathbb{D}_{\lambda+1}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}\right) > 0 \implies \Re p(z) = \Re\frac{\mathbb{D}_{\lambda}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)} > 0.$$

□

Theorem 2.11. *Let $\nu > -1, n \in \mathbb{N}_0, -\infty < \lambda < 0, M > 1, m \ge 1, \theta \in \mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\left|\phi\left(Me^{i\theta}, \frac{1}{\lambda}\big(1 - (1-\lambda)Me^{i\theta} - m\big)\right)\right| \ge M.$$

Also, let $f \in \mathcal{A}$ with $\mathbb{D}_{\lambda}^{\nu,n} f(z) \ne 0$ and $\mathbb{D}_{\lambda+1}^{\nu,n} f(z) \ne 0$ for $z \in U \setminus \{0\}$.

The inequality

$$\left|\phi\left(\frac{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}, \frac{\mathbb{D}_{\lambda+2}^{\nu,n} f(z)}{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}\right)\right| < M, z \in U,$$

implies

$$\left|\frac{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}\right| < M, z \in U.$$

Proof. Let

$$p(z) = \frac{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}, \tag{2.27}$$

then $p \in \mathcal{H}[1,1]$. Replacing λ by $\lambda+1$ in the relation (1.4), we obtain:

$$\mathbb{D}_{\lambda+2}^{\nu,n} f(z) = \frac{\lambda+1}{\lambda}\mathbb{D}_{\lambda+1}^{\nu,n} f(z) - \frac{1}{\lambda} z\big(\mathbb{D}_{\lambda+1}^{\nu,n} f(z)\big)',$$

and using (2.27), we have

$$\mathbb{D}_{\lambda+2}^{\nu,n} f(z) = \frac{\lambda+1}{\lambda} p(z)\mathbb{D}_{\lambda}^{\nu,n} f(z) - \frac{1}{\lambda} zp'(z)\mathbb{D}_{\lambda}^{\nu,n} f(z) - \frac{1}{\lambda} zp(z)\big(\mathbb{D}_{\lambda}^{\nu,n} f(z)\big)'. \tag{2.28}$$

Now, using (2.27) and (2.28), we get

$$\frac{\mathbb{D}_{\lambda+2}^{\nu,n} f(z)}{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)} = \frac{\lambda+1}{\lambda} - \frac{1}{\lambda}\frac{zp'(z)}{p(z)} - \frac{1}{\lambda}\frac{z\big(\mathbb{D}_{\lambda}^{\nu,n} f(z)\big)'}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}. \tag{2.29}$$

We have

$$p(z) = -\frac{\lambda}{1-\lambda} + \frac{1}{1-\lambda}\frac{z\big(\mathbb{D}_{\lambda}^{\nu,n} f(z)\big)'}{\mathbb{D}_{\lambda}^{\nu,n} f(z)},$$

which gives

$$\frac{z\big(\mathbb{D}_{\lambda}^{\nu,n} f(z)\big)'}{\mathbb{D}_{\lambda}^{\nu,n} f(z)} = (1-\lambda)p(z) + \lambda. \tag{2.30}$$

Using (2.30), (2.29) becomes

$$\frac{\mathbb{D}_{\lambda+2}^{\nu,n}f(z)}{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}=\frac{1}{\lambda}\left(1-(1-\lambda)p(z)-\frac{zp'(z)}{p(z)}\right).$$

If we put

$$p(z)=r, zp'(z)=s,$$

and use the transformation

$$\begin{cases} b_1=r \\ c_1=\frac{1}{\lambda}\left(1-(1-\lambda)r-\frac{s}{r}\right) \end{cases},$$

we get

$$\phi(b_1,c_1)=\psi(r,s),$$

where $\psi(r,s)=\psi(r,s,0;z)$.

Let $M>1, m\geq 1$ and $\theta\in\mathbb{R}$. If we put $r=Me^{i\theta}, s=mMe^{i\theta}$ in the above transformation, we get $\psi\in\Psi[M,1]$. Applying Lemma 1.1, we have

$$\left|\psi(p(z),zp'(z))\right|=\left|\phi\left(\frac{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}{\mathbb{D}_{\lambda}^{\nu,n}f(z)},\frac{\mathbb{D}_{\lambda+2}^{\nu,n}f(z)}{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}\right)\right|<M \implies \left|p(z)\right|=\left|\frac{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}{\mathbb{D}_{\lambda}^{\nu,n}f(z)}\right|<M.$$

□

Theorem 2.12. *Let $\nu>-1, n\in\mathbb{N}_0, -\infty<\lambda<0, \rho,\sigma\in\mathbb{R}$ and let $\phi:\mathbb{C}^2\to\mathbb{C}$ be an admissible function that satisfies the condition*

$$\Re\phi\left(\rho i,\frac{1}{\lambda}\left(1-(1-\lambda)\rho i-\frac{\sigma}{\rho i}\right)\right)\leq 0,$$

where $\sigma\leq-\frac{1}{2}(1+\rho^2)$. Also, let $f\in\mathcal{A}$ with $\mathbb{D}_{\lambda}^{\nu,n}f(z)\neq 0$ and $\mathbb{D}_{\lambda+1}^{\nu,n}f(z)\neq 0$ for $z\in U\setminus\{0\}$.

The inequality

$$\Re\phi\left(\frac{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}{\mathbb{D}_{\lambda}^{\nu,n}f(z)},\frac{\mathbb{D}_{\lambda+2}^{\nu,n}f(z)}{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}\right)>0, z\in U,$$

implies

$$\Re\frac{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}{\mathbb{D}_{\lambda}^{\nu,n}f(z)}>0, z\in U.$$

Proof. Proceeding like in the proof of Theorem 2.11, and using the transformation

$$\begin{cases} b_1=r \\ c_1=\frac{1}{\lambda}\left(1-(1-\lambda)r-\frac{s}{r}\right) \end{cases},$$

we get

$$\phi(b_1,c_1)=\psi(r,s),$$

where $\psi(r,s)=\psi(r,s,0;z)$.

Putting $r = \rho i, s = \sigma$ in the above transformation, we get $\psi \in \Psi_1[1]$ and

$$b_1 = \rho i,$$

$$c_1 = \frac{1}{\lambda}\left(1 - (1-\lambda)\rho i - \frac{\sigma}{\rho i}\right),$$

where $\sigma \leq -\frac{1}{2}(1+\rho^2)$.

Applying Lemma 1.2, we have

$$\Re\psi\big(p(z), zp'(z)\big) = \Re\phi\left(\frac{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}{\mathbb{D}_{\lambda}^{\nu,n}f(z)}, \frac{\mathbb{D}_{\lambda+2}^{\nu,n}f(z)}{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}\right) > 0 \implies \Re p(z) = \Re\frac{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}{\mathbb{D}_{\lambda}^{\nu,n}f(z)} > 0.$$

□

We now give our next results which can be proved by following similar lines of proofs as given for the above theorems.

Theorem 2.13. *Let $\nu > -1, n \in \mathbb{N}_0, -\infty < \lambda < 2, M > 1, m \geq 1, \theta \in \mathbb{R}$ and let $\phi : \mathbb{C}^3 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\left|\phi\Big(Me^{i\theta}, (1+m)Me^{i\theta}, (1+3m)Me^{i\theta} + L\Big)\right| \geq M,$$

where $\Re(Le^{-i\theta}) \geq (m-1)mM$. Also, let $f \in \mathcal{A}$.

The inequality

$$\left|\phi\left(\frac{\mathbb{D}_{\lambda}^{\nu,n}f(z)}{z}, \frac{\mathbb{D}_{\lambda}^{\nu,n+1}f(z)}{z}, \frac{\mathbb{D}_{\lambda}^{\nu,n+2}f(z)}{z}\right)\right| < M, z \in U,$$

implies

$$\left|\frac{\mathbb{D}_{\lambda}^{\nu,n}f(z)}{z}\right| < M, z \in U.$$

Theorem 2.14. *Let $\nu > -1, n \in \mathbb{N}_0, -\infty < \lambda < 2, \rho, \sigma, \mu, \nu \in \mathbb{R},$ and let $\phi : \mathbb{C}^3 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\Re\phi\big(\rho i, \rho i + \sigma, \rho i + 3\sigma + \mu + i\nu\big) \leq 0,$$

where $\sigma \leq -\frac{1}{2}(1+\rho^2), \sigma + \mu \leq 0$. Also, let $f \in \mathcal{A}$.

The inequality

$$\Re\phi\left(\frac{\mathbb{D}_{\lambda}^{\nu,n}f(z)}{z}, \frac{\mathbb{D}_{\lambda}^{\nu,n+1}f(z)}{z}, \frac{\mathbb{D}_{\lambda}^{\nu,n+2}f(z)}{z}\right) > 0, z \in U,$$

implies

$$\Re\frac{\mathbb{D}_{\lambda}^{\nu,n}f(z)}{z} > 0, z \in U.$$

Theorem 2.15. *Let $\nu > -1, n \in \mathbb{N}_0, -\infty < \lambda < 1, M > 1, m \geq 1, \theta \in \mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\left|\phi\left(Me^{i\theta}, \left(1 + \frac{m}{1-\lambda}\right)Me^{i\theta}\right)\right| \geq M.$$

Also, let $f \in \mathcal{A}$.

The inequality

$$\left|\phi\left(\frac{\mathbb{D}_\lambda^{\nu,n} f(z)}{z}, \frac{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}{z}\right)\right| < M, z \in U,$$

implies

$$\left|\frac{\mathbb{D}_\lambda^{\nu,n} f(z)}{z}\right| < M, z \in U.$$

Theorem 2.16. *Let $\nu > -1, n \in \mathbb{N}_0, -\infty < \lambda < 1, \rho, \sigma \in \mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\Re\phi\left(\rho i, \rho i + \frac{\sigma}{1-\lambda}\right) \leq 0,$$

where $\sigma \leq -\frac{1}{2}(1+\rho^2)$. Also, let $f \in \mathcal{A}$.

The inequality

$$\Re\phi\left(\frac{\mathbb{D}_\lambda^{\nu,n} f(z)}{z}, \frac{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}{z}\right) > 0, z \in U,$$

implies

$$\Re\frac{\mathbb{D}_\lambda^{\nu,n} f(z)}{z} > 0, z \in U.$$

Theorem 2.17. *Let $\nu > -1, n \in \mathbb{N}_0, -\infty < \lambda < 2, M > 1, m \geq 1, \theta \in \mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\left|\phi\left(Me^{i\theta}, Me^{i\theta}\left(1 + \frac{m}{(\nu+1)Me^{i\theta} - \nu}\right)\right)\right| \geq M.$$

Also, let $f \in \mathcal{A}$ with $\mathbb{D}_\lambda^{\nu,n} f(z) \neq 0$ and $\mathbb{D}_\lambda^{\nu,n+1} f(z) \neq 0$ for $z \in U \setminus \{0\}$.

The inequality

$$\left|\phi\left(\frac{\mathbb{D}_\lambda^{\nu+1,n} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)}, \frac{\mathbb{D}_\lambda^{\nu+1,n+1} f(z)}{\mathbb{D}_\lambda^{\nu,n+1} f(z)}\right)\right| < M, z \in U,$$

implies

$$\left|\frac{\mathbb{D}_\lambda^{\nu+1,n} f(z)}{\mathbb{D}_\lambda^{\nu,n} f(z)}\right| < M, z \in U.$$

Theorem 2.18. *Let $\nu > -1, n \in \mathbb{N}_0, -\infty < \lambda < 2, \rho, \sigma \in \mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\Re\phi\left(\rho i, \rho i + \frac{\sigma}{(\nu+1)\rho i - \nu}\right) \leq 0,$$

where $\sigma \leq -\frac{1}{2}(1+\rho^2)$. Also, let $f \in \mathcal{A}$ with $\mathbb{D}_\lambda^{\nu,n} f(z) \neq 0$ and $\mathbb{D}_\lambda^{\nu,n+1} f(z) \neq 0$ for $z \in U \setminus \{0\}$.

The inequality

$$\Re\phi\left(\frac{\mathbb{D}_{\lambda}^{\nu+1,n}f(z)}{\mathbb{D}_{\lambda}^{\nu,n}f(z)},\frac{\mathbb{D}_{\lambda}^{\nu+1,n+1}f(z)}{\mathbb{D}_{\lambda}^{\nu,n+1}f(z)}\right)>0, z\in U,$$

implies

$$\Re\frac{\mathbb{D}_{\lambda}^{\nu+1,n}f(z)}{\mathbb{D}_{\lambda}^{\nu,n}f(z)}>0, z\in U.$$

Theorem 2.19. *Let $\nu>-1, n\in\mathbb{N}_0, -\infty<\lambda<1, M>1, m\geq 1, \theta\in\mathbb{R}$ and let $\phi:\mathbb{C}^2\to\mathbb{C}$ be an admissible function that satisfies the condition*

$$\left|\phi\left(Me^{i\theta}, Me^{i\theta}\left(1+\frac{m}{(1-\lambda)Me^{i\theta}+\nu+\lambda}\right)\right)\right|\geq M.$$

Also, let $f\in\mathcal{A}$ with $\mathbb{D}_{\lambda}^{\nu,n}f(z)\neq 0$ and $\mathbb{D}_{\lambda}^{\nu+1,n}f(z)\neq 0$ for $z\in U\setminus\{0\}$.

The inequality

$$\left|\phi\left(\frac{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}{\mathbb{D}_{\lambda}^{\nu,n}f(z)},\frac{\mathbb{D}_{\lambda+1}^{\nu+1,n}f(z)}{\mathbb{D}_{\lambda}^{\nu+1,n}f(z)}\right)\right|<M, z\in U,$$

implies

$$\left|\frac{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}{\mathbb{D}_{\lambda}^{\nu,n}f(z)}\right|<M, z\in U.$$

Theorem 2.20. *Let $\nu>-1, n\in\mathbb{N}_0, -\infty<\lambda<1, \rho,\sigma\in\mathbb{R}$ and let $\phi:\mathbb{C}^2\to\mathbb{C}$ be an admissible function that satisfies the condition*

$$\Re\phi\left(\rho i, \rho i+\frac{\sigma}{(1-\lambda)\rho i+\nu+\lambda}\right)\leq 0,$$

where $\sigma\leq-\frac{1}{2}(1+\rho^2)$. Also, let $f\in\mathcal{A}$ with $\mathbb{D}_{\lambda}^{\nu,n}f(z)\neq 0$ and $\mathbb{D}_{\lambda}^{\nu+1,n}f(z)\neq 0$ for $z\in U\setminus\{0\}$.

The inequality

$$\Re\phi\left(\frac{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}{\mathbb{D}_{\lambda}^{\nu,n}f(z)},\frac{\mathbb{D}_{\lambda+1}^{\nu+1,n}f(z)}{\mathbb{D}_{\lambda}^{\nu+1,n}f(z)}\right)>0, z\in U,$$

implies

$$\Re\frac{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}{\mathbb{D}_{\lambda}^{\nu,n}f(z)}>0, z\in U.$$

Theorem 2.21. *Let $\nu>-1, n\in\mathbb{N}_0, -\infty<\lambda<1, M>1, m\geq 1, \theta\in\mathbb{R}$ and let $\phi:\mathbb{C}^2\to\mathbb{C}$ be an admissible function that satisfies the condition*

$$\left|\phi\left(Me^{i\theta}, Me^{i\theta}\left(1+\frac{m}{(1-\lambda)Me^{i\theta}+\lambda}\right)\right)\right|\geq M.$$

Also, let $f\in\mathcal{A}$ with $\mathbb{D}_{\lambda}^{\nu,n}f(z)\neq 0$ and $\mathbb{D}_{\lambda}^{\nu,n+1}f(z)\neq 0$ for $z\in U\setminus\{0\}$.

The inequality

$$\left|\phi\left(\frac{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}{\mathbb{D}_{\lambda}^{\nu,n}f(z)},\frac{\mathbb{D}_{\lambda+1}^{\nu,n+1}f(z)}{\mathbb{D}_{\lambda}^{\nu,n+1}f(z)}\right)\right|<M, z\in U,$$

implies

$$\left|\frac{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}{\mathbb{D}_{\lambda}^{\nu,n}f(z)}\right| < M, z \in U.$$

Theorem 2.22. *Let $\nu > -1, n \in \mathbb{N}_0, -\infty < \lambda < 1, \rho, \sigma \in \mathbb{R}$ and let $\phi : \mathbb{C}^2 \to \mathbb{C}$ be an admissible function that satisfies the condition*

$$\Re\phi\left(\rho i, \rho i + \frac{\sigma}{(1-\lambda)\rho i + \lambda}\right) \le 0,$$

where $\sigma \le -\frac{1}{2}(1+\rho^2)$. Also, let $f \in \mathcal{A}$ with $\mathbb{D}_{\lambda}^{\nu,n}f(z) \neq 0$ and $\mathbb{D}_{\lambda}^{\nu,n+1}f(z) \neq 0$ for $z \in U \setminus \{0\}$.

The inequality

$$\Re\phi\left(\frac{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}{\mathbb{D}_{\lambda}^{\nu,n}f(z)}, \frac{\mathbb{D}_{\lambda+1}^{\nu,n+1}f(z)}{\mathbb{D}_{\lambda}^{\nu,n+1}f(z)}\right) > 0, z \in U,$$

implies

$$\Re\frac{\mathbb{D}_{\lambda+1}^{\nu,n}f(z)}{\mathbb{D}_{\lambda}^{\nu,n}f(z)} > 0, z \in U.$$

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