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TRIPOLAR FUZZY INTERIOR IDEALS OF A $\Gamma-$ SEMIRING

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Abstract. In this paper, we introduce the notion of tripolar fuzzy set to be able to deal with tripolar information as a generalization of fuzzy set, bipolar fuzzy set and intuitionistic fuzzy set. The tripolar fuzzy set representation is very useful in discriminating relevant elements, irrelevant elements and contrary elements. We also introduce the notion of tripolar fuzzy ideal and tripolar fuzzy interior ideal of $\Gamma-$semiring. We study some of their algebraic properties, relations between them and characterization of tripolar fuzzy interior ideals are given.

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1. Introduction

In 1995, Murali Krishna Rao [9, 10, 11] introduced the notion of a $\Gamma-$semiring as a generalization of $\Gamma-$ring, ring, ternary semiring and semiring.

Historically semirings first appear implicitly in Dedekind and later in Macaulay, Neither and Lorenzen in connection with the study of a ring. However semirings first appear explicitly in Vandiver [27], also in connection with the axiomatization of arithmetic of natural numbers. Semirings have been studied by various researchers in an attempt to broaden techniques coming from semigroup theory, ring theory or in connection with applications. The developments of the theory in semirings have been taking place since 1950. Semirings abound in the mathematical world around us. A semiring is one of the fundamental structures in mathematics. Indeed the first mathematical structure we encounter the set of natural numbers is a semiring. Other semirings arise naturally in such diverse areas of mathematics as combinatorics, functional analysis, topology, graph theory, Euclidean geometry, probability theory, commutative ring theory, the mathematical modeling of quantum physics and parallel computation system.

As a generalization of ring, the notion of a $\Gamma-$ring was introduced by Nobusawa [24] in 1964. In 1981 Sen [26] introduced the notion of $\Gamma-$semigroup as a generalization of semigroup. The notion of ternary algebraic system was introduced by Lehmer [7] in 1932. Lister [8] introduced ternary ring. The set of all negative integers Z is not a semiring with respect to usual addition and multiplication but Z forms a $\Gamma-$semiring where $\Gamma = Z$. Murali Krishna Rao and Venkateswarlu [13] introduced the notion of $\Gamma-$incline and field $\Gamma-$semiring and studied properties of regular $\Gamma-$incline and field $\Gamma-$semiring. The theory of fuzzy sets most appropriate theory for dealing with uncertainty was first introduced by Zadeh [28] in 1965. There are many extensions of fuzzy sets, for example, intuitionistic fuzzy sets, interval valued fuzzy sets, vague sets, bipolar fuzzy sets, cubic sets etc. The concept of fuzzy set was applied to theory of subgroups by Rosenfeld [25]. The notion of an intuitionistic fuzzy set was first introduced by Atanassov [4] as a generalization of notion of fuzzy set. Murali Krishna Rao [12, 14-19] studied $T-$fuzzy ideals and ideals in semirgorups and ordered $\Gamma-$semirings. Murali Krishna Rao [20-23] studied fuzzy ideals and fuzzy soft ideals in $\Gamma-$semiring.

Bipolar fuzzy sets are an extension of fuzzy sets whose membership degree range is $[-1, 1]$. In 1994, Zhang [29] initiated the concept bipolar fuzzy sets as a generalization of fuzzy sets. In 2000, Lee [5, 6] used the term bipolar valued fuzzy sets and applied it to algebraic structure. Jun et al. [2, 3] studied intuitionistic fuzzy interior ideals in semigroups, introduced the notion of bipolar fuzzy ideals and bipolar fuzzy filters in CI-algebras.

In this paper, we introduce the notion of tripolar fuzzy set to be able to deal with tripolar information as a generalization of fuzzy set, bipolar fuzzy set and intuitionistic fuzzy set. The tripolar fuzzy set representation is very useful in discriminating relevant elements, irrelevant elements and contrary elements. We introduce the notion of tripolar fuzzy ideals and tripolar fuzzy interior ideals of $\Gamma-$semiring and study some of their algebraic properties and relations between them. Characterization of tripolar fuzzy interior ideals are given. We also prove that for any homomorphism ϕ from a $\Gamma-$semiring M to a $\Gamma-$semiring N, if A is a tripolar fuzzy interior ideal of M then the homomorphic image $\phi(A)$ is a tripolar fuzzy interior ideal of N and B is a tripolar fuzzy interior ideal of N then the homomorphic pre-image $\phi^{-1}(B)$ is a tripolar fuzzy interior ideal of M.

2. Preliminaries

In this section, we recall some definitions introduced by the pioneers in this field earlier.

Definition 2.1. Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. Then we call M as a $\Gamma-$semiring, if there exists a mapping $M \times \Gamma \times M \rightarrow M$ is written (x, α, y) as $x\alpha y$ such that it satisfies the following axioms

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$

(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$

(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$

(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

Every semiring R is a $\Gamma-$semiring with $\Gamma = R$ and ternary operation $x\gamma y$ as the usual semiring multiplication.

Definition 2.2. A $\Gamma-$semiring M is said to have a zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0,$ for all $x \in M, \alpha \in \Gamma$.

Definition 2.3. A $\Gamma-$semiring M is said to be commutative $\Gamma-$semiring if $x\alpha y = y\alpha x$, for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 2.4. A non-empty subset A of a $\Gamma-$semiring M is called a $\Gamma-$subsemiring M if $(A, +)$ is a subsemigroup of $(M, +)$ and $a\alpha b \in A$, for all $a, b \in A$ and $\alpha \in \Gamma$.

Definition 2.5. An additive subsemigroup I of a $\Gamma-$semiring M is said to be left (right) ideal of M if $M\Gamma I \subseteq I$ $(I\Gamma M \subseteq I)$. If I is both a left and a right ideal of M then I is called an ideal of $\Gamma-$semiring M.

Definition 2.6. A non-empty subset A of a $\Gamma-$semiring M is called an interior ideal of M if A is a $\Gamma-$subsemiring of M and $M\Gamma A\Gamma M \subseteq A$.

Definition 2.7. A function $f : R \rightarrow M$, where R and M are $\Gamma-$semirings, is said to be $\Gamma-$semiring homomorphism if $f(a + b) = f(a) + f(b)$ and $f(a\alpha b) = f(a)\alpha f(b)$, for all $a, b \in R, \alpha \in \Gamma$.

Definition 2.8. Let M be a non-empty set. Then a mapping $f : M \rightarrow [0, 1]$ is called a fuzzy subset of M.

Definition 2.9. Let M be a $\Gamma-$semiring. A fuzzy subset μ of M is said to be fuzzy $\Gamma-$subsemiring of M if it satisfying the following conditions

(i) $\mu(x + y) \geq min\ \{\mu(x), \mu(y)\}$

(ii) $\mu(x\alpha y) \geq min\ \{\mu(x), \mu(y)\}$, for all $x, y \in M, \alpha \in \Gamma$.

Definition 2.10. A fuzzy subset μ of a $\Gamma-$semiring M is called a fuzzy left (right) ideal of M if it satisfying the following conditions

(i) $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$

(ii) $\mu(x\alpha y) \geq \mu(y)$ $(\mu(x))$, for all $x, y \in M, \alpha \in \Gamma$.

Definition 2.11. A fuzzy subset μ of a $\Gamma-$semiring M is called a fuzzy ideal of M if it satisfying the following conditions
(i) $\mu(x+y) \geq min\{\mu(x), \mu(y)\}$
(ii) $\mu(x\alpha y) \geq max\ \{\mu(x), \mu(y)\}$, for all $x, y \in M, \alpha \in \Gamma$.

Definition 2.12. A fuzzy subset μ of $\Gamma-$semiring M is called a fuzzy interior ideal if

(i) $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$
(ii) $\mu(x\alpha y\beta z) \geq \mu(y)$, for all $x, y, z \in M, \alpha, \beta \in \Gamma$.

Definition 2.13. Let $\phi : M \to M'$ be a homomorphism of$\Gamma-$ semirings and f be a fuzzy subset of M. We define a fuzzy subset $\phi(f)$ of M' by

$$\phi(f)(x) = \begin{cases} \sup\limits_{y \in \phi^{-1}(x)} f(y), & \text{if } \phi^{-1}(x) \neq \emptyset, \\ 0, & \text{otherwise} \end{cases}.$$

Definition 2.14. An intuitionistic fuzzy set f of a non-empty set X is an object having the form $f = (\mu_f, \lambda_f) = \{x, \mu_f(x), \lambda_f(x) \mid x \in X\}$, where $\mu_f : X \to [0,1], \lambda_f : X \to [0,1]$ are membership functions, $\mu_f(x)$ is a degree of membership, $\lambda_f(x)$ is a degree of non membership and $0 \leq \mu_f(x) + \lambda_f(x) \leq 1$, for all $x \in X$.

Definition 2.15. A bipolar fuzzy set A of a non-empty set X is an object having the form $A = \{(x, \mu_A(x), \delta_A(x) \mid x \in X)$, where $\mu_A : X \to [0,1]; \delta_A : X \to [-1,0]$. $\mu_A(x)$ represents degree of satisfaction of an element x to the property corresponding to fuzzy set A and $\delta_A(x)$ represents degree of satisfaction of an element x to the implicit counter property of fuzzy set A.

Definition 2.16. Let $\phi : M \to N$ be a homomorphism of $\Gamma-$semirings M, N and μ be a fuzzy subset of M. Then μ is said to be ϕ homomorphism invariant if $\phi(a) = \phi(b)$ then $\mu(a) = \mu(b)$, for all $a, b \in M$.

Definition 2.17. A fuzzy subset $\mu : S \to [0,1]$ is non-empty if μ is not the constant function.

Definition 2.18. For any two fuzzy subsets λ and μ of S, $\lambda \subseteq \mu$ means $\lambda(x) \leq \mu(x)$, for all $x \in S$.

Definition 2.19. The complement of a fuzzy subset μ of a $\Gamma-$semiring M is denoted by $\overline{\mu}$ and it is defined as $\overline{\mu}(x) = 1 - \mu(x)$, for all $x \in M$.

3. TRIPOLAR FUZZY INTERIOR IDEALS OF A $\Gamma-$SEMIRING

In this section, we introduce the notion of tripolar fuzzy set to be able to deal with tripolar information as a generalization of fuzzy set, bipolar fuzzy set and intuitionistic fuzzy set. We also introduce the notion of tripolar fuzzy ideals and interior ideals of $\Gamma-$semiring.

Definition 3.1. A tripolar fuzzy set A in a universe set X is an object having the form $A = \{(x, \mu_A(x), \lambda_A(x), \delta_A(x)) \mid x \in X \text{ and } 0 \leq \mu_A(x) + \lambda_A(x) \leq 1\}$, where $\mu_A : X \to [0,1]; \lambda_A : X \to [0,1]; \delta_A : X \to [-1,0]$ such that $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$. The membership degree $\mu_A(x)$ characterizes the extent that the element x satisfies to the property corresponding to tripolar fuzzy set $A, \lambda_A(x)$ characterizes the extent that the element x satisfies to the not property (irrelevant) corresponding to tripolar fuzzy set A and $\delta_A(x)$ characterizes the extent that the element x satisfies to the implicit counter property of tripolar fuzzy set A. For simplicity $A = (\mu_A, \lambda_A, \delta_A)$ has been used for $A = \{(x, \mu_A(x), \lambda_A(x), \delta_A(x)) \mid x \in X, 0 \leq \mu_A(x) + \lambda_A(x) \leq 1\}$.

Remark 3.2. A tripolar fuzzy set A is a generalization of a bipolar fuzzy set and an intuitionistic fuzzy set. A tripolar fuzzy set $A = \{(x, \mu_A(x), \lambda_A(x), \delta_A(x)) \mid x \in X\}$ represents the sweet taste of food stuffs. Assuming the sweet taste of food stuff as a positive membership value $\mu_A(x)$ i.e. the element x is satisfying the sweet property. Then bitter taste of food stuff as a negative membership value $\delta_A(x)$ i.e. the element x is satisfying the bitter property, and the remaining tastes of food stuffs like acidic, chilly etc., as a non memberships value $\lambda_A(x)$ i.e., the element is satisfying irrelevant to the sweet property.

Definition 3.3. A tripolar fuzzy set $A = (\mu_A, \lambda_A, \delta_A)$ of a $\Gamma-$semiring M is called a tripolar fuzzy $\Gamma-$subsemiring of M if A satisfies the following conditions

(i) $\mu_A(x + y) \geq \min\{\mu_A(x), \mu_A(x)\}$
(ii) $\lambda_A(x + y) \leq \max\{\lambda_A(x), \lambda_A(y)\}$
(iii) $\delta_A(x + y) \leq \max\{\delta_A(x), \delta_A(y)\}$
(iv) $\mu_A(x\alpha y) \geq \min\{\mu_A(x), \mu_A(x)\}$
(v) $\lambda_A(x\alpha y) \leq \max\{\lambda_A(x), \lambda_A(y)\}$
(vi) $\delta_A(x\alpha y) \leq \max\{\delta_A(x), \delta_A(y)\}$, for all $x, y, z \in M, \alpha \in \Gamma$.

Definition 3.4. A tripolar fuzzy $\Gamma-$subsemiring $A = (\mu_A, \lambda_A, \delta_A)$ of a $\Gamma-$semiring M is called a tripolar fuzzy ideal of M if A satisfies the following conditions

(i) $\mu_A(x\alpha y) \geq \max\{\mu_A(x), \mu_{A)}(x)\}$
(ii) $\lambda_A(x\alpha y) \leq \min\{\lambda_A(x), \lambda_A(y)\}$
(iii) $\delta_A(x\alpha y) \leq \min\{\delta_A(x), \delta_A(y)\}$, for all $x, y, z \in M, \alpha \in \Gamma$

Definition 3.5. A tripolar fuzzy $\Gamma-$subsemiring $A = (\mu_A, \lambda_A, \delta_A)$ of the $\Gamma-$semiring M is called a tripolar fuzzy interior ideal of M if A satisfies the following conditions

(i) $\mu_A(x\alpha z\beta y) \geq \mu_A(z)$

(ii) $\lambda_A(x\alpha z\beta y) \leq \lambda_A(z)$

(iii) $\delta_A(x\alpha z\beta y) \leq \delta_A(z)$, for all $x, y, z \in M, \alpha, \beta \in \Gamma$.

Theorem 3.6. *Every tripolar fuzzy ideal of a $\Gamma-$semiring M is a tripolar fuzzy interior ideal of a $\Gamma-$semiring M.*

Proof. Let $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar fuzzy ideal of a $\Gamma-$semiring M. Then

(i) $\mu_A(x\alpha z\beta y) \geq \mu_A(x\alpha z) \geq \mu_A(z)$

(ii) $\lambda_A(x\alpha z\beta y) \leq \lambda_A(x\alpha z) \leq \lambda_A(z)$

(iii) $\delta_A(x\alpha z\beta y) \leq \delta_A(x\alpha z) \leq \delta_A(z)$, for all $x, y, z \in M, \alpha, \beta \in \Gamma$.

Hence A is a tripolar fuzzy interior ideal of M. □

Remark 3.7. Every tripolar fuzzy ideal A of a $\Gamma-$semiring M is a tripolar fuzzy $\Gamma-$subsemiring M, but the converse is not true.

Example 3.8. Let $M = \{x_1, x_2, x_3\}, \Gamma = \{\alpha, \beta\}$. We define operations with the following tables:

+	α	β
α	α	α
β	α	β

;

+	x_1	x_2	x_3
x_1	x_1	x_2	x_3
x_2	x_2	x_2	x_3
x_3	x_3	x_3	x_2

;

α	x_1	x_2	x_3
x_1	x_1	x_3	x_3
x_2	x_3	x_2	x_3
x_3	x_3	x_3	x_3

and

β	x_1	x_2	x_3
x_1	x_1	x_2	x_3
x_2	x_3	x_2	x_3
x_3	x_3	x_3	x_3

.

Then M is a $\Gamma-$semiring. B is a tripolar fuzzy set defined as

$B = \{(x_1, 0.2, 0.7, -0.2), (x_2, 0.3, 0.6, -0.3), (x_3, 0.6, 0.3, -0.3)\}$.

Then $B$is a tripolar fuzzy $\Gamma-$subsemiring of M and B is not a tripolar fuzzy ideal of M. B is a tripolar fuzzy interior ideal of M.

Theorem 3.9. *Every tripolar fuzzy interior ideal over a regular $\Gamma-$semiring M is a tripolar fuzzy ideal of M.*

Proof. Let $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar fuzzy interior ideal of the regular$\Gamma-$semiring M. Suppose $x, y \in M, \alpha \in \Gamma$. Then there exist $\beta, \gamma \in \Gamma, z \in M$ such that $x\alpha y = x\alpha y\beta z\gamma x\alpha y$.

$$\mu_A(x\alpha y) = \mu_A(x\alpha y\beta z\gamma x\alpha y)$$
$$= \mu_A\big(x\alpha y\beta(z\gamma x\alpha y)\big) \geq \mu_A(y)$$
$$\mu_{A)}(x\alpha y) = \mu_A\big((x\alpha y\beta z)\gamma x\alpha y\big) \geq \mu_A(x).$$

Therefore μ_A is a fuzzy ideal of M.

$$\lambda_A(x\alpha y) = \lambda_A(x\alpha y\beta z\gamma x\alpha y) \leq \lambda_A(y)$$
$$\lambda_A(x\alpha y) = \lambda_{A)}\big((x\alpha y\beta z)\gamma x\alpha y\big) \leq \lambda_A(x).$$

Therefore λ_A is a fuzzy ideal of M.

$$\delta_A(x\alpha y) = \delta_A(x\alpha y\beta z\gamma x\alpha y) \leq \delta_A(y)$$
$$\delta_A(x\alpha y) \leq \delta_A(x).$$

Therefore δ_A is a fuzzy ideal of M.
Hence A is a tripolar fuzzy ideal of the $\Gamma-$semiring M. □

Theorem 3.10. *If a tripolar fuzzy set $A = (\mu_A, \lambda_A, \delta_A)$ of a $\Gamma-$semiring M is an interior ideal of a $\Gamma-$semiring M then $(\mu_A, \overline{\mu}_A, \delta_A)$ is a tripolar fuzzy interior ideal of a $\Gamma-$semiring M.*

Proof. Let $x, y \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned}
\overline{\mu}_A(x\alpha y) &= 1 - \mu_A(x\alpha y) \\
&\leq 1 - \min\{\mu_A(x), \mu_A(y)\} \\
&= \max\{1 - \mu_A(x), 1 - \mu_A(y)\} \\
&= \max\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}. \\
\overline{\mu}_A(x\alpha z\beta y) &= 1 - \mu_A(x\alpha z\beta y) \\
&\leq 1 - \mu_A(z) \\
&= \overline{\mu}_A(z).
\end{aligned}$$

Therefore $(\mu_A, \overline{\mu}_A, \delta_A)$ is a tripolar fuzzy interior ideal of the $\Gamma-$semiring M. □

Definition 3.11. Let $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar fuzzy set of a $\Gamma-$semiring M and $\alpha \in [0, 1]$. Then the sets $\mu_{A,\alpha} = \{x \in M \mid \mu_A(x) \geq \alpha\}; \lambda_{A,\alpha} = \{x \in M \mid \lambda_A(x) \leq \alpha\}; \delta_{A,-\alpha} = \{x \in M \mid \delta_A(x) \geq -\alpha\}$ are called a $\mu-$level $\alpha-$cut, $\lambda-$level $\alpha-$cut and $\delta-$level $-\alpha-$cut of A respectively.

Theorem 3.12. *If $A = (\mu_A, \lambda_A, \delta_A)$ is a tripolar fuzzy interior ideal of the $\Gamma-$semiring M then $\mu-$level $t-$cut, $\lambda-$level $t-$cut and $\delta-$level $-t-$cut of A are interior ideals of a $\Gamma-$semiring M, for all $t \in Im(\mu_A) \cap Im(\lambda_A) \subseteq [0, 1]$ and $-t \in Im(\delta_A)$.*

Proof. Let $t \in Im(\mu_A) \cap Im(\lambda_A) \subseteq [0,1]$, $-t \in Im(\delta_A)$ and $x, y \in \mu_{A,t},\ \alpha \in \Gamma$. Then $\mu_A(x) \geq t$ and $\mu_A(y) \geq t$.

$$\mu_A(x+y) \geq \min\{\mu_A(x), \mu_A(y)\} \geq t.$$
$$\text{Therefore}\ \ x+y \in \mu_{A,t}.$$
$$\mu_A(x\alpha y) \geq \min\{\mu_A(x), \mu_A(y)\} \geq t.$$
$$\text{Therefore}\ \ x\alpha y \in \mu_{A,t}.$$

Hence $\mu_{A,t}$ is the $\Gamma-$subsemiring of M.

Let $x, y \in M, z \in \mu_{A,t}$ and $\alpha, \beta \in M$. Then $\mu_A(x\alpha z\beta y) \geq \mu_A(z) \geq t$. Therefore $x\alpha z\beta y \in \mu_{A,t}$. Hence $\mu_{A,t}$ is an interior ideal of the $\Gamma-$semiring M. Suppose $x, y \in \lambda_{A,t}$ and $\alpha \in \Gamma$. Then $\lambda_A(x) \leq t, \lambda_A(y) \leq t$

$$\Rightarrow \lambda_A(x+y) \leq \max\{\lambda_A(x), \lambda_A(y)\} \leq t.$$
$$\text{Therefore } x+y \in \lambda_{A,t}.$$
$$\lambda_A(x\alpha y) \leq \max\{\lambda_A(x), \lambda_A(y)\} \leq t.$$
$$\text{Therefore } x\alpha y \in \lambda_{A,t}.$$

Hence $\lambda_{A,t}$ is the $\Gamma-$subsemiring of M.

Let $x, y \in M, z \in \lambda_{A,t}$ and $\alpha, \beta \in \Gamma$. Then $\lambda_A(x\alpha z\beta y) \leq \lambda_A(z) \leq t$. Therefore $x\alpha z\beta y \in \lambda_{A,t}$. Suppose $x, y \in \delta_{A,-t}$ and $\alpha \in \Gamma$. Then $\delta_A(x) \leq -t, \delta_A(y) \leq -t$.

$$\delta_A(x+y) \leq \max\{\delta_A(x), \delta_A(y)\} \leq -t.$$
$$\text{Therefore } x+y \in \delta_{A,-t}$$
$$\delta_A(x\alpha y) \leq \max\{\delta_A(x), \delta_A(y)\} \leq -t.$$
$$\text{Therefore } x\alpha y \in \delta_{A,-t}.$$

Let $x, y \in M, z \in \delta_{A,-t}$ and $\alpha \in \Gamma$. Then $\delta_A(x\alpha z\beta y) \leq \delta_a(z) \leq -t$. Therefore $x\alpha z\beta y \in \delta_{A,-t}$. Hence $\delta_{A,-t}$ is an interior ideal of the $\Gamma-$semiring M. □

Theorem 3.13. *If $A = (\mu_A, \lambda_A, \delta_A)$ is a tripolar fuzzy interior ideal of a $\Gamma-$semiring M, then*

(1) $\mu_A(x) = \sup\{\alpha \in [0,1] \mid x \in \mu_{A,\alpha}\}$
(2) $\lambda_A(x) = \inf\{\alpha \in [0,1] \mid x \in \lambda_{A,\alpha}\}$
(3) $\delta_A(x) = \inf\{\alpha \in [-1,0] \mid x \in \delta_{A,\alpha}\}$, *for all $x \in M, \alpha \in \Gamma$.*

Proof. Proofs of (1) and (2) are similar to proof of Theorem 3.12 in [3], so we omit the proof. Let $\eta = \inf\{\alpha \in [-1, 0] \mid x \in \delta_{A,\alpha}\}$. Then

$$\begin{aligned}
&\inf\{\alpha \in [-1, 0] \mid x \in \delta_{A,\alpha}\} < \eta + \epsilon, \quad \text{for any } \ \epsilon > 0 \\
\Rightarrow\ &\alpha < \eta + \epsilon, \ \text{for some } \ \alpha \in [-1, 0], \ x \in \delta_{A,\alpha} \\
\Rightarrow &\delta_A(x) \leq \eta, \quad \text{since } \ \delta_A(x) \leq \alpha. \\
&\text{Let } \delta_A(x) = \beta. \text{ Then } \ x \in \delta_{A,\beta} \\
\Rightarrow &\beta \in \{\alpha \in [-1, 0] \mid x \in \delta_{A,\alpha}\} \\
\Rightarrow &\inf\{\alpha \in [-1, 0] \mid x \in \delta_{A,\alpha}\} \leq \beta \\
\Rightarrow &\eta \leq \beta = \delta_A(x). \\
\Rightarrow &\delta_A(x) = \eta.
\end{aligned}$$

Hence $\delta_A(x) = \inf\{\alpha \in [-1, 0] \mid x \in \delta_{A,\alpha}\}$. Hence the theorem. □

The following proof of the theorem is similar to proof of the Theorem 3.9 in [3]. Hence we omit the proof of the following theorem.

Theorem 3.14. *Let $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar fuzzy set in the $\Gamma-$semiring M such that non-empty sets $\mu_{A,\alpha}, \lambda_{A,\alpha}, \delta_{A,-\alpha}$ are interior ideals of M, for all $\alpha \in [0, 1]$. Then A is a tripolar fuzzy interior ideal of M.*

Theorem 3.15. *A tripolar fuzzy set $A = (\mu_A, \lambda_A, \delta_A)$ is a fuzzy interior ideal of $\Gamma-$semiring M if and only if fuzzy subsets $\mu_A, \overline{\lambda}_A, \delta_A$ are fuzzy interior ideals of a $\Gamma-$semiring M.*

Proof. Suppose $A = (\mu_A, \lambda_A, \delta_A)$ is a tripolar fuzzy interior ideal of the $\Gamma-$semiring M. Then obviously μ_A, δ_A are fuzzy interior ideals of M. Let $x, y \in M, \alpha \in \Gamma$.

$$\begin{aligned}
\overline{\lambda}_A(x + y) &= 1 - \lambda_A(x + y) \\
&\geq 1 - \max\{\lambda_A(x), \lambda_A(y)\} \\
&= \min\{1 - \lambda_A(x), 1 - \lambda_A(y)\} \\
&= \min\{\overline{\lambda}_A(x), \overline{\lambda}_A(y)\}. \\
\overline{\lambda}_A(x\alpha y) &= 1 - \lambda_A(x\alpha y) \\
&\geq 1 - \max\{\lambda_A(x), \lambda_A(y)\} \\
&= \min\{1 - \lambda_A(x), 1 - \lambda_A(y)\} \\
&= \min\{\overline{\lambda}_A(x), \overline{\lambda}_A(y)\}, \ \text{for all } x, y \in M, \alpha \in \Gamma.
\end{aligned}$$

Suppose $x, y, z \in M, \alpha, \beta \in \Gamma$. Then $\overline{\lambda}_A(x\alpha z\beta y) = 1 - \lambda_A(x\alpha z\beta y) \geq 1 - \lambda_A(z) = \overline{\lambda}_A(z)$. Hence $\overline{\lambda}$ is an fuzzy interior ideal of M.

Conversely suppose that $\mu_A, \overline{\lambda}_A, \delta_A$ are fuzzy interior ideals of the $\Gamma-$semring M.
Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

$$\begin{aligned}
\lambda_A(x+y) &= 1 - \overline{\lambda}_A(x+y) \\
&\geq \max\{1 - \overline{\lambda}_A(x), 1 - \overline{\lambda}_A(y)\} \\
&= \max\{\lambda_A(x), \lambda_A(y)\} \\
\lambda_A(x\alpha y) &= 1 - \overline{\lambda}_A(x\alpha y) \\
&\geq \max\{1 - \overline{\lambda}_A(x), 1 - \overline{\lambda}_A(y)\} \\
&= \max\{\lambda_A(x), \lambda_A(y)\} \\
\overline{\lambda}_A(x\alpha z\beta y) &\geq \overline{\lambda}_A(z) \\
\Rightarrow 1 - \lambda_A(x\alpha z\beta y) &\geq 1 - \lambda_A(z) \\
\Rightarrow \lambda_A(x\alpha z\beta y) &\leq \lambda_A(z).
\end{aligned}$$

Hence the theorem. □

Corollary 3.16. *A tripolar fuzzy set $A = (\mu_A, \lambda_A, \delta_A)$ is a tripolar fuzzy interior ideal of a $\Gamma-$semiring M if and only if the subsets $(\mu_A, \overline{\mu}_A, \delta_A)$ and $(\overline{\lambda}_A, \lambda_A, \delta_A)$ are tripolar fuzzy interior ideals of a $\Gamma-$semiring M.*

Definition 3.17. Let $f : X \to Y$ be a map. If $A = (\mu_A, \lambda_A, \delta_A)$ and $B = (\mu_B, \lambda_B, \delta_B)$ are tripolar fuzzy sets in X and Y respectively. Then pre-image of B under f, denoted by $f^{-1}(B)$ is a tripolar fuzzy set in X defined by $f^{-1} = \big(f^{-1}(\mu_B), f^{-1}(\lambda_B), f^{-1}(\delta_B)\big)$,
where $f^{-1}(\mu_B) = \mu_B(f), f^{-1}(\lambda_B) = \lambda_B(f)$ and $f^{-1}(\delta_B) = \delta_B(f)$.

Theorem 3.18. *Let $f : M \to N$ be a homomorphism of $\Gamma-$semrings. If $B = (\mu_B, \lambda_B, \delta_B)$ is a tripolar fuzzy interior ideal of $\Gamma-$semiring N. Then $f^{-1}(B) = \big(f^{-1}(\mu_B), f^{-1}(\lambda_B), f^{-1}(\delta_B)\big)$ is a tripolar fuzzy interior ideal of $\Gamma-$semiring M.*

Proof. Suppose $B = (\mu_B, \lambda_B, \delta_B)$ is a tripolar fuzzy interior ideal of the $\Gamma-$semiring N and $x, y \in M, \alpha \in \Gamma$. Then

$$\begin{aligned}
f^{-1}\big(\mu_B(x+y)\big) &= \mu_B(f(x+y)) \\
&= \mu_B\big(f(x) + f(y)\big) \\
&\geq \min\{\mu_B(f(x)), \mu_B(f(y))\} \\
&= \min\{f^{-1}(\mu_B(x)), f^{-1}(\mu_B(y))\}.
\end{aligned}$$

$$\begin{aligned} f^{-1}\big(\mu_B(x\alpha y)\big) &= \mu_B(f(x\alpha y)) \\ &= \mu_B\big(f(x)\alpha f(y)\big) \\ &\geq \min\{\mu_B(f(x)), \mu_B(f(y))\} \\ &= \min\{f^{-1}(\mu_B(x)), f^{-1}(\mu_B(y))\}. \end{aligned}$$

Suppose $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, then we have

$$\begin{aligned} f^{-1}\big(\mu_B(x\alpha z\beta y)\big) &= \mu_B(f(x\alpha z\beta y)) \\ &= \mu_B\big(f(x)\alpha f(z)\beta f(y)\big) \\ &\geq \mu_B f(z) \\ &= f^{-1}(\mu_B f(z)). \\ f^{-1}\big(\lambda_B(x+y)\big) &= \lambda_B(f(x+y)) \\ &= \lambda_B\big(f(x)+f(y)\big) \\ &\leq \max\{\lambda_B(f(x)), \lambda_B(f(y))\} \\ &= \max\{f^{-1}(\lambda_B(x)), f^{-1}(\lambda_B(y))\} \\ f^{-1}\big(\lambda_B(x\alpha y)\big) &= \lambda_B(f(x\alpha y)) \\ &= \lambda_B\big(f(x)\alpha f(y)\big) \\ &\leq \max\{\lambda_B(f(x)), \lambda_B(f(y))\} \\ &= \max\{f^{-1}(\lambda_B(x)), f^{-1}(\lambda_B(y))\}. \\ f^{-1}\big(\lambda_B(x\alpha z\beta y)\big) &= \lambda_B(f(x\alpha z\beta y)) \\ &= \lambda_B\big(f(x)\alpha f(z)\beta f(y)\big) \\ &\leq \lambda_B(f(z)) \\ &= f^{-1}(\lambda_B(f(z))). \\ f^{-1}\big(\delta_B(x+y)\big) &= \delta_B(f(x+y)) \\ &= \delta_B\big(f(x)+f(y)\big) \\ &\leq \max\{\delta_B(f(x)), \delta_B(f(y))\} \\ &= \max\{f^{-1}(\delta_B(x)), f^{-1}(\delta_B(y))\} \\ f^{-1}\big(\delta_B(x\alpha y)\big) &= \delta_B(f(x\alpha y)) \\ &= \delta_B\big(f(x)\alpha f(y)\big) \\ &\leq \max\{\delta_B(f(x)), \delta_B(f(y))\} \\ &= \max\{f^{-1}(\delta_B(x)), f^{-1}(\delta_B(y))\} \\ .f^{-1}\big(\delta_B(x\alpha z\beta y)\big) &= \delta_B(f(x\alpha z\beta y)) \\ &= \delta_B\big(f(x)\alpha f(z)\beta f(y)\big) \\ &\leq \delta_B(f(z)) \end{aligned}$$

Hence $f^{-1}(B) = \big(f^{-1}(\mu_B), f^{-1}(\lambda_B), f^{-1}(\delta_B)\big)$ is a tripolar fuzzy interior ideal of the $\Gamma-$semiring M. □

Theorem 3.19. *Let I be an interior ideal of a $\Gamma-$semiring M and $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar fuzzy subset of M defined by*

$$\mu_A(x) = \begin{cases} \alpha_0, & \text{if } x \in I; \\ \alpha_1, & \text{otherwise} \end{cases} ; \ \lambda_A(x) = \begin{cases} \beta_0, & \text{if } x \in I; \\ \beta_1, & \text{otherwise} \end{cases} ; \delta_A(x) = \begin{cases} \gamma_0, & \text{if } x \in I; \\ \gamma_1, & \text{otherwise} \end{cases} ,$$

for all $x \in M$ and $\alpha_i, \beta_i \in [0,1]$ such that $\alpha_0 > \alpha_1, \beta_0 > \beta_1$ and $\alpha_i + \beta_i \leq 1$, for $i = 0, 1, \gamma_0 > \gamma_1, \gamma_0, \gamma_1 \in [-1, 0]$. Then $A = (\mu_A, \lambda_A, \delta_A)$ is a tripolar fuzzy interior ideal of the $\Gamma-$semiring M and $\mu_{A,\alpha_0} = \lambda_{A,\beta_0} = \delta_{A,\gamma_0} = I$.

Proof. Obviously $A = (\mu_A, \lambda_A, \delta_A)$ is tripolar fuzzy $\Gamma-$subsemiring of M. Suppose $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.
case (i) : If $z \notin I$ then

$$\mu_A(x\alpha z\beta y) \geq \alpha_1 = \mu_A(z)$$
$$\lambda_A(x\alpha z\beta y) \leq \beta_1 = \lambda_A(z)$$
$$\delta_A(x\alpha z\beta y) \leq \gamma_1 = \delta_A(z).$$

case (ii) : If $z \in I$ then

$$\mu_A(x\alpha z\beta y) = \alpha_0 = \mu_A(z)$$
$$\lambda_A(x\alpha z\beta y) = \beta_0 = \lambda_A(z)$$
$$\delta_A(x\alpha z\beta y) = \gamma_0 = \delta_A(z).$$

Therefore $A = (\mu_A, \lambda_A, \delta_A)$ is tripolar fuzzy interior ideal of the $\Gamma-$semiring M.
Obviously, by definitions of μ_A, λ_A and δ_A, $\mu_{A,\alpha_0} = \lambda_{A,\beta_0} = \delta_{A,\gamma_0} = I$.
Hence the theorem. □

Corollary 3.20. *Let χ_I be the characteristic function of an interior ideal I of a $\Gamma-$semiring M. Then tripolar fuzzy set $(\chi_I, \overline{\chi}_I, \delta_I)$, where*

$$\delta_I(x) = \begin{cases} 0, & \text{if } x \in I; \\ -1, & \text{otherwise} \end{cases}$$

is tripolar interior fuzzy ideal of a $\Gamma-$semring M.

Theorem 3.21. *Let M and N be $\Gamma-$semirings, $\phi : M \to N$ be a homomorphism and f be a ϕ invariant fuzzy ideal of a $\Gamma-$semiring M. If $x = \phi(a)$, then $\phi(f)(x) = f(a), a \in M$.*

Proof. Let M and N be $\Gamma-$semirings, $a \in M, x \in N$, $x = \phi(a)$.
Then $a \in \phi^{-1}(x)$ and $t \in \phi^{-1}(x)$.
Therefore $\phi(t) = x = \phi(a)$, since f is ϕ invariant.
$f(t) = f(a) \Rightarrow \phi(f)(x) = \sup_{t\in\phi^{-1}(x)} \{f(t)\} = f(a)$.
Hence $\phi(f)(x) = f(a)$. □

Definition 3.22. Let a functions $\phi : M \to N$ be a homomorphism of $\Gamma-$semirings M, N and $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar fuzzy set of a $\Gamma-$semiring M. Then A is said to be ϕ homomorphism invariant if $\phi(a) = \phi(b)$

(i) $\mu_A(x) = \mu_A(y)$
(ii) $\lambda_A(x) = \lambda_A(y)$
(iii) $\delta_A(x) = \delta_A(y)$, for all $x, y \in M$.

Theorem 3.23. *Let M and N be $\Gamma-$semirings and $\phi : M \to N$ be an onto homomorphism. If A is a homomorphism ϕ invariant tripolar interior ideal of a $\Gamma-$semiring M, then image of A under homomorphism ϕ is a tripolar fuzzy interior ideal of a $\Gamma-$semiring M.*

Proof. Let $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar interior ideal of the$\Gamma-$semiring M and $x, y \in N, \alpha \in \Gamma$. Then there exist $a, b \in M$ such that $\phi(a) = x, \phi(b) = y$

$$\begin{aligned}
\Rightarrow \phi(a+b) &= x + y \\
\Rightarrow \phi(\mu_A)(x+y) &= \mu_A(a+b) \\
&\geq \min\{\mu_A(a), \mu_A(b)\} \\
&= \min\{\phi(\mu_A)(x), \phi(\mu_A)(y)\}. \\
\phi(\mu_A(x\alpha y)) &= \mu_A(a\alpha b) \\
&\geq \min\{\mu_A(x), \mu_A(b)\} \\
&= \min\{\phi(\mu_A)(x), \phi(\mu_A)(y)\}.
\end{aligned}$$

Suppose $x, y, z \in N, \alpha, \beta \in \Gamma$. Then there exist $a, b, c \in N$ such that $\phi(a) = x, \phi(b) = y$ and $\phi(c) = z$. Then

$$\begin{aligned}
\phi(\mu_A(x\alpha z\beta y)) &= \mu_A(a\alpha c\beta b) \\
&\geq \mu_A(c) \\
&= \phi(\mu_A(z)).
\end{aligned}$$

Hence $\phi(\mu_A)$ is a fuzzy interior ideal of the$\Gamma-$semiring M.

$$\begin{aligned}
\phi(\lambda_A)(x+y) &= \lambda_A(a+b) \\
&\leq \max\{\lambda_A(a), \lambda_A(b)\} \\
&= \max\{\phi(\lambda_A)(x), \phi(\lambda_A)(y)\}. \\
\phi(\lambda_A(x\alpha y)) &= \lambda_A(a\alpha b) \\
&\leq \min\{\lambda_A(x), \lambda_A(b)\} \\
&= \min\{phi(\lambda_A)(x), \phi(\lambda_A)(y)\}. \\
\phi(\lambda_A(x\alpha z\beta y) &= \lambda_A(a\alpha c\beta b) \\
&\leq \lambda_A(c) \\
&= \phi(\lambda_A(z)).
\end{aligned}$$

Therefore $\phi(\lambda_A)$ is a fuzzy interior ideal of the $\Gamma-$semiring M.

$$\begin{aligned}
\phi(\delta_A)(x+y) &= \delta_A(a+b) \\
&\leq \max\{\delta_A(a), \delta_A(b)\} \\
&= \max\{\phi(\delta_A)(x), \phi(\delta_A)(y)\}. \\
\phi(\delta_A(x\alpha y)) &= \delta_A(a\alpha b) \\
&\leq \min\{\delta_A(x), \delta_A(b)\} \\
&= \min\{phi(\delta_A)(x), \phi(\delta_A)(y)\}. \\
\phi(\delta_A(x\alpha z\beta y) &= \delta_A(a\alpha c\beta b) \\
&\leq \delta_A(c) \\
&= \phi(\delta_A(z)).
\end{aligned}$$

Therefore $\phi(\delta_A)$ is a fuzzy interior ideal of the $\Gamma-$semiring M. Hence $\phi(A)$ is a tripolar fuzzy interior ideal of the $\Gamma-$semiring M. □

4. Conclusion

In this paper, we introduced the notion of a tripolar fuzzy set to be able to deal with tripolar information as a generalization of a fuzzy set,a bipolar fuzzy set and an intuitionistic fuzzy set. We also introduced the notion of tripolar fuzzy ideals and tripolar fuzzy interior ideals of a $\Gamma-$semiring. We studied some of their algebraic properties and relations between them.

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