A NEW SUBCLASS OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH CHEBYSHEV POLYNOMIAL

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ABSTRACT. In this paper we introduce a new subclass $H_\Sigma(k, \lambda, t)$ of analytic bi-univalent functions by using the Chebyshev polynomials. We obtain estimates on initial coefficient bounds for functions in this subclass.

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1. Introduction

Let A denote the class of functions of the form:

(1.1)
$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

which are analytic in the open unit disc $U = \{z : z \in \mathbb{C} \quad and \quad |z| < 1\}$ and satisfy the normalization condition $g(0) = g'(0) = 1$.

Let $f(z)$ and $g(z)$ be analytic functions in U , we say that $f(z)$ is subordinate to $g(z)$, written as

$$f(z) \prec g(z)$$

if there exist a Schwarz functions $w(z)$ in U, such that $f(z) = g(w(z))$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ [6]

In particular, if the function $g(z)$ is univalent in U, then the above subordination is equivalent to

$$f(0) = g(0)$$

and

$$f(U) \subseteq g(U).$$

Let S be the class of A consisting of the functions of the form (1.1) which are also univalent in U. According to Koebe one quarter theorem [6], it is well known fact that every function $g \in S$ has an inverse g^{-1}, which is defined by

$$g^{-1}(g(z)) = z(z \in U)$$

and

$$g(g^{-1}(w)) = w(|w| < r_0(g); r_0(g) \geq \frac{1}{4}),$$

where

(1.2) $\qquad g^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + ...$

A function $f(z) \in A$ is said to be bi-univalent in U if both $g(z)$ and $g^{-1}(z)$ are univalent in U. Let Σ denote the class of all bi-univalent functions in U given by the Taylor-Maclaurin series expansion (1.1). Several recent researchers [2],[7],[5],[8],[9],[3] used Chebyshev polynomial expansions for finding the estimates on the coefficient bounds for the subclasses of biunivalent functions.Hence motivated by their work, we also use Chebyshev polynomial expansion to find the coefficient estimate of our new subclass of biunivalent functions. The significance of Chebyshev polynomial in numerical analysis is increased in both theoretical and practical points of view. Out of four kinds of Chebyshev polynomials, many researchers dealing with orthogonal polynomials of Chebyshev.For a brief history of Chebyshev polynomials of first kind $T_n(t)$, the second kind $U_n(t)$ and their applications one can refer[1],[4]. The Chebyshev polynomials of the first and second kinds are well known and they are defined by

$$T_n(t) = cosn\theta$$

and

$$U_n(t) = \frac{sin(n+1)\theta}{sin\theta} \qquad (-1 < t < 1)$$

where n denotes the polynomial degree and $t = cos\theta$.

We define for $g(z)$ in the form(1.1), the following:

$$F_\lambda^0 g(z) = g(z)$$

$$F_0^k g(z) = g(z)$$

$$F_\lambda^1 g(z) = F_\lambda g(z) = (1-\lambda)g(z) + \lambda z g'(z) = z + \sum_{n=2}^{\infty}(1 + (n-1)\lambda)b_n z^n$$

and

$$F_\lambda^2 g(z) = F_\lambda[F_\lambda g(z)] = z + \sum_{n=2}^{\infty}((1 + (n-1)\lambda)^2 b_n z^n$$

$$\left(0 \leq \lambda < \frac{1}{1-n}, k \in \mathbb{N}_0 = 0, 1, 2, ...\right)$$

Hence, it can be easily seen that

$$(1.3) \qquad F_\lambda^k g(z) = z + \sum_{n=2}^{\infty} (1 + (n-1)\lambda)^k b_n z^n$$

In the present paper, we introduce the subclass $H_\Sigma(k, \lambda, t)$ of bi-univalent functions associated with the linear operator $F_\lambda^k g(z)$. Also estimates on the coefficients of functions in this class are obtained using Chebyshev polynomial expansions.

Definition 1.1. *For $t \in (1/2, 1]$,$0 \le \lambda < \dfrac{1}{1-n}$ a function $g(z) \in \Sigma$ is said to be in the class $H_\Sigma(k, \lambda, t)$,if the following condition are satisfied:*

$$(1.4) \qquad \frac{z(F_\lambda^k g(z))'}{F_\lambda^k g(z)} \prec H(z, t) := \frac{1}{1 - 2tz + z^2} \qquad (z \in U) \, (k = 0, 1, 2, ...z \in U)$$

and

$$(1.5) \qquad \frac{w(F_\lambda^k h(w))'}{F_\lambda^k h(w)} \prec H(w, t) := \frac{1}{1 - 2tw + w^2}, \qquad (w \in U), (k = 0, 1, 2, ...w \in U)$$

where $h = g^{-1}$.

We note that if $t = cos\alpha$, where $\alpha \in (-\pi/3, \pi/3)$, then

$$H(z, t) = \frac{1}{1 - 2cos\alpha z + z^2} = 1 + \sum_{n=1}^{\infty} \frac{sin(n+1)\alpha}{sin\alpha} z^n \qquad (z \in U).$$

Thus

$$H(z, t) = 1 + 2cos\alpha z + (3cos^2\alpha - sin^2\alpha)z^2 + ...(z \in U)$$

From [10], we can write

$$(1.6) \qquad H(z, t) = 1 + U_1(t)z + U_2(t)z^2 + ... \qquad (z \in U, t \in (-1, 1))$$

where

$$U_{n-1}(t) = \frac{sin(n \arccos t)}{\sqrt{1 - t^2}} \qquad (n \in N)$$

are the Chebyshev polynomials of the second kind and we have

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t),$$

and

$$(1.7) \qquad U_1(t) = 2t, \quad U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t, \quad U_4(t) = 16t^4 - 12t^2 + 1, ...$$

The generating function of the first kind of Chebyshev polynomial $T_n(t), t \in [-1, 1]$, is given by

$$\sum_{n=0}^{\infty} T_n(t)z^n = \frac{1 - tz}{1 - 2tz + z^2} \qquad (z \in U).$$

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The first kind of Chebyshev polynomial $T_n(t)$ and the second kind of Chebyshev polynomial $U_n(t)$ are connected by:

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t); \quad T_n(t) = U_n(t) - tU_{n-1}(t); \quad 2T_n(t) = U_n(t) - U_{n-2}(t).$$

In this present paper, we use the Chebyshev polynomials expansions to provide the initial coefficients of bi-univalent functions in $H_\Sigma(k, \lambda, t)$.

2. Coefficient bounds for the class $H_\Sigma(k, \lambda, t)$

Theorem 2.1. *For $0 \leq \lambda < \dfrac{1}{1-n}$ and $t \in (1/2, 1]$, let the function $g(z) \in \Sigma$ given by (1.1) be in the class $H_\Sigma(k, \lambda, t)$. Then*

$$(2.1) \qquad |b_2| \leq \frac{2t\sqrt{2t}}{\sqrt{(2(1+2\lambda)^k - 2(1+\lambda)^{2k})4t^2 + (1+\lambda)^{2k}}}$$

$$(2.2) \qquad |b_3| \leq \frac{4t^2}{(1+\lambda)^{2k}} + \frac{t}{(1+2\lambda)^k}$$

and for some $\mu \in R$,

$$(2.3) \qquad |b_3 - \mu b_2^2| \leq \begin{cases} \dfrac{2t}{2(1+2\lambda)^k}, \\ \qquad for \, |\mu - 1| \leq \dfrac{[(1+2\lambda)^k - (1+\lambda)^{2k}]8t^2 + (1+\lambda)^{2k}}{8t^2(1+2\lambda)^k} \\ \dfrac{8(1-\mu)t^3}{[(1+2\lambda)^k - (1+\lambda)^{2k}]8t^2 + (1+\lambda)^{2k}}, \\ \qquad for \, |\mu - 1| \geq \dfrac{[(1+2\lambda)^k - (1+\lambda)^{2k}]8t^2 + (1+\lambda)^{2k}}{8t^2(1+2\lambda)^k} \end{cases}$$

Proof. Let $g(z) \in \Sigma$ given by (1.1) be in the class $H_\Sigma(k, \lambda, t)$. From (1.3) and (1.4), we have

$$(2.4) \qquad \frac{z(F_\lambda^k g(z))'}{F_\lambda^k g(z)} = 1 + U_1(t)p(z) + U_2(t)p^2(z) + \ldots$$

and for its inverse map, $h = g^{-1}$,

$$(2.5) \qquad \frac{w(F_\lambda^k h(w))'}{F_\lambda^k h(w)} = 1 + U_1(t)q(w) + U_2(t)q^2(w) + \ldots$$

for some analytic function

$$(2.6) \qquad p(z) = c_1 z + c_2 z^2 + c_3 z^3 + \ldots \qquad (z \in U)$$

and

$$(2.7) \qquad q(w) = d_1 w + d_2 w^2 + d_3 w^3 + \ldots \qquad (w \in U),$$

such that $p(0) = q(0) = 0, |p(z)| < 1 (z \in U)$ and $|q(w)| < 1 (w \in U)$. It is well-known that if $|p(z)| < 1$ and $|q(w)| < 1$, then

(2.8) $$|c_j| \leq 1 \quad and \quad |d_j| \leq 1 \quad for\,all \quad j \in N.$$

From (2.4),(2.5),(2.6)and (2.7), we have

(2.9) $$\frac{z(F_\lambda^k g(z))'}{F_\lambda^k g(z)} = 1 + U_1(t)c_1(z) + [U_1(t)c_2 + U_2(t)c_1^2]z^2 + ...$$

(2.10) $$\frac{w(F_\lambda^k h(w))'}{F_\lambda^k h(w)} = 1 + U_1(t)d_1(w) + [U_1(t)d_2 + U_2(t)d_1^2]w^2 + ...$$

Equating the coefficients in (2.9) and (2.10), we get

(2.11) $$(1 + \lambda)^k b_2 = U_1(t)c_1$$

(2.12) $$2(1 + 2\lambda)^k b_3 - (1 + \lambda)^{2k} b_2^2 = U_1(t)c_2 + U_2(t)c_1^2$$

(2.13) $$-(1 + \lambda)^k b_2 = U_1(t)d_1$$

(2.14) $$(4b_2^2 - 2b_3)(1 + 2\lambda)^{2k} - (1 + \lambda)^{2k} b_2^2 = U_1(t)d_2 + U_2(t)d_1^2$$

From (2.11) and (2.13), we obtain

(2.15) $$c_1 = -d_1$$

and

(2.16) $$2(1 + \lambda)^{2k} b_2^2 = U_1^2(t)(c_1^2 + d_1^2)$$

Also, by using (2.12) and (2.14) we obtain

(2.17) $$2[2(1 + 2\lambda)^k - (1 + \lambda)^{2k}]b_2^2 = U_1(t)(c_2 + d_2) + U_2(t)(c_1^2 + d_1^2)$$

By using (2.16)in (2.17), we get

(2.18) $$\left(2(1 + 2\lambda)^k - (1 + \lambda)^{2k} - \frac{U_2(t)}{U_1^2(t)}(1 + \lambda)^{2k}\right) 2b_2^2 = U_1(t)(c_2 + d_2).$$

From (1.6),(2.8) and (2.18), we have the desired in equality (2.1). Next, by subtracting (2.14) from (2.12), we have

(2.19) $$2a_3[(1 - \lambda)g_3 + \lambda h_3] - 2a_2^2[(1 - \lambda)g_3 + \lambda h_3] = U_1(t)(c_2 - d_2) + U_2(t)(c_1^2 - d_1^2).$$

Further, in view of (2.15), we obtain

(2.20) $$b_3 = b_2^2 + \frac{U_1(t)(c_2 - d_2)}{4(1 + 2\lambda)^k}$$

Hence using (2.16)and applying (1.6), we get desired inequality (2.2). Now, by using (2.18) and (2.20) for some $\mu \in R$, we get

$$b_3 - \mu b_2^2 = (1-\mu)\left[\frac{U_1^3(t)(c_2+d_2)}{2([2(1+2\lambda)^k - (1+\lambda)^{2k}]U_1^2(t) - U_2(t)(1+\lambda)^{2k})}\right] + \frac{U_1(t)(c_2-d_2)}{4(1+2\lambda)^k}$$

$$= \frac{U_1(t)}{2}\left[\left(h(\mu) + \frac{1}{2(1+2\lambda)^k}\right)c_2 + \left(h(\mu) - \frac{1}{2(1+2\lambda)^k}\right)d_2\right],$$

where

$$h(\mu) = \frac{U_1^2(t)(1-\mu)}{[2(1+2\lambda)^k - (1+\lambda)^{2k}]U_1^2(t) - U_2(t)(1+\lambda)^{2k}}.$$

So,we conclude that

$$|b_3 - \mu b_2^2| \leq \begin{cases} \dfrac{2t}{2(1+2\lambda)^k}, & 0 \leq |h(\mu)| \leq \dfrac{1}{2(1+2\lambda)^k} \\ 2|h(\mu)|(t), & |h(\mu)| \geq \dfrac{1}{2(1+2\lambda)^k} \end{cases}$$

This completes the proof.

Taking $\mu = 1$ in Theorem (2.1), we get the following consequence.

Corollary 2.2. *For $0 \leq \lambda \leq \frac{1}{1-n}$ and $t \in (1/2, 1]$, let the function $f \in \Sigma$ given by (1.1) be in the class $H_\Sigma(k, \lambda, t)$ .Then*

$$|b_3 - b_2^2| \leq \frac{2t}{2(1+2\lambda)^k}$$

Taking $\lambda = 1$ and k=1 in Theorem (2.1), we get the following consequence.

Corollary 2.3. *For $t \in (1/2, 1]$, let the function $f \in \Sigma$ given by (1.1) be in the class $H_\Sigma(t)$ Then*

$$|b_2| \leq \frac{t\sqrt{2t}}{\sqrt{1-2t^2}}$$

$$|b_3| \leq 4t^2 + \frac{t}{3}$$

and for some $\mu \in R$,

$$|b_3 - \mu b_2^2| \leq \begin{cases} \dfrac{t}{3}, & |\mu - 1| \leq \dfrac{1-2t^2}{6t^2} \\ \dfrac{2(1-\mu)t^3}{1-2t^2}, & |\mu - 1| \geq \dfrac{1-2t^2}{6t^2} \end{cases}$$

Taking $\mu = 1$ in Corollary(2.3) , we get the following consequence

Corollary 2.4. *For $t \in (1/2, 1]$, let the function $f \in \Sigma$ given by(1.1) be in the class $H_\Sigma(t)$. Then*

$$|b_3 - b_2^2| \leq \frac{t}{3}.$$

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