1-CONVERGENT TRIPLE SEQUENCE SPACES OVER $n$-NORMED SPACE

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Abstract. The concept of 2-normed spaces was initially developed by Gähler [10], which was extend to $n$-norm by Misiak [17] for single sequence space. The main objective of this paper is to study triple sequence spaces over $n$-norm via the sequence of modulus functions. 2010 Mathematics Subject Classification. 40A05, 40C05, 46A45.

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1. Introduction

A triple sequence (real or complex) is a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ are the set of natural numbers, real numbers, and complex numbers respectively. We denote by $\omega'''$ the class of all complex triple sequence $(x_{pqr})$, where $p, q, r \in \mathbb{N}$. Then under the coordinate wise addition and scalar multiplication $\omega'''$ is a linear space. A triple sequence can be represented by a matrix, in case of double sequences we write in the form of a square. In case of triple sequence it will be in the form of a box in three dimensions.

The different types of notions of triple sequences and their statistical convergence were introduced and investigated initially by Sahiner et. al [22]. Later Debnath et.al [2, 3], Esi et.al [4, 5, 6], Tripathy [24] and many others authors have studied it further and obtained various results.

Statistical convergence was introduced by Fast [7] and later on it was studied by Fridy [8, 9] from the sequence space point of view and linked it with summability theory. The notion of statistical convergent double sequence was introduced by Mursaleen and Edely [18].
I-convergence is a generalization of the statistical convergence. Kostyrko et. al. [15] introduced the notion of I-convergence of real sequence and studied its several properties. Later Jalal [11, 12, 13], Salat et. al. [20] and many other researchers contributed in its study. Sahiner and Tripathy [22] studied I-related properties in triple sequence spaces and showed some interesting results. Tripathy [24] extended the concept of I-convergent to double sequence and later Kumar [16] obtained some results on I-convergent double sequence. Recently Jalal and Malik [14] extended the concept of n-norms to triple sequence spaces and proved several algebraic and topological properties.

In this paper we define the spaces \( c_3[\mathcal{F},\|\cdot\|,\ldots,\|\cdot\|]^I, c_3^0[\mathcal{F},\|\cdot\|,\ldots,\|\cdot\|]^I, \ell_\infty^3[\mathcal{F},\|\cdot\|,\ldots,\|\cdot\|]^I, M_3^3[\mathcal{F},\|\cdot\|,\ldots,\|\cdot\|]^I, M_3^3_0[\mathcal{F},\|\cdot\|,\ldots,\|\cdot\|]^I \) by using the concept of n-normed space via the sequence of modulii functions \( F = (f_{pqr}) \). We study some algebraic and topological properties of these sequence spaces and some inclusion relations are obtained.

2. Definitions and preliminaries

**Definition 2.1.** Let \( X \neq \emptyset \). A class \( I \subset 2^X \) (Power set of \( X \)) is said to be an ideal in \( X \) if the following conditions holds good:

(i) \( I \) is additive that is if \( A,B \in I \) then \( A \cup B \in I \);

(ii) \( I \) is hereditary that is if \( A \in I \), and \( B \subset A \) then \( B \in I \).

\( I \) is called non-trivial ideal if \( X \notin I \)

**Definition 2.2.** [21, 22] A triple sequence \( (x_{pqr}) \) is said to be convergent to \( L \) in Pringsheim’s sense if for every \( \epsilon > 0 \), there exists \( N \in \mathbb{N} \) such that

\[ |x_{pqr} - L| < \epsilon \]

whenever \( p \geq N, q \geq N, r \geq N \)

and write as \( \lim_{p,p,r \to \infty} x_{pqr} = L \).

**Note:** A triple sequence is convergent in Pringsheim’s sense may not be bounded [21, 22].

**Example** Consider the sequence \( (x_{pqr}) \) defined by

\[ x_{pqr} = \begin{cases} p + q & \text{for all } p = q \text{ and } r = 1 \\ \frac{1}{p^qr} & \text{otherwise} \end{cases} \]

Then \( x_{pqr} \to 0 \) in Pringsheim’s sense but is unbounded.

**Definition 2.3.** A triple sequence \( (x_{pqr}) \) is said to be I-convergence to a number \( L \) if for every \( \epsilon > 0 \),

\[ \{(p,q,r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{pqr} - L| \geq \epsilon \} \in I. \]

In this case we write \( I - \lim x_{pqr} = L \).
Definition 2.4. A triple sequence \((x_{pqr})\) is said to be \(I\)-null if \(L = 0\). In this case we write 
\(I - \lim x_{pqr} = 0\).

Definition 2.5. [21, 22] A triple sequence \((x_{pqr})\) is said to be Cauchy sequence if for every 
\(\epsilon > 0\), there exists \(N \in \mathbb{N}\) such that
\[|x_{pqr} - x_{lmn}| < \epsilon \quad \text{whenever} \quad p \geq l \geq N, q \geq m \geq N, r \geq n \geq N\]

Definition 2.6. A triple sequence \((x_{pqr})\) is said to be \(I\)-Cauchy sequence if for every 
\(\epsilon > 0\), there exists \(N \in \mathbb{N}\) such that
\[\{(p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{pqr} - a_{lmn}| \geq \epsilon\} \in I\]
whenever \(p \geq l \geq N, q \geq m \geq N, r \geq n \geq N\)

Definition 2.7. [21, 22] A triple sequence \((x_{pqr})\) is said to be bounded if there exists \(M > 0\), such that
\[|x_{pqr}| < M \quad \text{for all} \quad p, q, r \in \mathbb{N}\]

Definition 2.8. A triple sequence \((x_{pqr})\) is said to be \(I\)-bounded if there exists \(M > 0\), such that
\[\{(p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{pqr}| \geq M\} \in I \quad \text{for all} \quad p, q, r \in \mathbb{N}\]

Definition 2.9. A triple sequence space \(E\) is said to be solid if \((\alpha_{pqr} x_{pqr}) \in E\) whenever 
\((x_{pqr}) \in E\) and for all sequences \((\alpha_{pqr})\) of scalars with \(|\alpha_{pqr}| \leq 1\), for all \(p, q, r \in \mathbb{N}\).

Definition 2.10. Let \(E\) be a triple sequence space and \(x = (x_{pqr}) \in E\). Define the set \(S(x)\) as
\[S(x) = \{(x_{\pi(pqr)}) : \pi \text{ is a permutations of } \mathbb{N}\}\]
If \(S(x) \subseteq E\) for all \(x \in E\), then \(E\) is said to be symmetric.

Definition 2.11. A triple sequence space \(E\) is said to be convergence free if \((y_{pqr}) \in E\) whenever 
\((x_{pqr}) \in E\) and \(x_{pqr} = 0\) implies \(y_{pqr} = 0\) for all \(p, q, r \in \mathbb{N}\).

Definition 2.12. A triple sequence space \(E\) is said to be sequence algebra if \(x \cdot y \in E\)
whenever \(x = (x_{pqr}) \in E\) and \(y = (y_{pqr}) \in E\), that is product of any two sequences is also in the space.

Definition 2.13. \((n\text{-Normed Space})\) Let \(n \in \mathbb{N}\) and \(X\) be a linear space over the field \(\mathbb{R}\) of reals of dimension \(d\), where \(2 \leq d \leq n\). A real valued function \(\|\cdot, \ldots, \cdot\|\) on \(X^n\) satisfying the following four conditions:

1. \(\|x_1, x_2, \ldots, x_n\| = 0\) if and only if \(x_1, x_2, \ldots, x_n\) are linearly dependent in \(X\);
2. \(\|x_1, x_2, \ldots, x_n\|\) is invariant under permutation;
3. \(\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \|x_1, x_2, \ldots, x_n\|\) for any \(\alpha \in \mathbb{R}\);
Definition 2.14. (Modulus Function) A function \( f : [0, \infty) \to [0, \infty) \) is called a modulus function if it satisfies the following conditions

(i) \( f(x) = 0 \) if and only if \( x = 0 \).

(ii) \( f(x + y) \leq f(x) + f(y) \) for all \( x \geq 0 \) and \( y \geq 0 \).

(iii) \( f \) is increasing.

(iv) \( f \) is continuous from the right at \( 0 \).
Since $|f(x) - f(y)| \leq f(|x - y|)$, it follows from condition (iv) that $f$ is continuous on $[0, \infty)$. Furthermore, from condition (2) we have $f(nx) \leq nf(x)$, for all $n \in \mathbb{N}$, and so $f(x) = f(n(x \frac{1}{n})) \leq nf(x)$. Hence $\frac{1}{n}f(x) \leq f(x)$ for all $n \in \mathbb{N}$.

Let $I$ be an admissible ideal, $F = (f_{pqr})$ be a sequence of modulus functions and $(X, \|\cdot, \ldots, \cdot\|)$ be a $n$-normed space. By $\omega''(n - X)$ we denote the space of all triple sequences defined over $(X, \|\cdot, \ldots, \cdot\|)$. In the present paper we define the following sequence spaces

$$c^3[F, \|\cdot, \ldots, \cdot\|]^I = \left\{ x = x_{pqr} \in \omega''(n - X) : \forall \epsilon > 0, \text{ the set } \left\{ (p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : f_{pqr}(\|x_{pqr} - L, z_1, \ldots, z_{n-1}\|) \geq \epsilon, \text{ for some } L \in \mathbb{C} \text{ and } z_1, \ldots, z_{n-1} \in X \right\} \in I \right\}$$

$$c_0^3[F, \|\cdot, \ldots, \cdot\|]^I = \left\{ x = x_{pqr} \in \omega''(n - X) : \forall \epsilon > 0, \text{ the set } \left\{ (p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : f_{pqr}(\|x_{pqr}, z_1, \ldots, z_{n-1}\|) \geq \epsilon, \text{ } z_1, \ldots, z_{n-1} \in X \right\} \in I \right\}$$

$$\ell_\infty^3[F, \|\cdot, \ldots, \cdot\|]^I = \left\{ x = x_{pqr} \in \omega''(n - X) : \exists K > 0 \text{ such that } \left\{ (p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sup_{p, q, r \geq 1} \{ f_{pqr}(\|x_{pqr}, z_1, \ldots, z_{n-1}\|) \geq K, \text{ } z_1, \ldots, z_{n-1} \in X \right\} \in I \right\}$$

and

$$M_0^3[F, \|\cdot, \ldots, \cdot\|]^I = c^3[F, \|\cdot, \ldots, \cdot\|]^I \cap \ell_\infty^3[F, \|\cdot, \ldots, \cdot\|]^I$$

$$M_0^3[F, \|\cdot, \ldots, \cdot\|]^I = c_0^3[F, \|\cdot, \ldots, \cdot\|]^I \cap \ell_\infty^3[F, \|\cdot, \ldots, \cdot\|]^I$$

For $F(x) = x$ we have

$$c^3[\|\cdot, \ldots, \cdot\|]^I = \left\{ x = x_{pqr} \in \omega''(n - X) : \forall \epsilon > 0, \text{ the set } \left\{ (p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \|x_{pqr} - L, z_1, \ldots, z_{n-1}\| \geq \epsilon, \text{ for some } L \in \mathbb{C} \text{ and } z_1, \ldots, z_{n-1} \in X \right\} \in I \right\}$$

$$c_0^3[\|\cdot, \ldots, \cdot\|]^I = \left\{ x = x_{pqr} \in \omega''(n - X) : \forall \epsilon > 0, \text{ the set } \left\{ (p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \|x_{pqr}, z_1, \ldots, z_{n-1}\| \geq \epsilon, \text{ } z_1, \ldots, z_{n-1} \in X \right\} \in I \right\}$$

$$\ell_\infty^3[\|\cdot, \ldots, \cdot\|]^I = \left\{ x = x_{pqr} \in \omega''(n - X) : \exists K > 0 \text{ such that } \left\{ (p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sup_{p, q, r \geq 1} \{ \|x_{pqr}, z_1, \ldots, z_{n-1}\| \geq K, \text{ } z_1, \ldots, z_{n-1} \in X \right\} \in I \right\}$$
and

\[ M^3[\|\cdot\|, \ldots, ||\|] = c^3[\|\cdot\|, \ldots, ||\|] \cap \ell^3_{\infty}[\|\cdot\|, \ldots, ||\|] \]

\[ M_0^3[\|\cdot\|, \ldots, ||\|] = c_0^3[\|\cdot\|, \ldots, ||\|] \cap \ell^3_{\infty}[\|\cdot\|, \ldots, ||\|] \]

3. Algebraic and Topological Properties of the New Sequence Spaces

**Theorem 3.1.** Let \( F = (f_{pqr}) \) be a sequence of modulus functions then the triple sequence spaces \( c_0^3[F, ||\cdot||, \ldots, ||\|] \), \( c^3[F, ||\cdot||, \ldots, ||\|] \), \( \ell^3_{\infty}[F, ||\cdot||, \ldots, ||\|] \), \( M^3[F, ||\cdot||, \ldots, ||\|] \) and \( M_0^3[F, ||\cdot||, \ldots, ||\|] \)

all linear over the field \( \mathbb{C} \) of complex numbers.

**Proof.** We prove the result for the sequence space \( c^3[F, ||\cdot||, \ldots, ||\|] \).

Let \( x = (x_{pqr}), y = (y_{pqr}) \in c^3[F, ||\cdot||, \ldots, ||\|] \) and \( \alpha, \beta \in \mathbb{C} \), then there exist positive integers \( m_\alpha \) and \( n_\beta \) such that \( |\alpha| \leq m_\alpha \) and \( |\beta| \leq n_\beta \), then for \( z_1, z_2, \ldots, z_{n-1} \in X \)

\[ I - \lim f_{pqr}(||x_{pqr} - L_1, z_1, \ldots, z_{n-1}||) = 0, \text{ for some } L_1 \in \mathbb{C} \]

\[ I - \lim f_{pqr}(||x_{pqr} - L_2, z_1, \ldots, z_{n-1}||) = 0, \text{ for some } L_2 \in \mathbb{C} \]

Now for a given \( \epsilon > 0 \) we set

\[ C_1 = \left\{ (p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : f_{pqr}(||x_{pqr} - L_1, z_1, \ldots, z_{n-1}||) > \frac{\epsilon}{2} \right\} \in I \quad (2.1) \]

\[ C_2 = \left\{ (p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : f_{pqr}(||y_{pqr} - L_2, z_1, \ldots, z_{n-1}||) > \frac{\epsilon}{2} \right\} \in I \quad (2.2) \]

Since \( f_{pqr} \) is a modulus function, so it is non-decreasing and convex, hence we get

\[
\begin{align*}
    f_{pqr}(||\alpha x_{pqr} + \beta y_{pqr} - (\alpha L_1 + \beta L_2), z_1, \ldots, z_{n-1}||) & = f_{pqr}(||\alpha x_{pqr} - \alpha L_1 + (\beta y_{pqr} - \beta L_2), z_1, \ldots, z_{n-1}||) \\
    & \leq f_{pqr}(||\alpha||x_{pqr} - L_1, z_1, \ldots, z_{n-1}||) + f_{pqr}(||\beta||y_{pqr} - L_2, z_1, \ldots, z_{n-1}||) \\
    & = |\alpha|f_{pqr}(||x_{pqr} - L_1||) + |\beta|f_{pqr}(||y_{pqr} - L_2||) \\
    & \leq m_\alpha f_{pqr}(||x_{pqr} - L_1, z_1, \ldots, z_{n-1}||) + n_\beta f_{pqr}(||y_{pqr} - L_2, z_1, \ldots, z_{n-1}||) 
\end{align*}
\]

From (2.1) and (2.2) we can write

\[ \{(p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : f_{pqr}(||\alpha x_{pqr} + \beta y_{pqr} - (\alpha L_1 + \beta L_2), z_1, \ldots, z_{n-1}||) > \epsilon \} \subseteq C_1 \cup C_2 \]

Thus \( \alpha x + \beta y \in c^3[F, ||\cdot||, \ldots, ||\|] \).

Therefore \( c^3[F, ||\cdot||, \ldots, ||\|] \) is a linear space.

In the same way we can show that other spaces are linear as well. \[\square\]
Hence the sequence $(c_0[F, ||\cdot, \cdot, \cdot||]^I) \subset (c_0[F, ||\cdot, \cdot, \cdot||]^I) \subset (c^3[F, ||\cdot, \cdot, \cdot||]^I)$ holds.

**Proof.** The inclusion $(c_0[F, ||\cdot, \cdot, \cdot||]^I) \subset (c^3[F, ||\cdot, \cdot, \cdot||]^I)$ is obvious. We prove $(c^3[F, ||\cdot, \cdot, \cdot||]^I) \subset (c^3[F, ||\cdot, \cdot, \cdot||]^I)$.

Let $x = (x_{pqr}) \in (c^3[F, ||\cdot, \cdot, \cdot||]^I)$ then there exists $L \in \mathbb{C}$ such that $I - \lim f_{pqr}(\|x_{pqr} - L, z_1, \ldots, z_{n-1}\|) = 0$, $z_1, \ldots, z_{n-1} \in X$

Since $F = (f_{pqr})$ is a sequence of modulus functions so

$$f_{pqr}(\|x_{pqr}, z_1, \ldots, z_{n-1}\|) \leq f_{pqr}(\|x_{pqr} - L, z_1, \ldots, z_{n-1}\|) + f_{pqr}(\|L, z_1, \ldots, z_{n-1}\|)$$

On taking supremum over $p, q$ and $r$ on both sides gives $x = (x_{pqr}) \in (c^3[F, ||\cdot, \cdot, \cdot||]^I)$

Hence the inclusion $(c_0[F, ||\cdot, \cdot, \cdot||]^I) \subset (c^3[F, ||\cdot, \cdot, \cdot||]^I) \subset (c^3[F, ||\cdot, \cdot, \cdot||]^I)$ holds.

**Theorem 3.3.** The triple sequence spaces $c_0^3[F, ||\cdot, \cdot, \cdot||]^I$ and $M_0^3[F, ||\cdot, \cdot, \cdot||]^I$ are solid.

**Proof.** We prove the result for $c_0^3[F, ||\cdot, \cdot, \cdot||]^I$.

Consider $x = (x_{pqr}) \in c_0^3[F, ||\cdot, \cdot, \cdot||]^I$, then

$I - \lim_{p,q,r} f_{pqr}(\|x_{pqr}, z_1, \ldots, z_{n-1}\|) = 0$

Consider a sequence of scalar $(\alpha_{pqr})$ such that $|\alpha_{pqr}| \leq 1$ for all $p, q, r \in \mathbb{N}$.

Then we have

$$I - \lim_{p,q,r} f_{pqr}(\|\alpha_{pqr} x_{pqr}, z_1, \ldots, z_{n-1}\|) \leq I - |\alpha_{pqr}| \lim_{p,q,r} f_{pqr}(\|x_{pqr}, z_1, \ldots, z_{n-1}\|)$$

$$\leq I - \lim_{p,q,r} f_{pqr}(\|x_{pqr}, z_1, \ldots, z_{n-1}\|) = 0$$

Hence $I - \lim_{p,q,r} f_{pqr}(\|\alpha_{pqr} x_{pqr}, z_1, \ldots, z_{n-1}\|) = 0$ for all $p, q, r \in \mathbb{N}$

Which gives $(\alpha_{pqr} x_{pqr}) \in c_0^3[F, ||\cdot, \cdot, \cdot||]^I$

Hence the sequence space $c_0^3[F, ||\cdot, \cdot, \cdot||]^I$ is solid.

The result for $M_0^3[F, ||\cdot, \cdot, \cdot||]^I$ can be similarly proved.

**Theorem 3.4.** The triple sequence spaces $c_0^3[F, ||\cdot, \cdot, \cdot||]^I$, $c^3[F, ||\cdot, \cdot, \cdot||]^I$, $\ell_\infty^3[F, ||\cdot, \cdot, \cdot||]^I$, $M^3[F, ||\cdot, \cdot, \cdot||]^I$ and $M_0^3[F, ||\cdot, \cdot, \cdot||]^I$ are sequence algebras.

**Proof.** We prove the result for $c_0^3[F, ||\cdot, \cdot, \cdot||]^I$.

Let $x = (x_{pqr}), y = (y_{pqr}) \in c_0^3[F, ||\cdot, \cdot, \cdot||]^I$

Then we have $I - \lim f_{pqr}(\|x_{pqr}, z_1, \ldots, z_{n-1}\|) = 0$ and $I - \lim f_{pqr}(\|x_{pqr}, z_1, \ldots, z_{n-1}\|) = 0$

Using definition modulus functions we have $I - \lim f_{pqr}(\|(x_{pqr} \cdot y_{pqr}), z_1, \ldots, z_{n-1}\|) = 0$. It
implies that $x \cdot y \in c_0^3[F, ||\cdot||, \ldots, ||\cdot||]$. 

The result can be proved for the spaces $c_3[F, ||\cdot||, \ldots, ||\cdot||]$, $\ell_3^\infty[F, ||\cdot||, \ldots, ||\cdot||]$, and $M_3[F, ||\cdot||, \ldots, ||\cdot||]$ in the same way. □

**Theorem 3.5.** In general the sequence spaces $c_3[F, ||\cdot||, \ldots, ||\cdot||]$, $c_3[F, ||\cdot||, \ldots, ||\cdot||]$, and $\ell_3^\infty[F, ||\cdot||, \ldots, ||\cdot||]$ are not convergence free.

**Proof.** We prove the result for the sequence space $c_3[F, ||\cdot||, \ldots, ||\cdot||]$ using an example.

**Example:** Let $I = I_f$ define the triple sequence $x = (x_{pqr})$ as

$$x_{pqr} = \begin{cases} 
0 & \text{if } p = q = r \\
1 & \text{otherwise}
\end{cases}$$

Then if $f_{pqr}(x) = x_{pqr} \forall p, q, r \in \mathbb{N}$, we have $x = (x_{pqr}) \in c_3[F, ||\cdot||, \ldots, ||\cdot||]$.

Now define the sequence $y = y_{pqr}$ as

$$y_{pqr} = \begin{cases} 
0 & \text{if } r \text{ is odd, and } p, q \in \mathbb{N} \\
0 & \text{otherwise}
\end{cases}$$

Then for $f_{pqr}(x) = x_{pqr} \forall p, q, r \in \mathbb{N}$, it is clear that $y = (y_{pqr}) \notin c_3[F, ||\cdot||, \ldots, ||\cdot||]$.

Hence the sequence spaces $c_3[F, ||\cdot||, \ldots, ||\cdot||]$ is not convergence free.

The space $c_3[F, ||\cdot||, \ldots, ||\cdot||]$ and $\ell_3^\infty[F, ||\cdot||, \ldots, ||\cdot||]$ are not convergence free in general can be proved in the same fashion. □

**Theorem 3.6.** In general the triple sequences $c_0^3[F, ||\cdot||, \ldots, ||\cdot||]$ and $c_3[F, ||\cdot||, \ldots, ||\cdot||]$ are not symmetric if $I$ is neither maximal nor $I = I_f$.

**Proof.** We prove the result for the sequence space $c_0^3[F, ||\cdot||, \ldots, ||\cdot||]$ using an example.

**Example:** Define the triple sequence $x = (x_{pqr})$ as

$$x_{pqr} = \begin{cases} 
0 & \text{if } r = 1, \text{ for all } p, q \in \mathbb{N} \\
1 & \text{otherwise}
\end{cases}$$

Then if $f_{pqr}(x) = x_{pqr} \forall p, q, r \in \mathbb{N}$, we have $x = (x_{pqr}) \in c_0^3[F, ||\cdot||, \ldots, ||\cdot||]$.

Now if $x_{\pi(pqr)}$ be a rearrangement of $x = (x_{pqr})$ defined as

$$x_{\pi(pqr)} = \begin{cases} 
1 & \text{for } p, q, r \text{ even } \in K \\
0 & \text{otherwise}
\end{cases}$$

Then $\{x_{\pi(p,q,r)}\} \notin c_0^3[F, ||\cdot||, \ldots, ||\cdot||]$ as $x_{\pi(pqr)} = 1$

Hence the sequence spaces $c_0^3[F, ||\cdot||, \ldots, ||\cdot||]$ is not symmetric in general.

The space $c_3[F, ||\cdot||, \ldots, ||\cdot||]$ is not symmetric in general can be proved in the same fashion. □
Theorem 3.7. Let $F = (f_{pqr})$ and $G = (g_{pqr})$ be two sequences of modulus functions. Then

$$\mathcal{T}^3[F, ||\cdot,\cdot,\cdot||] \cap \mathcal{T}^3[G, ||\cdot,\cdot,\cdot||] \subseteq \mathcal{T}^3[F + G, ||\cdot,\cdot,\cdot||]$$

where $\mathcal{T} = c, c_0$, or $\ell_\infty$

Proof. We prove the result for $\mathcal{T} = \ell_\infty$. Let $x = (x_{ik}) \in \ell_\infty^3[F, ||\cdot,\cdot,\cdot||] \cap \ell_\infty^3[G, ||\cdot,\cdot,\cdot||]$.

Then for $z_1, \ldots, z_{n-1} \in X$ we have

$$\left\{ (p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sup_{pqr \geq 1} \{ f_{pqr}(\|x_{pqr}, z_1, \ldots, z_{n-1}\|) \} \geq K_1 \right\} \in I \text{ for some } K_1 > 0$$

and

$$\left\{ (p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sup_{pqr \geq 1} \{ g_{pqr}(\|x_{pqr}, z_1, \ldots, z_{n-1}\|) \} \geq K_2 \right\} \in I \text{ for some } K_2 > 0$$

Now since

$$\sup_{pqr \geq 1} \{ (f_{pqr} + g_{pqr})(\|x_{pqr}, z_1, \ldots, z_{n-1}\|) \} = \sup_{pqr \geq 1} \left\{ f_{pqr}(\|x_{pqr}, z_1, \ldots, z_{n-1}\|) + g_{pqr}(\|x_{pqr}, z_1, \ldots, z_{n-1}\|) \right\} \leq \sup_{pqr \geq 1} \left\{ f_{pqr}(\|x_{pqr}, z_1, \ldots, z_{n-1}\|) \right\} + \sup_{pqr \geq 1} \left\{ g_{pqr}(\|x_{pqr}, z_1, \ldots, z_{n-1}\|) \right\}$$

Hence for $K = \max\{K_1, K_2\}$ we have

$$\left\{ (p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \sup_{pqr \geq 1} \{ (f_{pqr} + g_{pqr})(\|x_{pqr}, z_1, \ldots, z_{n-1}\|) \} \geq K \right\} \in I$$

Therefore $x \in \ell_\infty^3[F + G, ||\cdot,\cdot,\cdot||]$.

Hence

$$\ell_\infty^3[F, ||\cdot,\cdot,\cdot||] \cap \ell_\infty^3[G, ||\cdot,\cdot,\cdot||] \subseteq \ell_\infty^3[F + G, ||\cdot,\cdot,\cdot||]$$

In the same way the inclusion for $\mathcal{T} = c, c_0$ can be proved. \qed

References


