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I-CONVERGENT TRIPLE SEQUENCE SPACES OVER n-NORMED SPACE

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Abstract. The concept of 2-normed spaces was initially developed by Gähler [10], which was extend to n-norm by Misiak [17] for single sequence space. The main objective of this paper is to study triple sequence spaces over n-norm via the sequence of modulus functions.
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1. Introduction

A triple sequence (real or complex) is a function $x : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ are the set of natural numbers, real numbers, and complex numbers respectively. We denote by ω''' the class of all complex triple sequence (x_{pqr}), where $p, q, r \in \mathbb{N}$. Then under the coordinate wise addition and scalar multiplication ω''' is a linear space. A triple sequence can be represented by a matrix, in case of double sequences we write in the form of a square. In case of triple sequence it will be in the form of a box in three dimensions.

The different types of notions of triple sequences and their statistical convergence were introduced and investigated initially by Sahiner et. al [22]. Later Debnath et.al [2, 3], Esi et.al [4, 5, 6], Tripathy [24] and many others authors have studied it further and obtained various results.

Statistical convergence was introduced by Fast [7] and later on it was studied by Fridy [8, 9] from the sequence space point of view and linked it with summability theory. The notion of statistical convergent double sequence was introduced by Mursaleen and Edely [18].

I-convergence is a generalization of the statistical convergence. Kostyrko et. al. [15] introduced the notion of I-convergence of real sequence and studied its several properties. Later Jalal [11, 12, 13], Salat et. al. [20] and many other researchers contributed in its study. Sahiner and Tripathy [22] studied I-related properties in triple sequence spaces and showed some interesting results. Tripathy [24] extended the concept of I-convergent to double sequence and later Kumar [16] obtained some results on I-convergent double sequence. Recently Jalal and Malik [14] extended the concept of n-norms to triple sequence spaces and proved several algebraic and topological properties.

In this paper we define the spaces $c^3[\mathcal{F}, \|\cdot, \ldots, \cdot\|]^I$, $c_0^3[\mathcal{F}, \|\cdot, \ldots, \cdot\|]^I$, $\ell_\infty^3[\mathcal{F}, \|\cdot, \ldots, \cdot\|]^I$, $M_I^3[\mathcal{F}, \|\cdot, \ldots, \cdot\|]^I$ and $M_{0I}^3[\mathcal{F}, \|\cdot, \ldots, \cdot\|]^I$ by using the concept of n-normed space via the sequence of modulii functions $F = (f_{pqr})$. We study some algebraic and topological properties of these sequence spaces and some inclusion relations are obtained.

2. Definitions and preliminaries

Definition 2.1. *Let $X \neq \phi$. A class $I \subset 2^X$ (Power set of X) is said to be an ideal in X if the following conditions holds good:*

(i) *I is additive that is if $A, B \in I$ then $A \cup B \in I$;*
(ii) *I is hereditary that is if $A \in I$, and $B \subset A$ then $B \in I$.*

I is called non-trivial ideal if $X \notin I$

Definition 2.2. [21, 22] *A triple sequence (x_{pqr}) is said to be convergent to L in Pringsheim's sense if for every $\epsilon > 0$, there exists $\mathbf{N} \in \mathbb{N}$ such that*

$$|x_{pqr} - L| < \epsilon \quad \textit{whenever} \quad p \geq \mathbf{N}, q \geq \mathbf{N}, r \geq \mathbf{N}$$

and write as $\lim_{p,p,r\to\infty} x_{pqr} = L$.

Note: A triple sequence is convergent in Pringsheim's sense may not be bounded [21, 22].
Example Consider the sequence (x_{pqr}) defined by

$$x_{pqr} = \begin{cases} p+q & \text{for all } p = q \text{ and } r = 1 \\ \frac{1}{p^2qr} & \text{otherwise} \end{cases}$$

Then $x_{pqr} \to 0$ in Pringsheim's sense but is unbounded.

Definition 2.3. *A triple sequence (x_{pqr}) is said to be I-convergence to a number L if for every $\epsilon > 0$,*

$$\{(p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{pqr} - L| \geq \epsilon\} \in I.$$

In this case we write $I - \lim x_{pqr} = L$.

Definition 2.4. *A triple sequence (x_{pqr}) is said to be I-null if $L = 0$. In this case we write* $I - \lim x_{pqr} = 0$.

Definition 2.5. [21, 22] *A triple sequence (x_{pqr}) is said to be Cauchy sequence if for every $\epsilon > 0$, there exists $\mathbf{N} \in \mathbb{N}$ such that*

$$|x_{pqr} - x_{lmn}| < \epsilon \;\; \textit{whenever} \;\; p \geq l \geq \mathbf{N}, q \geq m \geq \mathbf{N}, r \geq n \geq \mathbf{N}$$

Definition 2.6. *A triple sequence (x_{pqr}) is said to be $I-$Cauchy sequence if for every $\epsilon > 0$, there exists $\mathbf{N} \in \mathbb{N}$ such that*

$$\{(p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{pqr} - a_{lmn}| \geq \epsilon\} \in I$$

whenever $p \geq l \geq \mathbf{N}, q \geq m \geq \mathbf{N}, r \geq n \geq \mathbf{N}$

Definition 2.7. [21, 22] *A triple sequence (x_{pqr}) is said to be bounded if there exists $M > 0$, such that $|x_{pqr}| < M$ for all $p, q, r \in \mathbb{N}$.*

Definition 2.8. *A triple sequence (x_{pqr}) is said to be $I-$bounded if there exists $M > 0$, such that $\{(p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{pqr}| \geq M\} \in I$ for all $p, q, r \in \mathbb{N}$.*

Definition 2.9. *A triple sequence space E is said to be solid if $(\alpha_{pqr} x_{pqr}) \in E$ whenever $(x_{pqr}) \in E$ and for all sequences (α_{pqr}) of scalars with $|\alpha_{pqr}| \leq 1$, for all $p, q, r \in \mathbb{N}$.*

Definition 2.10. *Let E be a triple sequence space and $x = (x_{pqr}) \in E$. Define the set $S(x)$ as*

$$S(x) = \left\{ \left(x_{\pi(pqr)}\right) : \pi \textit{ is a permutations of } \mathbb{N} \right\}$$

If $S(x) \subseteq E$ for all $x \in E$, then E is said to be symmetric.

Definition 2.11. *A triple sequence space E is said to be convergence free if $(y_{pqr}) \in E$ whenever $(x_{pqr}) \in E$ and $x_{pqr} = 0$ implies $y_{pqr} = 0$ for all $p, q, r \in \mathbb{N}$.*

Definition 2.12. *A triple sequence space E is said to be sequence algebra if $x \cdot y \in E$, whenever $x = (x_{pqr}) \in E$ and $y = (y_{pqr}) \in E$, that is product of any two sequences is also in the space.*

Definition 2.13. *(n-**Normed Space**) Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{R}$ of reals of dimension d, where $2 \leq d \leq n$. A real valued function $\| \cdot, ..., \cdot \|$ on X^n satisfying the following four conditions:*

(1) $\|x_1, x_2, ..., x_n\| = 0$ *if and only if* $x_1, x_2, ..., x_n$ *are linearly dependent in* X*;*
(2) $\|x_1, x_2, ..., x_n\|$ *is invariant under permutation;*
(3) $\|\alpha x_1, x_2, ..., x_n\| = |\alpha| \|x_1, x_2, ..., x_n\|$ *for any* $\alpha \in \mathbb{R}$*;*

(4) $\|x_1 + x_1', x_2, ..., x_n\| \leq \|x_1, x_2, ..., x_n\| + \|x_1', x_2, ..., x_n\|$;

is called an n-norm on X and $(X, \|\cdot, ..., \cdot\|)$ is called an $n-$normed space over the field $\mathbb{R}$. For example $(\mathbb{R}^n, \|\cdot, ..., \cdot\|_E)$ where

$\|x_1, x_2, ..., x_n\|_E =$ *the volume of the n-dimensional parallelopiped spanned by the vectors* $x_1, x_2, ..., x_n$

Which can also be written as

$$\|x_1, x_2, ..., x_n\|_E = |\det(x_{ij})|$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, \|\cdot, ..., \cdot\|)$ be an $n-$normed space of dimension $2 \leq n \leq d$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $\|\cdot, ..., \cdot\|_\infty$ on X^{n-1} defined by

$$\|x_1, x_2, ..., x_{n-1}\|_\infty = \max\{\|x_1, x_2, ..., x_{n-1}, a_i\| : i = 1, 2, ..., n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, ..., a_n\}$.
The standard n-norm on X, a real inner product space of dimension $d \leq n$ is as follows:

$$\|x_1, x_2, \cdots, x_n\|_S = \begin{vmatrix} \langle x_1, x_1\rangle & \cdot & \cdot & \cdot & \langle x_1, x_n\rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle x_n, x_1\rangle & \cdot & \cdot & \cdot & \langle x_n, x_n\rangle \end{vmatrix}^{\frac{1}{2}}$$

where $\langle\cdot, \cdot\rangle$ denotes the inner product on X. For $n = 1$ this n-norm is the usual norm $\|x\| = \langle x_1, x_1\rangle^{\frac{1}{2}}$.
A sequence (x_k) in a n-normed space $(X, \|\cdot, ..., \cdot\|)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} \|x_k - L, z_1, ..., z_{n-1}\| = 0 \text{ for every } z_1, ..., z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, \|\cdot, ..., \cdot\|)$ is said to be Cauchy if

$$\lim_{k,p\to\infty} \|x_k - x_p, z_1, ..., z_{n-1}\| = 0 \text{ for every } z_1, ..., z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-complete n-normed space is said to be n-Banach space. The n-normed space has been studied in stretch [6, 19, 23].

Definition 2.14. ***(Modulus Function)*** *A function $f : [0, \infty) \to [0, \infty)$ is called a modulus function if it satisfies the following conditions*

(i) $f(x) = 0$ *if and only if* $x = 0$.
(ii) $f(x + y) \leq f(x) + f(y)$ *for all* $x \geq 0$ *and* $y \geq 0$.
(iii) f *is increasing.*
(iv) f *is continuous from the right at 0.*

Since $|f(x)-f(y)| \leq f(|x-y|)$, *it follows from condition (iv) that* f *is continuous on* $[0,\infty)$. *Furthermore, from condition (2) we have* $f(nx) \leq nf(x)$, *for all* $n \in \mathbb{N}$, *and so*
$f(x) = f\big(nx(\frac{1}{n})\big) \leq nf\big(\frac{x}{n}\big)$.
Hence $\frac{1}{n}f(x) \leq f(\frac{x}{n})$ *for all* $n \in \mathbb{N}$

Let I be an admissible ideal, $F = (f_{pqr})$ be a sequence of modulus functions and $(X, \|\cdot,\ldots,\cdot\|)$ be a n-normed space. By $\omega'''(n-X)$ we denote the space of all triple sequences defined over $(X, \|\cdot,\ldots,\cdot\|)$. In the present paper we define the following sequence spaces

$$c^3[F,\|\cdot,\ldots,\cdot\|]^I = \Big\{x = x_{pqr} \in \omega'''(n-X) : \forall\ \epsilon > 0,\ \text{the set } \Big\{(p,q,r) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : f_{pqr}\left(\|x_{pqr} - L, z_1, \cdots, z_{n-1}\|\right) \geq \epsilon,\ \text{for some } L \in \mathbb{C} \text{ and } z_1,\ldots,z_{n-1} \in X\Big\} \in I\Big\}$$

$$c_0^3[F,\|\cdot,\ldots,\cdot\|]^I = \Big\{x = x_{pqr} \in \omega'''(n-X) : \forall\ \epsilon > 0,\ \text{the set } \Big\{(p,q,r) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : f_{pqr}\left(\|x_{pqr}, z_1, \cdots, z_{n-1}\|\right) \geq \epsilon,\ z_1,\ldots,z_{n-1} \in X\Big\} \in I\Big\}$$

$$\ell_\infty^3[F,\|\cdot,\ldots,\cdot\|]^I = \Big\{x = x_{pqr} \in \omega'''(n-X) : \exists\ K > 0 \text{ such that } \Big\{(p,q,r) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \sup_{p,q,r\geq 1}\{f_{pqr}\left(\|x_{pqr}, z_1, \cdots, z_{n-1}\|\right)\} \geq K,\ z_1,\ldots,z_{n-1} \in X\Big\} \in I\Big\}$$

and

$$M^3[F,\|\cdot,\ldots,\cdot\|]^I = c^3[F,\|\cdot,\ldots,\cdot\|]^I \cap \ell_\infty^3[F,\|\cdot,\ldots,\cdot\|]^I$$
$$M_0^3[F,\|\cdot,\ldots,\cdot\|]^I = c_0^3[F,\|\cdot,\ldots,\cdot\|]^I \cap \ell_\infty^3[F,\|\cdot,\ldots,\cdot\|]^I$$

For $F(x) = x$ we have

$$c^3[\|\cdot,\ldots,\cdot\|]^I = \Big\{x = x_{pqr} \in \omega'''(n-X) : \forall\ \epsilon > 0,\ \text{the set } \Big\{(p,q,r) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \|x_{pqr} - L, z_1, \cdots, z_{n-1}\| \geq \epsilon,\ \text{for some } L \in \mathbb{C} \text{ and } z_1,\ldots,z_{n-1} \in X\Big\} \in I\Big\}$$

$$c_0^3[\|\cdot,\ldots,\cdot\|]^I = \Big\{x = x_{pqr} \in \omega'''(n-X) : \forall\ \epsilon > 0,\ \text{the set } \Big\{(p,q,r) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \|x_{pqr}, z_1, \cdots, z_{n-1}\| \geq \epsilon,\ z_1,\ldots,z_{n-1} \in X\Big\} \in I\Big\}$$

$$\ell_{\infty I}^3[\|\cdot,\ldots,\cdot\|]^I = \Big\{x = x_{pqr} \in \omega'''(n-X) : \exists\ K > 0 \text{ such that } \Big\{(p,q,r) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \sup_{p,q,r\geq 1}\left(\|x_{pqr}, z_1, \cdots, z_{n-1}\|\right) \geq K,\ z_1,\ldots,z_{n-1} \in X\Big\} \in I\Big\}$$

and

$$M^3[\|\cdot,\ldots,\cdot\|]^I = c^3[\|\cdot,\ldots,\cdot\|]^I \cap \ell^3_\infty[\|\cdot,\ldots,\cdot\|]^I$$
$$M^3_0[\|\cdot,\ldots,\cdot\|]^I = c^3_0[\|\cdot,\ldots,\cdot\|]^I \cap \ell^3_\infty[\|\cdot,\ldots,\cdot\|]^I$$

3. Algebraic and Topological Properties of the new Sequence spaces

Theorem 3.1. *Let $F = (f_{pqr})$ be a sequence of modulus functions then the triple sequence spaces $c^3_0[F,\|\cdot,\ldots,\cdot\|]^I$, $c^3[F,\|\cdot,\ldots,\cdot\|]^I$, $\ell^3_\infty[F,\|\cdot,\ldots,\cdot\|]^I$, $M^3[F,\|\cdot,\ldots,\cdot\|]^I$ and $M^3_0[F,\|\cdot,\ldots,\cdot\|]^I$ all linear over the field $\mathbb{C}$ of complex numbers.*

Proof. We prove the result for the sequence space $c^3[F,\|\cdot,\ldots,\cdot\|]^I$.
Let $x = (x_{pqr}), y = (y_{pqr}) \in c^3[F,\|\cdot,\ldots,\cdot\|]^I$ and $\alpha, \beta \in \mathbb{C}$, then there exist positive integers m_α and n_β such that $|\alpha| \le m_\alpha$ and $|\beta| \le n_\beta$, then for $z_1, z_2, \ldots, z_{n-1} \in X$

$$I - \lim f_{pqr}(\|x_{pqr} - L_1, z_1, \ldots, z_{n-1}\|) = 0, \text{ for some } L_1 \in \mathbb{C}$$
$$I - \lim f_{pqr}(\|x_{pqr} - L_2, z_1, \ldots, z_{n-1}\|) = 0, \text{ for some } L_2 \in \mathbb{C}$$

Now for a given $\epsilon > 0$ we set

$$C_1 = \left\{(p,q,r) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : f_{pqr}(\|x_{pqr} - L_1, z_1, \ldots, z_{n-1}\|) > \frac{\epsilon}{2}\right\} \in I \qquad (2.1)$$
$$C_2 = \left\{(p,q,r) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : f_{pqr}(\|y_{pqr} - L_2, z_1, \ldots, z_{n-1}\|) > \frac{\epsilon}{2}\right\} \in I \qquad (2.2)$$

Since f_{pqr} is a modulus function, so it is non-decreasing and convex, hence we get

$$\begin{aligned} f_{pqr}(\|(\alpha x_{pqr} + \beta y_{pqr}) - (\alpha L_1 + \beta L_2), z_1, \ldots, z_{n-1}\|) \\ &= f_{pqr}(\|(\alpha x_{pqr} - \alpha L_1) + (\beta y_{pqr} - \beta L_2), z_1, \ldots, z_{n-1}\|) \\ &\le f_{pqr}(|\alpha|\|x_{pqr} - L_1, z_1, \ldots, z_{n-1}\|) + f_{pqr}(|\beta|\|y_{pqr} - L_2, z_1, \ldots, z_{n-1}\|) \\ &= |\alpha| f_{pqr}(|x_{pqr} - L_1|) + |\beta| f_{pqr}(|y_{pqr} - L_2|) \\ &\le m_\alpha f_{pqr}(\|x_{pqr} - L_1, z_1, \ldots, z_{n-1}\|) + n_\beta f_{pqr}(\|y_{pqr} - L_2, z_1, \ldots, z_{n-1}\|) \end{aligned}$$

From (2.1) and (2.2) we can write

$$\{(p,q,r) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : f_{pqr}(\|(\alpha x_{pqr} + \beta y_{pqr}) - (\alpha L_1 + \beta L_2), z_1, \ldots, z_{n-1}\|) > \epsilon\} \subseteq C_1 \cup C_2$$

Thus $\alpha x + \beta y \in c^3[F,\|\cdot,\ldots,\cdot\|]^I$.
Therefore $c^3[F,\|\cdot,\ldots,\cdot\|]^I$ is a linear space.
In the same way we can show that other spaces are linear as well. □

Theorem 3.2. *Let $F = (f_{pqr})$ be a sequence of modulus functions then the inclusions $c_0^3[F, \|\cdot, \ldots, \cdot\|]^I \subset c^3[F, \|\cdot, \ldots, \cdot\|]^I \subset \ell_\infty^3[F, \|\cdot, \ldots, \cdot\|]^I$ holds .*

Proof. The inclusion $c_0^3[F, \|\cdot, \ldots, \cdot\|]^I \subset c^3[F, \|\cdot, \ldots, \cdot\|]^I$ is obvious.
We prove $c^3[F, \|\cdot, \ldots, \cdot\|]^I \subset \ell_\infty^3[F, \|\cdot, \ldots, \cdot\|]^I$.
Let $x = (x_{pqr}) \in c^3[F, \|\cdot, \ldots, \cdot\|]^I$ then there exists $L \in \mathbb{C}$ such that
$I - \lim f_{pqr}(\|x_{pqr} - L, z_1, \ldots, z_{n-1}\|) = 0$, $z_1, \ldots, z_{n-1} \in X$
Since $F = (f_{pqr})$ is a sequence of modulus functions so

$$f_{pqr}(\|x_{pqr}, z_1, \ldots, z_{n-1}\|) \leq f_{pqr}(\|x_{pqr} - L, z_1, \ldots, z_{n-1}\|) + f_{pqr}(\|L, z_1, \ldots, z_{n-1}\|)$$

On taking supremum over p, q and r on both sides gives $x = (x_{pqr}) \in \ell_\infty^3[F, \|\cdot, \ldots, \cdot\|]^I$
Hence the inclusion $c_0^3[F, \|\cdot, \ldots, \cdot\|]^I \subset c^3[F, \|\cdot, \ldots, \cdot\|]^I$
$\subset \ell_\infty^3[F, \|\cdot, \ldots, \cdot\|]^I$ holds. □

Theorem 3.3. *The triple sequence $c_0^3[F, \|\cdot, \ldots, \cdot\|]^I$ and $M_0^3[F, \|\cdot, \ldots, \cdot\|]^I$ are solid.*

Proof. We prove the result for $c_0^3[F, \|\cdot, \ldots, \cdot\|]^I$.
Consider $x = (x_{pqr}) \in c_0^3[F, \|\cdot, \ldots, \cdot\|]^I$,
then $I - \lim_{p,q,r} f_{pqr}(\|x_{pqr}, z_1, \ldots, z_{n-1}\|) = 0$
Consider a sequence of scalar (α_{pqr}) such that $|\alpha_{pqr}| \leq 1$ for all $p, q, r \in \mathbb{N}$.
Then we have

$$\begin{aligned} I - \lim_{p,q,r} f_{pqr}(|\alpha_{pqr}(x_{pqr}), z_1, \ldots, z_{n-1}\|) &\leq I - |\alpha_{pqr}| \lim_{p,q,r} f_{pqr}(\|x_{pqr}, z_1, \ldots, z_{n-1}\|) \\ &\leq I - \lim_{p,q,r} f_{pqr}(\|x_{pqr}, z_1, \ldots, z_{n-1}\|) \\ &= 0 \end{aligned}$$

Hence $I - \lim_{p,q,r} f_{pqr}(\|\alpha_{pqr} x_{pqr}, z_1, \ldots, z_{n-1}\|) = 0$ for all $p, q, r \in \mathbb{N}$
Which gives $(\alpha_{pqr} x_{pqr}) \in c_0^3[F, \|\cdot, \ldots, \cdot\|]^I$
Hence the sequence space $c_0^3[F, \|\cdot, \ldots, \cdot\|]^I$ is solid.
The result for $M_0^3[F, \|\cdot, \ldots, \cdot\|]^I$ can be similarly proved. □

Theorem 3.4. *The triple sequence spaces $c_0^3[F, \|\cdot, \ldots, \cdot\|]^I$, $c^3[F, \|\cdot, \ldots, \cdot\|]^I$, $\ell_\infty^3[F, \|\cdot, \ldots, \cdot\|]^I$, $M^3[F, \|\cdot, \ldots, \cdot\|]^I$ and $M_0^3[F, \|\cdot, \ldots, \cdot\|]^I$ are sequence algebras.*

Proof. We prove the result for $c_0^3[F, \|\cdot, \ldots, \cdot\|]^I$.
Let $x = (x_{pqr}), y = (y_{pqr}) \in c_0^3[F, \|\cdot, \ldots, \cdot\|]^I$
Then we have $I - \lim f_{pqr}(\|x_{pqr}, z_1, \ldots, z_{n-1}\|) = 0$ and $I - \lim f_{pqr}(\|x_{pqr}, z_1, \ldots, z_{n-1}\|) = 0$
Using definition modulus functions we have $I - \lim f_{pqr}(\|(x_{pqr} \cdot y_{pqr}), z_1, \ldots, z_{n-1}\|) = 0$. It

implies that $x \cdot y \in c_0^3[F, \|\cdot, \ldots, \cdot\|]^I$

The result can be proved for the spaces $c^3[F, \|\cdot, \ldots, \cdot\|]^I$, $\ell_\infty^3[F, \|\cdot, \ldots, \cdot\|]^I$, $M^3[F, \|\cdot, \ldots, \cdot\|]^I$ and $M_0^3[F, \|\cdot, \ldots, \cdot\|]^I$ in the same way. □

Theorem 3.5. *In general the sequence spaces*$c_0^3[F, \|\cdot, \ldots, \cdot\|]^I$, $c^3[F, \|\cdot, \ldots, \cdot\|]^I$ *and* $\ell_\infty^3[F, \|\cdot, \ldots, \cdot\|]^I$ *are not convergence free.*

Proof. We prove the result for the sequence space $c_I^3[F, \|\cdot, \ldots, \cdot\|]^I$ using an example.

Example: Let $I = I_f$ define the triple sequence $x = (x_{pqr})$ as

$$x_{pqr} = \begin{cases} 0 & \text{if } p = q = r \\ 1 & \text{otherwise} \end{cases}$$

Then if $f_{pqr}(x) = x_{pqr}$ $\forall$ $p, q, r \in \mathbb{N}$, we have $x = (x_{pqr}) \in c^3[F, \|\cdot, \ldots, \cdot\|]^I$.

Now define the sequence $y = y_{pqr}$ as

$$y_{pqr} = \begin{cases} 0 & \text{if } r \text{ is odd , and } \; p, q \in \mathbb{N} \\ lmn & \text{otherwise} \end{cases}$$

Then for $f_{pqr}(x) = x_{pqr}$ $\forall$ $p, q, r \in \mathbb{N}$, it is clear that $y = (y_{pqr}) \notin c^3[F, \|\cdot, \ldots, \cdot\|]^I$

Hence the sequence spaces $c^3[F, \|\cdot, \ldots, \cdot\|]^I$ is not convergence free.

The space $c^3[F, \|\cdot, \ldots, \cdot\|]^I$ and $\ell_\infty^3[F, \|\cdot, \ldots, \cdot\|]^I$ are not convergence free in general can be proved in the same fashion. □

Theorem 3.6. *In general the triple sequences* $c_0^3[F, \|\cdot, \ldots, \cdot\|]^I$ *and* $c^3[F, \|\cdot, \ldots, \cdot\|]^I$ *are not symmetric if* I *is neither maximal nor* $I = I_f$.

Proof. We prove the result for the sequence space $c_0^3[F, \|\cdot, \ldots, \cdot\|]^I$ using an example.

Example: Define the triple sequence $x = (x_{pqr})$ as

$$x_{pqr} = \begin{cases} 0 & \text{if } r = 1, \;\; \text{for all } p, q \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases}$$

Then if $f_{pqr}(x) = x_{pqr}$ $\forall$ $p, q, r \in \mathbb{N}$, we have $x = (x_{pqr}) \in c_0^3[F, \|\cdot, \ldots, \cdot\|]^I$.

Now if $x_{\pi(pqr)}$ be a rearrangement of $x = (x_{pqr})$ defined as

$$x_{\pi(pqr)} = \begin{cases} 1 & \text{for } \; p, q, r \text{ even} \in K \\ 0 & \text{otherwise} \end{cases}$$

Then $\{x_{\pi(p,q,r)}\} \notin c_0^3[F, \|\cdot, \ldots, \cdot\|]^I$ as $x_{\pi(pqr)} = 1$

Hence the sequence spaces $c_0^3[F, \|\cdot, \ldots, \cdot\|]^I$ is not symmetric in general.

The space $c^3[F, \|\cdot, \ldots, \cdot\|]^I$ is not symmetric in general can be proved in the same fashion. □

Theorem 3.7. *Let $F = (f_{pqr})$ and $G = (g_{pqr})$ be two sequences of modulus functions. Then*

$$\mathcal{T}^3[F, \|\cdot, \ldots, \cdot\|]^I \cap \mathcal{T}^3[G, \|\cdot, \ldots, \cdot\|]^I \subseteq \mathcal{T}^3[F+G, \|\cdot, \ldots, \cdot\|]^I$$

where $\mathcal{T} = c, c_0,$ or ℓ_∞

Proof. We prove the result for $\mathcal{T} = \ell_\infty$. Let $x = (x_{ijk}) \in \ell^3_\infty[F, \|\cdot, \ldots, \cdot\|]^I \cap \ell^3_\infty[G, \|\cdot, \ldots, \cdot\|]^I$. Then for $z_1, \ldots, z_{n-1} \in X$ we have

$$\left\{(p,q,r) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \sup_{p,q,r\geq 1} \{f_{pqr}(\|x_{pqr}, z_1, \cdots, z_{n-1}\|)\} \geq K_1\right\} \in I \quad \text{for some } K_1 > 0$$

and

$$\left\{(p,q,r) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \sup_{p,q,r\geq 1} \{g_{pqr}(\|x_{pqr}, z_1, \cdots, z_{n-1}\|)\} \geq K_2\right\} \in I \quad \text{for some } K_2 > 0$$

Now since

$$\sup_{p,q,r\geq 1}\left\{(f_{pqr}+g_{pqr})(\|x_{pqr}, z_1, \cdots, z_{n-1}\|)\right\} = \sup_{p,q,r\geq 1}\left\{f_{pqr}(\|x_{pqr}, z_1, \cdots, z_{n-1}\|) + g_{pqr}(\|x_{pqr}, z_1, \cdots, z_{n-1}\|)\right\}$$
$$\leq \sup_{p,q,r\geq 1}\left\{f_{pqr}(\|x_{pqr}, z_1, \cdots, z_{n-1}\|)\right\} + \sup_{p,q,r\geq 1}\left\{g_{pqr}(\|x_{pqr}, z_1, \cdots, z_{n-1}\|)\right\}$$

Hence for $K = \max\{K_1, K_2\}$ we have

$$\left\{(p,q,r) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \sup_{p,q,r\geq 1}\left\{(f_{pqr}+g_{pqr})(\|x_{pqr}, z_1, \cdots, z_{n-1}\|)\right\} \geq K\right\} \in I$$

Therefore $x \in \ell^3_\infty[F+G, \|\cdot, \ldots, \cdot\|]^I$.
Hence

$$\ell^3_\infty[F, \|\cdot, \ldots, \cdot\|]^I \cap \ell^3_\infty[G, \|\cdot, \ldots, \cdot\|]^I \subseteq \ell^3_\infty[F+G, \|\cdot, \ldots, \cdot\|]^I$$

In the same way the inclusion for $\mathcal{T} = c, c_0$ can be proved. □

References

[1] A. Alotaibi, M. Mursaleen and S. K. Sharma, *Double sequence spaces over n-normed spaces defined by a sequence of orlicz functions*, J. Inequal. Appl., **2014** (2014), Article ID 216.

[2] S. Debnath, B. Sarma, B.C. Das, *Some generalized triple sequence spaces of real numbers* J. Nonlinear Anal. Optim. **6** (2015), 71–79.

[3] S. Debnath and N. Subramanian, *Generalized rough lacunary statistical triple difference sequence spaces in probability of fractional order defined by musielak-orlicz function* Bol. Soc. Paran. Mat. (3s) Vol. **37**, (2019), 5562.

[4] A. Esi, *On some triple almost lacunary sequence spaces defined by orlicz functions,* Res. Rev., Discr. Math. Struct. 1 (2014), 16–25.

[5] A. Esi and M.N. Catalbas, *Almost convergence of triple sequences* G. J. Math. Anal. **2**, (2014), 6–10.

[6] A. Esi and E. Savas *On lacunary statically convergent triple sequences in probabilistic normed space* Appl Mathand Inf Sci **9**, (2015), 2529–2534.

[7] H. Fast, *Surla convergence statistique*, Colloq. Math., **2**, (1951), 241–244.

[8] J.A. Fridy, *On statistical convergence*, Analysis, **5**, (1985), 301–313.

[9] J.A. Fridy, *Statistical limit points*, Proc. Amer. Math. Soc., **11**, (1993), 1187–1192.

[10] S. Gähler, *Linear 2-normietre Rume*, Math. Nachr., 28 (1965), 1–43.

[11] T. Jalal, *Some new I-convergent sequence spaces defined by using a sequence of modulus functions in n-normed spaces*, Int. J. Math. Archive, **5**(9) (2014), 202–209.

[12] T. Jalal, *Some new I-lacunary generalized difference sequence spaces defined in n-normed spaces*, Springer Proc. Math. Sat.,**171** (2016), 249–258.

[13] T. Jalal, *Some new lacunary sequence spaces of invariant means defined by musielak-Orlicz functions on n-normed space*, Int. J. P. Appl. Math., **119**(7) (2018), 1–11.

[14] T. Jalal and I.A. Malik, *Some new triple sequence spaces over n-normed space*, Proyecciones, (2018), (to appear).

[15] P. Kostyrko, T. Salat, W. Wilczynski, *I-convergence*, Real Anal. Exch. **26**(2) (2000), 669–686.

[16] V. Kumar, *On I-convergence of double sequences*, Math. Commun.,**12** (2007), 171–181.

[17] A. Misiak, *$n-$Inner product spaces*, Math. Nachr., **140** (1989), 299–319.

[18] M. Mursaleen and O. H. H. Edely, *Statistical convergence of double sequences*, J. Math. Anal. Appl., **288** (2003), 223–231.

[19] M. Mursaleen, K. Raj and S. K. Sharma, *Some spaces of differences and lacunary statistical convergence in n-normed space defined by sequence of orlicz functions*, Miskolc Math. Notes, **16**(1) (2015), 283–304.

[20] T. Salat, B. C. Tripathy and M. Ziman, *On some properties of I-convergence*, Tatra Mountain Math. Publ., (2000) 669–686.

[21] A. Sahiner, M. Gurdal and F.K. Duden, *Triple Sequences and their statistical convergence*, Selcuk J. Appl Math **8**, (2007), 49–55.

[22] A. Sahiner and B.C. Tripathy, *Some I related properties of triple sequences*, Selcuk J. Appl. Math. **9** (2008), 9–18.

[23] S.K. Sharma and Ayhan Esi, *Some I-convergent sequence spaces defined by using sequence of moduli and n-normed space*, J. Egyp. Math. Soc. **21**(2) (2013), 103–107.

[24] B. C. Tripathy, *Statistically convergent double sequence*, Tamkang. J. Math., **34**(3) (2003), 231–237.