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STABILITY ANALYSIS OF LINEAR θ-METHOD FOR COUPLED DIFFERENTIAL EQUATION WITH ONE DELAY

QI WANG*, QIAN YANG

School of Applied Mathematics, Guangdong University of Technology, Guangzhou, 510006, P.R. China

Corresponding author: bmwzwq@126.com

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ABSTRACT. The purpose of this paper is to study stability of numerical solutions for a coupled differential equation with piecewise constant arguments with one delay. A sufficient condition such that the equation is asymptotically stable is derived. Moreover, when the linear θ-method is applied to this equation, it is proved that the linear θ-method is asymptotically stable if and only if $1/2 < \theta \leq 1$. Some numerical examples to confirm the theoretical results are given.
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Key words and phrases. piecewise constant arguments; coupled differential equation; linear θ-method; numerical solution; Stability.

1. INTRODUCTION

In present paper, we consider the following three-dimensional coupled differential equation with piecewise constant arguments (abbreviated as EPCA)

$$\begin{aligned} x'(t) &= ax(t) + by([t]), \\ y'(t) &= cy(t) + dz([t]), \\ z'(t) &= ez(t) + fx([t]), \\ x(0) &= x_0, y(0) = y_0, z(0) = z_0, \end{aligned} \tag{1.1}$$

where $a, b, c, d, e, f \in \mathbb{R}$, $x_0, y_0, z_0 \in \mathbb{R}$ are given initial values and $[\cdot]$ denotes the greatest integer function. (1.1) can be rewritten as

$$\begin{aligned} X'(t) &= AX(t) + BX([t]), \quad t \geq 0, \\ X(0) &= X_0, \end{aligned} \tag{1.2}$$

where $X(t) = (x(t), y(t), z(t))^T$, $X_0 = (x_0, y_0, z_0)^T$ and

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & e \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & d \\ f & 0 & 0 \end{pmatrix}.$$

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The general form of this kind of equation is

$$X'(t) = f(t, X(t), X(\alpha(t))), \quad t \geq 0, \\ X(0) = X_0, \tag{1.3}$$

where the argument $\alpha(t)$ has intervals of constancy. In the past three decades, the theory of EPCA has been intensively studied. Some properties such as stability [1], oscillation [2,3], periodicity [4], bifurcation [5] and asymptotic behavior [6] are included. EPCA has been widely found in scientific and engineering applications such as biology, ecology, control science and so on. The general theory and basic results for EPCA have by now been thoroughly investigated in the book of Wiener [7]. It is known that the investigation on the numerical solution of EPCA has been conducted for many years. Liu et al. [8] studied the stability of the Runge-Kutta methods for EPCA $u'(t) = au(t) + a_0 u([t])$. In [9, 10], oscillations of numerical solution in θ-methods and Runge-Kutta methods for EPCA $x'(t) + ax(t) + a_1 x([t-1]) = 0$ were considered, respectively. Wen et al. [11] considered the dissipativity of analytic solution and numerical solution of a class of nonlinear EPCA. More results of numerical treatment for EPCA, see [12–14]. To the best of our knowledge, until now very few results dealing with the numerical solution of multi-dimensional EPCA have been reported except for [15]. Apart from [15], in this paper, we will consider the numerical stability of (1.1) in a more direct manner.

In next section, we will give the expression of analytic solution of (1.1) and a sufficient condition under which the analytic solution of (1.1) is asymptotically stable will be obtained.

2. ANALYTICAL STABILITY

Definition 1. *[7] A solution of (1.2) on $[0, \infty)$ is a function $X(t)$ satisfies the conditions:*

(i) *$X(t)$ is continuous on $[0, \infty)$;*
(ii) *The derivative $X'(t)$ exists at each point $t \in [0, \infty)$, with the possible exception of the points $[t] \in [0, \infty)$ where one-sided derivatives exist;*
(iii) *(1.1) is satisfied on each interval $[k, k+1) \subset [0, \infty)$ with integral end-points.*

Theorem 1. *[16] The analytic solution of (1.2) is*

$$X(t) = Q_0(\{t\}) B_0^{[t]} X_0,$$

where

$$Q_0(t) = e^{At} + (e^{At} - I)A^{-1}B, \quad B_0 = e^{A} + (e^{A} - I)A^{-1}B,$$

where $\{t\}$ is the fractional part of t.

Definition 2. *If any solution $X(t)$ of (1.2) satisfies*

$$\lim_{t \to \infty} X(t) = 0,$$

then the zero solution of (1.2) is called asymptotically stable.

Lemma 1. *[7] The zero solution of (1.2) is asymptotically stable if and only if $\rho(B_0) < 1$, where $\rho(B_0) = \max\limits_i(|\lambda_i|)$, λ_i, $i = 1, 2, 3$ are the eigenvalues of B_0.*

From [15], we can obtain the following result for three-dimensional case.

Theorem 2. *The zero solution of (1.2) is asymptotically stable if*

$$\begin{cases} \mu[A] < 0, \\ \|B\| < -\mu[A], \end{cases}$$

where $\|\cdot\|$ denotes the matrix norm induced by a vector norm on C^3 and $\mu[\cdot]$ denotes the logarithmic norm of the matrix, defined by

$$\mu[L] = \lim_{\Delta \to 0^+} \frac{\|I_3 + \Delta L\| - 1}{\Delta},$$

here I_3 is the 3×3 identity matrix.

Corollary 1. *Eq. (1.1) is asymptotically stable, if*

$$\max\{a, c, e\} < 0, \quad \max\{|b|, |d|, |f|\} < -\max\{a, c, e\}. \tag{2.1}$$

Proof: We know that (1.1) is asymptotically stable if and only if (1.2) is asymptotically stable. So we only need to study the stability of (1.2). From

$$\mu[L] = \lambda_{max}\left(\frac{L + L^*}{2}\right), \quad \|L\| = \sqrt{\rho(L^* L)},$$

we have

$$\mu[A] = \max\{a, c, e\}, \quad \|B\| = \max\{|b|, |d|, |f|\},$$

then by Theorem 2, (2.1) is obtained.

Remark 1. *From Corollary 1, the following inequality can be easily got*

$$\left|\frac{bdf}{ace}\right| < 1. \tag{2.2}$$

In fact, there are nine cases in (2.1):

(i) *If $|b| > |d|$, $|b| > |f|$, $0 > a > c$ and $0 > a > e$ then $|b| < -a$;*
(ii) *If $|b| > |d|$, $|b| > |f|$, $0 > c > a$ and $0 > c > e$ then $|b| < -c$;*
(iii) *If $|b| > |d|$, $|b| > |f|$, $0 > e > a$ and $0 > e > c$ then $|b| < -e$;*
(iv) *If $|d| > |b|$, $|d| > |f|$, $0 > a > c$ and $0 > a > e$ then $|d| < -a$;*
(v) *If $|d| > |b|$, $|d| > |f|$, $0 > c > a$ and $0 > c > e$ then $|d| < -c$;*
(vi) *If $|d| > |b|$, $|d| > |f|$, $0 > e > a$ and $0 > e > c$ then $|d| < -e$;*
(vii) *If $|f| > |b|$, $|f| > |d|$, $0 > a > c$ and $0 > a > e$ then $|f| < -a$;*
(viii) *If $|f| > |b|$, $|f| > |d|$, $0 > c > a$ and $0 > c > e$ then $|f| < -c$;*
(ix) *If $|f| > |b|$, $|f| > |d|$, $0 > e > a$ and $0 > e > c$ then $|f| < -e$,*

in a word, the coefficients a, b, c, d, e and f are all satisfy (2.2).

3. NUMERICAL STABILITY

Let $h = 1/m$ $(m \geq 1)$ be stepsize and the gridpoints $t_n = nh$ $(n = 1, 2, \cdots)$, application of the linear θ-method with $0 \leq \theta \leq 1$ to (1.1),

$$\begin{aligned} x_{n+1} &= x_n + h\{\theta(ax_{n+1} + by^h([(n+1)h])) + (1-\theta)(ax_n + by^h([nh]))\}, \\ y_{n+1} &= y_n + h\{\theta(cy_{n+1} + dz^h([(n+1)h])) + (1-\theta)(cy_n + dz^h([nh]))\}, \\ z_{n+1} &= z_n + h\{\theta(ez_{n+1} + fx^h([(n+1)h])) + (1-\theta)(ez_n + fx^h([nh]))\}, \end{aligned} \tag{3.1}$$

where $x^h([nh])$ and $x^h([(n+1)h])$ are approximations to $x([t])$ of (1.1) at t_n and t_{n+1}, respectively. In the same way, $y^h([nh])$ and $y^h([(n+1)h])$ are approximations to $y([t])$ of (1.1) at t_n and t_{n+1}, respectively, $z^h([nh])$ and $z^h([(n+1)h])$ are approximations to $z([t])$ of (1.1) at t_n and t_{n+1}, respectively. Let $n = km+l$ $(l = 0, 1, 2, \cdots, m-1)$, then by Definition 1, $x^h([t_n + \delta h])$, $y^h([t_n + \delta h])$ and $z^h([t_n + \delta h])$, $0 \leq \delta \leq 1$ can be defined as x_{km}, y_{km} and z_{km}, respectively. So (3.1) turns into

$$\begin{aligned} x_{n+1} &= x_n + h\{\theta(ax_{n+1} + by_{km}) + (1-\theta)(ax_n + by_{km})\}, \\ y_{n+1} &= y_n + h\{\theta(cy_{n+1} + dz_{km}) + (1-\theta)(cy_n + dz_{km})\}, \\ z_{n+1} &= z_n + h\{\theta(ez_{n+1} + fx_{km}) + (1-\theta)(ez_n + fx_{km})\}, \end{aligned} \tag{3.2}$$

hence

$$\begin{aligned} x_{n+1} &= \tfrac{1+h(1-\theta)a}{1-h\theta a} x_n + \tfrac{hb}{1-h\theta a} y_{km}, \\ y_{n+1} &= \tfrac{1+h(1-\theta)c}{1-h\theta c} y_n + \tfrac{hd}{1-h\theta c} z_{km}, \\ z_{n+1} &= \tfrac{1+h(1-\theta)e}{1-h\theta e} z_n + \tfrac{hf}{1-h\theta e} x_{km}. \end{aligned} \tag{3.3}$$

Denote

$$M_1 = \begin{pmatrix} \frac{1+h(1-\theta)a}{1-h\theta a} & 0 & 0 \\ 0 & \frac{1+h(1-\theta)c}{1-h\theta c} & 0 \\ 0 & 0 & \frac{1+h(1-\theta)e}{1-h\theta e} \end{pmatrix}, M_2 = \begin{pmatrix} 0 & \frac{hb}{1-h\theta a} & 0 \\ 0 & 0 & \frac{hd}{1-h\theta c} \\ \frac{hf}{1-h\theta e} & 0 & 0 \end{pmatrix},$$

from (3.3) we have

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = M_1 \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} + M_2 \begin{pmatrix} x_{km} \\ y_{km} \\ z_{km} \end{pmatrix}. \tag{3.4}$$

Introducing

$$W_n = (x_n, y_n, z_n, x_{n-1}, y_{n-1}, z_{n-1}, \cdots, x_{km+1}, y_{km+1}, z_{km+1}, x_{km}, , y_{km}, z_{km})^T,$$

then (3.4) reads

$$W_{n+1} = RW_n, \tag{3.5}$$

where

$$R = \begin{pmatrix} M_1 & 0 & \cdots & 0 & M_2 \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ & \cdots & & \cdots & \\ 0 & 0 & \cdots & I & 0 \end{pmatrix},$$

with

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, 0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The following theorem give us the stability of the linear θ-method.

Theorem 3. *The linear θ-method applied to (1.1) with condition (2.1) is asymptotically stable if and only if $1/2 < \theta \leq 1$.*

Proof: Assume that $1/2 < \theta \leq 1$, let

$$R(x_1) = \frac{1 + x_1(1-\theta)}{1 - x_1\theta}, \quad R(x_2) = \frac{1 + x_2(1-\theta)}{1 - x_2\theta}, \quad R(x_3) = \frac{1 + x_3(1-\theta)}{1 - x_3\theta},$$

where

$$x_1 = ha, \quad x_2 = hc, \quad x_3 = he.$$

After some derivations, the following characteristic polynomial of R can be derived

$$\begin{aligned} P(r) &= \det[Ir^{l+1} - M_1 r^l - M_2] \\ &= r^{3l}(r - R(x_1))(r - R(x_2))(r - R(x_3)) - \tfrac{bdf}{ace}(R(x_1) - 1)(R(x_2) - 1)(R(x_3) - 1) \\ &= (r - R(x_1))(r - R(x_2))(r - R(x_3)) \left[r^{3l} - \tfrac{bdf}{ace} \tfrac{(R(x_1)-1)(R(x_2)-1)(R(x_3)-1)}{(r-R(x_1))(r-R(x_2))(r-R(x_3))} \right]. \end{aligned} \tag{3.6}$$

Obviously, $P(r)$ has three zeros $R(x_1)$, $R(x_2)$ and $R(x_3)$ of order 1, respectively. According to the property of stability function of the linear θ-method and [17] we get $|R(x_1)| < 1$, $|R(x_2)| < 1$ and $|R(x_3)| < 1$ for $1/2 < \theta \leq 1$.

Set

$$s(r) = r^{3l}, \quad g(r) = -\frac{bdf}{ace} \frac{(R(x_1) - 1)(R(x_2) - 1)(R(x_3) - 1)}{(r - R(x_1))(r - R(x_2))(r - R(x_3))},$$

so for any m and $|z| = 1$, from (2.2) we obtain

$$|g(r)| = \left| -\frac{bdf}{ace} \frac{(R(x_1) - 1)(R(x_2) - 1)(R(x_3) - 1)}{(r - R(x_1))(r - R(x_2))(r - R(x_3))} \right| \leq \left| \frac{bdf}{ace} \right| < 1 = |r^{3l}| = |s(r)|.$$

By Rouché's theorem, we know that $s(r)$ and $s(r) + g(r)$ have the same number of zeros inside the unit circle. It is observed that $s(r)$ has $3l$ zeros, so $s(r) + g(r)$ also has $3l$ zeros inside the unit circle. Thus, all roots of characteristic polynomial $P(r)$ have modulus less than 1, which implies that $\rho(R) < 1$, where $\rho(R)$ denotes the spectral radius of matrix R.

By Lemma 5.6.10 in [18], there exists a norm $\|\cdot\|$ such that $\|R\| < 1$. So from (3.5) we have

$$\|W_{n+1}\| = \|R\| \|W_n\| < \|W_n\|,$$

which means that the linear θ-method is asymptotically stable.

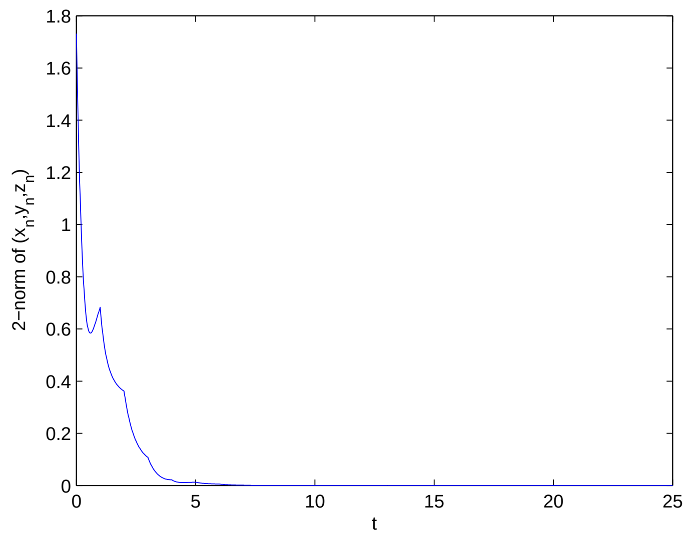

FIGURE 1. The numerical solution of (4.1) with θ=0.6 and m=50.

Conversely, for (3.1), the linear θ-method is asymptotically stable implies that

$$\lim_{n\to\infty} x_n = 0, \quad \lim_{n\to\infty} y_n = 0, \quad \lim_{n\to\infty} z_n = 0.$$

We focus on the special case that $b = d = f = 0$ and $a < 0$, $c < 0$, $e < 0$. Obviously, the condition (2.1) is satisfied in this case. Then (3.4) can be written as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = M_1 \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}. \tag{3.7}$$

From the similar analysis in [19], we can prove that the linear θ-method is asymptotically stable when $1/2 < \theta \leq 1$. This completes the proof.

4. NUMERICAL EXPERIMENTS

Example 1 Consider the following coupled EPCA

$$\begin{aligned} x'(t) &= -2x(t) - 1.5y([t]), \\ y'(t) &= -3y(t) + 1.2z([t]), \\ z'(t) &= -4z(t) + 0.5x([t]), \\ x(0) &= y(0) = z(0) = 1. \end{aligned} \tag{4.1}$$

It is easy to see that $\max\{a, c, e\} = -2 < 0$, $\max\{|b|, |d|, |f|\} = 1.5 < 2 = -\max\{a, c, e\}$, so Corollary 1 holds, then the analytic solution of (4.1) is asymptotically stable. In FIGURE 1, we draw the 2-norm of numerical solution of (4.1) with $\theta = 0.6$ and $m = 50$. From this figure we can see that the numerical solution of (4.1) is asymptotically stable.

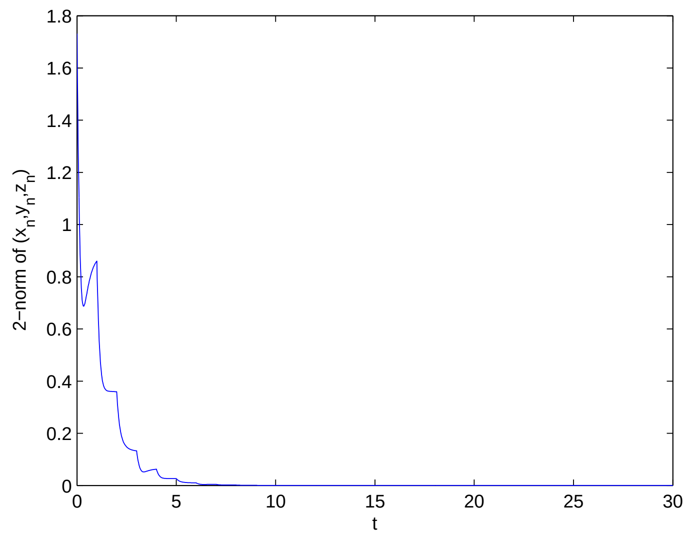

FIGURE 2. The numerical solution of (4.2) with θ=0.7 and m=50.

Example 2 Consider another three-dimensional EPCA

$$\begin{aligned} x'(t) &= -5x(t) + 2y([t]), \\ y'(t) &= -4y(t) - 3z([t]), \\ z'(t) &= -6z(t) + 1.5x([t]), \\ x(0) &= y(0) = z(0) = 1. \end{aligned} \tag{4.2}$$

We compute that $\max\{a, c, e\} = -4 < 0$, $\max\{|b|, |d|, |f|\} = 3 < 4 = -\max\{a, c, e\}$, so Corollary 1 is satisfied, hence the analytic solution of (4.2) is asymptotically stable. Set $\theta = 0.7$ and $m = 50$, we draw the 2-norm of numerical solution of (4.2) in FIGURE 2. We observe from this figure that the numerical solution of (4.2) is asymptotically stable.

Example 3 For the three-dimensional EPCA

$$\begin{aligned} x'(t) &= -10x(t) - 3y([t]), \\ y'(t) &= -9y(t) + 5z([t]), \\ z'(t) &= -8z(t) + 7x([t]), \\ x(0) &= y(0) = z(0) = 1, \end{aligned} \tag{4.3}$$

the coefficients $a = -10, b = -3, c = -9, d = 5, e = -8, f = 7$ satisfy $\max\{a, c, e\} = -8 < 0$ and $\max\{|b|, |d|, |f|\} = 7 < 8 = -\max\{a, c, e\}$, so Corollary 1 holds, thus the analytic solution of (4.3) is asymptotically stable. In FIGURE 3, we draw the 2-norm of numerical solution of (4.3) with $\theta = 0.8$ and $m = 50$. From this figure we can see that the numerical solution of (4.3) is asymptotically stable.

All the numerical examples are in agreement with the theoretical results in this paper.

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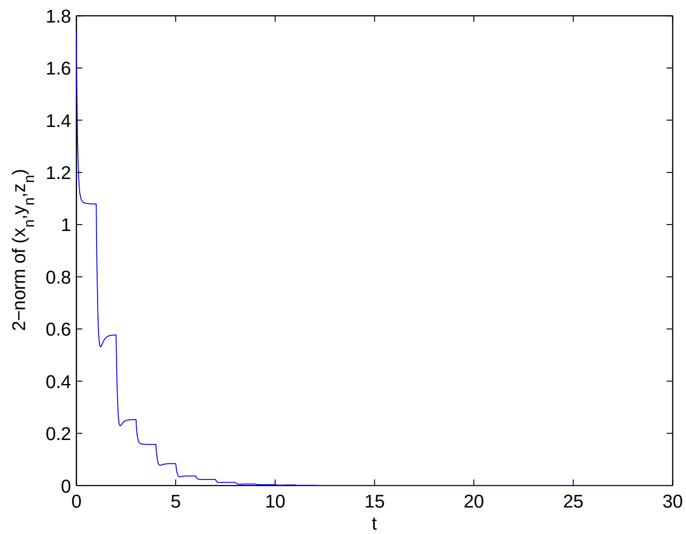

FIGURE 3. The numerical solution of (4.3) with θ=0.8 and m=50.

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