Acia Pac. J. Math. 2019 **6**:12

STABILITY AND SENSITIVITY ANALYSIS OF POVERTY-CRIME NPCJR MODEL

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Received Feb. 18, 2019

Abstract. The paper makes an examination into mathematical model of poverty-crime through stability and sensitivity analysis. The reproduction number R_0 has been determined in terms of the model parameters. The result has inicated that the CFE and CEE is locally and globally asymptotically stable whenever the reproduction number $R_0 < 1$ and $R_0 > 1$ respectively. The sensitivity analysis result indicates that, if efforts for combatting poverty-crime are geared towards checkmating the proportion of non-impoverished individuals α, the transmission coefficient β, the probability that a recovered individuals revert back to crime ϕ, the natural death rate μ_2, the proportion of captured criminals that are jailed ε and the rate at which criminals are being apprehended ρ; then, poverty-crime can be eliminated.

2010 Mathematics Subject Classification. Primary 30A10, Secondary 26D07.

Key words and phrases. modelling; poverty-crime; stability analysis; sensitivity analysis.

1. Introduction

Many scholars have extensively written about mathematical model that concerns crime and poverty [1–6]. In many instances, infectious disease modeling technique is being employed, especially by considering the fact that poverty-crime is being spread like a contagious disease [7], and it has eaten deep into almost every society. As such, if used correctly, mathematical model of poverty induced crime will offer gainful insight into the remote cause of criminality and aid our understanding of the underlying complex nature of the society as to how criminality thrives. Therefore, mathematical model can be such an important mechanism that can complements efforts that are geared towards the needed solution for societal social problems [7,8].

We are concerned about crimes that are induced by poverty, these may include, but not limited to, burglary, property crime, armed robbery, stealing, kidnapping and their likes. These kind of social problems are generally termed as poverty led crimes [3,5,7,8]; it is informed by the remote cause of abject poverty. As matter of a fact, a number of social scientists believe that most indicators of social problems have their roots in poverty which results into crime among other social vices [1,8]. Therefore, it is high time to use scientific approaches to explore avenues and to checkmate the activities of criminals in the society, especially with the use of mathematics.

In the past, man's quest for solution to social problems has resulted into so many explorations. For instance, using martial and physical strategies in combating crime has often being objected as being associated with

DOI: 10.28924/APJM/6-12

numerous challenges considering the risk and complexity of the society [2,9]. Perhaps, a better alternative rests on the use of technology as well as modern ICT devices. But even at that, in attempting to solve one problem other problems are undeniably being created, that is why modeling approaches is required to checkmate some of the avoidable problems even before hand.

In modern day society, it is a very well-known fact that the use of scientific methods to complement the use of direct physical or technology based methods in solving social problems is more preferred. Because models are abstractions of reality! As such, if a model has correctly captured the significant components of a society (system), in mathematical expressions, then such a model will be able to present solution(s) that can be used to address whatsoever issue that is at hand with outmost precision.

For that reason, the use of mathematical models in explaining social problems will not only continue to be an important tool but also a relevant scheme for crime prevention and control. In this regard, our work will builds on the existing works of [8] as a modification; and of course, we work in a practical manner by considering Nigerian data as obtained in [10].

2. The Model Formulation

We improved the model due to [8] by co-opting new set of assumptions and by incorporating the former assumptions. Firstly, the population is divided into five sub-classes: the non-impoverished class N, the poverty class P, the criminal class C, the jailed class J, and recovered class (from jail or from impoverished class) R. The total population is $T = N + P + C + J + R$. The population under consideration is homogeneous (well-mixed population) and is varying in size due to maturation, migration and death.

We assume that the probability of persons in poverty class resorting to crime is higher than those individuals in non-impoverished class and the tendency of becoming poor is dependent upon unemployment rate. Futhermore, those who have already gone to jail flow immediately into the recovered class and then due to contact with criminals revert back to criminality at some reduced rate, and some may also become members of non-impoverished class owing to genuine recovery that encompasses skills acquisition and entrepreneur training. The detail of the model is presented in Figure 1, while the meaning of parameter/state variables are given in Table 1.

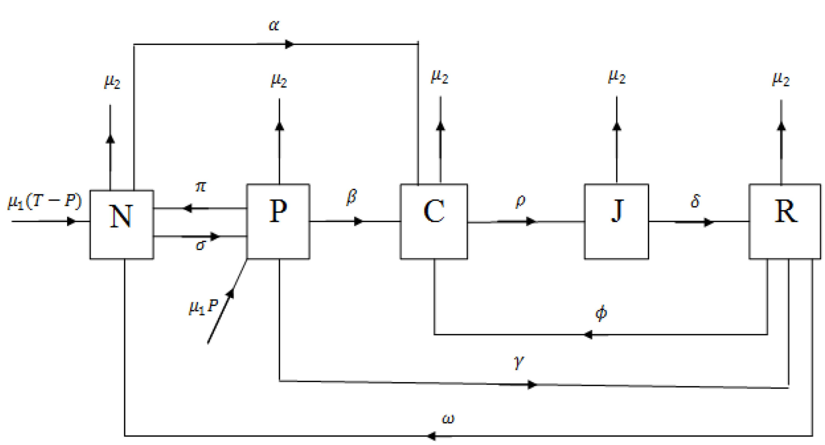

Figure 1. The Model Flow Diagram

Table1: State Variables and Parameters of the Model

Notations	Variable/Parameter Definition
$N(t)$	Fraction of Non-impoverished individuals.
$P(t)$	Fraction of individuals in the Poverty class.
$C(t)$	Fraction of individuals in the Criminal class.
$J(t)$	Fraction of individuals in the Jailed class.
$R(t)$	Fraction of individuals in the Recovered class.
$T(t)$	The population $N(t) + P(t) + C(t) + J(t) + R(t)$
t	Time
α	The proportion of non-impoverished individuals involved in crime.
β	The transmission coefficient due to contact (perweek) with criminals.
σ	The rate of flow from non-impoverished class to poverty class.
ρ	The rate at which criminals are bein gapprehended.
δ	The rate at which individuals get out of jail to recovered class.
ϕ	The probability that a recovered individuals revert back to crime $(0 \leq \phi \leq 1)$.
γ	The rate of converting those in poverty to recovered class.
ε	The proportion of captured criminals that are jailed.
π	The rate at which a formally poor person becomes wealthy by self effort.
μ_1	The rate of recruitment into Impoverished and Non-impoverished class.
μ_2	The natural death rate.
ω	The rate which rehabilitates recovered individual into non-impoverished class.

Under these assumptions, the interaction between poverty-crime is governed by the following system of nonlinear Ordinary Differential Equations (ODE):

$$N' = \mu_1(T-P) - (\sigma+\mu_2)N + \omega R + \pi P - \alpha\beta NC \tag{2.1}$$

$$P' = \mu_1 P + \sigma N - \beta PC - (\gamma+\mu_2+\pi)\,P \tag{2.2}$$

$$C' = \beta PC + \alpha\beta NC + \phi\beta RC - (\varepsilon\rho+\mu_2)\,C \tag{2.3}$$

$$J' = \varepsilon\rho C - (\delta+\mu_2)\,J \tag{2.4}$$

$$R' = \gamma P + \delta J - \phi\beta RC - (\omega+\mu_2)\,R \tag{2.5}$$

$$T = N + P + C + J + R \tag{2.6}$$

3. Methods

The method of linearization is adopted. We focus on the establishment of necessary the conditions for local and global stability of the crime-free equilibrium and crime-endemic equilibrium states.

Lemma 3.1. *The growth of the population is proportional to its size.*

Proof. It could be deduced from equation (2.1) to (2.5) that the rate of change in the whole population as expressed by equation (2.6) means

$$T'(t) = N'(t) + P'(t) + C'(t) + J'(t) + R'(t) \tag{3.1}$$

Hence, by simple algebraic addition we have $T'(t) = \mu_1 T(t) + \mu_2 T(t)$ which gives:

$$T'(t) = (\mu_1 - \mu_2) T(t) \tag{3.2}$$

This entails that $\frac{dT}{dt} \propto (\mu_1 - \mu_2) T$, which has three possible cases:

Case 1: $\mu_1 > \mu_2 = a$ i.e. $\frac{dT}{dt} = aT$

The solution is $T_0 e^{at}$ (where T_0 is the initial condition) inwhich the population is grows.

Case 2: $\mu_1 < \mu_2 = -a$ i.e. $\frac{dT}{dt} = -aT$

The solution is $T_0 e^{-at}$ (where T_0 is the initial condition) inwhich the population is decays.

Case 3: $\mu_1 = \mu_2 = 0$

$\frac{dT}{dt} = 0$ which is a steady state, the population is at fixed.

Hence, we conclude that this model makes sense biologically under **case 1** which in turn implies *Lemma 1*. □

4. Stability and Sensitivity Analysis

4.1. **Existence of CFE and CEE States.** It is clear that the system (see equation (2.1) to (2.6)) admits two steady states - a unique crime free equilibrium $E^* = (N^*, P^*, 0, 0, R^*)$ and existence of permanence crime state $E^{**} = (N^{**}, P^{**}, C^{**}, J^{**}, R^{**})$. This could be obtained by equating the RHS of the system of equations (2.1) - (2.5) to zero and then solve the result for $E^* = (N > 0, P > 0, C = 0, J = 0, R > 0)$ and $E^{**} = (N > 0, P > 0, C > 0, J > 0, R > 0)$ respectively.

4.2. **Local Stability.** The Jacobian of the system is given by

$$J = \begin{pmatrix} x & \pi - \mu_1 & -N\alpha\beta & 0 & \omega \\ \sigma & y & -P\beta & 0 & 0 \\ C\alpha\beta & C\beta & z & 0 & C\beta\phi \\ 0 & 0 & \varepsilon\rho & -\delta - \mu_2 & 0 \\ 0 & \gamma & -R\beta\phi & \delta & -\omega - \mu_2 - C\beta\phi \end{pmatrix} \tag{4.1}$$

where $x = -(\sigma + \mu_2 + \alpha\beta C)$, $y = (\mu_1 - \gamma - \pi - \mu_2 - C\beta)$ and $z = (P\beta - \mu_2 - \varepsilon\rho + N\alpha\beta + R\beta\phi)$.

Evaluating J at (CFE) which is $(N = N^*, P = P^*, C = J = 0, R = R^*)$, we have:

$$J_{(CFE)} = \begin{pmatrix} -\sigma-\mu_2 & \pi-\mu_1 & -\alpha\beta N^* & 0 & \omega \\ \sigma & \mu_1-\gamma-\pi-\mu_2 & -\beta P^* & 0 & 0 \\ 0 & 0 & H & 0 & 0 \\ 0 & 0 & \varepsilon\rho & -\delta-\mu_2 & 0 \\ 0 & \gamma & -\beta\phi R^* & \delta & -\omega-\mu_2 \end{pmatrix} \tag{4.2}$$

where $H = \beta P^* - \mu_2 - \varepsilon\rho + \alpha\beta N^* + \beta\phi R^*$.

We now set the characteristic polynomial $|J_{E^*} - \lambda I|$. From row three, we see that the characteristic equation is

$$H \begin{vmatrix} -\sigma-\mu_2 & \pi-\mu_1 & 0 & \omega \\ \sigma & \mu_1-\gamma-\pi-\mu_2 & 0 & 0 \\ 0 & 0 & -\delta-\mu_2 & 0 \\ 0 & \gamma & \delta & -\omega-\mu_2 \end{vmatrix} = 0 \tag{4.3}$$

Again, from the next row three, it gives:

$$[H-\lambda]\left[-(\delta+\mu_2)-\lambda\right] \begin{vmatrix} -(\sigma+\mu_2)-\lambda & \pi-\mu_1 & \omega \\ \sigma & -s-\lambda & 0 \\ 0 & \gamma & -(\omega+\mu_2)-\lambda \end{vmatrix} = 0 \tag{4.4}$$

where $s = (\gamma + \pi + \mu_2 - \mu_1)$.

By considering the last row, we get:

$$\gamma\left[-\sigma\omega\right][H-\lambda]\left[-(\delta+\mu_2)-\lambda\right]\left[-(\omega+\mu_2)-\lambda\right] \begin{vmatrix} -(\sigma+\mu_2)-\lambda & \pi-\mu_1 \\ \sigma & -s-\lambda \end{vmatrix} = 0 \tag{4.5}$$

By careful observation, the issue of local stability has sort itself out, since it obvious that the characteristic equation is:

$$-\sigma\omega\gamma\left[H-\lambda\right]\left[-(\delta+\mu_2)-\lambda\right]\left[-(\omega+\mu_2)-\lambda\right]\left[\left(-(\sigma+\mu_2)-\lambda\right)(-s-\lambda)-\sigma(\pi-\mu_1)\right] = 0 \tag{4.6}$$

Clearly, $\lambda_i < 0$, for $i = 1,2,3,4$. in $\left[-(\delta+\mu_2)-\lambda\right], \left[-(\omega+\mu_2)-\lambda\right], \left[-(\sigma+\mu_2)-\lambda\right]$ and $\left[(-s-\lambda)\right]$ which implies local stability.

And lastly, for $\lambda_5 < 0$ it requires $H < 0$. This means $(\beta P^* - \mu_2 - \varepsilon\rho + \alpha\beta N^* + \beta\phi R^*) < 0$.This can be possible under the condition $(\mu_2 + \varepsilon\rho) > \beta\left(\alpha N^* + P^* + +\phi R^*\right)$ which guarantees that the system is locally asymptotically stable whenever $R_0 < 1$.

We now proceed to obtain the basic reproduction number R_0 from the characteristic equation (4.6) . The lambda independent term, which in most cases is the dominant eigen value gives $R_0 = \frac{\sigma^2\beta\gamma\omega(\pi-\mu_1)(P^*+\alpha N^*+\phi R^*)}{\sigma^2\gamma\omega(\mu_2+\varepsilon\rho)(-\pi+\mu_1)} = \frac{\beta(\alpha N^*+P^*++\phi R^*)}{\mu_2+\varepsilon\rho}$.

4.3. **Global Stability.** We illustrate the application of Lyapunov's theorem by showing global stability of the crime-free equilibrium and the crime endemic equilibrium for the system in equation (2.1) to (2.5). We approach the problem by constructing a Lyapunov function [11]. We consider the model on the space of the first four variables only (N, P, C, R). It is clear that if the crime-free equilibrium for the first four equations is globally stable, then $R(t) \to 0$, and the crime-free equilibrium for the full model is globally stable.

Lemma 4.1. *The crime-free equilibrium is globally asymptotically stable when* $R_0 < 1$. .

Proof. Consider a composite quadratic function

$V : \{(N, P, C, J) \in D : N > 0, P > 0, C \geq 0, J \geq 0, R > 0\}$, defined by logarithmic Lyapunov function and common quadratic Lyapunov function of the form:

$$V(t) = c_1\left(N - N^* - N^* \ln\frac{N^*}{N}\right) + \frac{c_2}{2}(P - P^*)^2 + \frac{c_3}{2}(C - C^*)^2 + \left[\frac{c_4}{2}(J - J^*)\right]^2 \tag{4.7}$$

By simple expansion of (4.7), we have Lyapunov function is of the form:

$$V(t) = c_1\left(N - N^* - N^* \ln\frac{N^*}{N}\right) + \frac{c_2}{2}(P - 2PP^* + P^*) + \frac{c_3}{2}(C - 2CC^* + C^*) + \frac{c_4^2}{4}J^2 - \frac{c_4^2}{2}JJ^* + \frac{c_4^2}{4}J^{*2} \tag{4.8}$$

The time derivative of (4.8) along the solution of the model i.e. equation (2.1) to (2.5) yields:

(4.9)
$$\begin{aligned} \dot{V}(t) &= c_1\left\{\dot{N}(t) - N^*(t) - \dot{N}\ln\frac{N^*}{N(t)} - N\left[\frac{\dot{N^*}}{N(t)} - \frac{N^*(t)\dot{N}(t)}{(N(t))^2}\right]\right\} + c_2(P(t) - P^*(t))\left[\left(\dot{P}(t) - P^*(t)\right)\right] \\ &+ c_3(C(t) - C^*(t))\left[\left(\dot{C}(t) - \dot{C^*}(t)\right)\right] + \frac{c_4^2}{2}(J(t) - J^*(t))\left[\left(\dot{J}(t) - \dot{J^*}(t)\right)\right] \end{aligned}$$

By applying the CFE state E^* with the system of equations (2.1) to (2.5) together into (4.9) as detail in [12], we onserve that $\dot{V}(t) \leq 0$ for $R_0 < 1$ and $\dot{V}(t) = 0$ if and only if $C(t) = 0$. Therefore, one sees that $(N, P, C, J) \to (N^*, P^*, 0, 0, R^*)$ as $t \to \infty$ since $C(t) \to 0$ as $t \to \infty$. Consequently, the largest compact invariant set in $\left\{(N, P, C, J, R) \in D : \dot{V} = 0\right\}$ is the singleton $\{E^*\}$ and by Lasalle's invariance principle [13], E^* is globally asymptotically stable in D if $R_0 < 1$. □

Lemma 4.2. *The endemic equilibrium is globally asymptotically stable when* $R_0 > 1$. .

Now we shall consider the crime endemic equilibrium state. The endemic equilibrium is unique and locally stable whenever it exists [14]. For $R_0 > 1$, the only other equilibrium E^* is unstable. This suggests that the endemic equilibrium E^{**} may be globally stable. Thus, we establish this result of global stability via a Lyapunov function as contained [14].

Proof. We consider again only the first four components on the system (2.1) - (2.5), (N, P, C, J) we assume that they belong to the positive orthant D_+^4. We define a Lypunov function:

$$\begin{aligned} V(t) &= c_1\left(N - N^{**} - N^{**}\ln\frac{N^{**}}{N}\right) + c_2\left(P - P^{**} - P^{**}\ln\frac{P^{**}}{P}\right) + c_3\left(C - C^{**} - C^{**}\ln\frac{C^{**}}{C}\right) \\ &+ c_4\left(J - J^{**} - J^{**}\ln\frac{J^{**}}{J}\right) \end{aligned} \tag{4.10}$$

where $c_1 > 0, c_2 > 0, c_3 > 0$ and $c_4 > 0$ to be determined.

Notice that $V(t) = 0$ when $(N, P, C, J) = (N^{**}, P^{**}, C^{**}, J^{**})$ and $V\dot{(t)} > 0$ otherwise; V is also radially unbounded. What remains to be proved is that the derivative of $V(t)$ with respect to t is negative. We differentiate with respect to t and replace $N'(t), P'(t), C'(t)$ and $J'(t)$ with their equals in equation (2.1) to (2.5).

$$\frac{dV}{dt} = c_1\left(1-\frac{N^{**}}{N}\right)S' + c_2\left(1-\frac{P^{**}}{P}\right)P' + c_3\left(1-\frac{C^{**}}{C}\right)C' + c_4\left(1-\frac{J^{**}}{J}\right)J' \tag{4.11}$$

Substituting N', P', C', and J', we get:

$$\begin{aligned} \frac{dV}{dt} &= c_1\left(1-\frac{S^{**}}{S}\right)(\mu_1(T-P)-(\sigma+\mu_2)N+\omega R+\pi P-\alpha\beta NC) \\ &+ c_2\left(1-\frac{P^{**}}{P}\right)(\mu_1 P+\sigma N-\beta PC-(\gamma+\mu_2+\pi)P) \\ &+ c_3\left(1-\frac{C^{**}}{C}\right)(\beta PC+\alpha\beta NC+\phi\beta RC-(\varepsilon\rho+\mu_2)C) \\ &+ c_4\left(1-\frac{J^{**}}{J}\right)(\varepsilon\rho C-(\delta+\mu_2)J) \end{aligned} \tag{4.12}$$

After applying the CEE state E^{**} into (4.12) , we now have to apply the Krasovkii-LaSalle theorem. We consider the set where the Lyapunov function is equal to zero.

$$\ell = \{x \in R^n \,| V\text{"}\,(t) = 0\} \tag{4.13}$$

It is clear that $V' = 0$ if and only if $N = N^{**}, P = P^{**}, C = C^{**}$ and $J = J^{**}$for some constants c_1, c_2, c_3 and c_4.

Hence, the set ℓ consists of the sigleton $(N^{**}, P^{**}, C^{**}, J^{**})$. This concludes the proof. □

4.4. **Sensitivity Analysis.** This is the process of testing the robustness of model predictions by studying how the variation in the output of a mathematical model depends on different sources of variations in the input of the mathematical model. One common type of sensitivity analysis is to determine how a focused quantity (depending on variables) is related to perturbation of each parameter [15]. The goal of sensitivity analysis is to decide qualitatively which parameters are most influential in the model output. Thus, sensitivity analysis can be performed on a dynamic system or on static quantities such as the reproduction number or equilibrium prevalence.

Usually, a parameter is called sensitive if small changes in the value of the parameter produce large changes in the solution of the differential equations. To perform sensitivity analysis of a dynamical system, we assume that the differential equations depend on a parameter p:

$$y'(t) = f_i(y_1, ..., y_n, t, p) \qquad i = 1, ..., n. \tag{4.14}$$

The parameter may be one of the coefficients in the system or one of the initial conditions. The solution of the initial value problem can be thought of as a function of both the time variable t and the parameter $p : y_i(t, p), i = 1, ..., n$. The sensitivity is computed by finding the derivatives of each variable with respect to each parameter at any time t. To determine best control measures, knowledge of the relative importance of the different factors responsible for transmission is useful. The sensitivity index of R_0 with respect to a parameter say ω is $\frac{\partial R_0}{\partial \omega}$. From our study, six paramertes $\alpha, \beta, \phi, \mu_2, \varepsilon$ and ρ have impact on the dynamics of poverty-crime viz: $\frac{\beta N^*}{\mu_2+\varepsilon\rho}, \frac{(P^*+\alpha N^*+\phi R^*)}{\mu_2+\varepsilon\rho}, \frac{\beta R^*}{\mu_2+\varepsilon\rho}, -\frac{\beta(P^*+\alpha N^*+\phi R^*)}{(\mu_2+\varepsilon\rho)^2}, -\frac{\rho\beta(P^*+\alpha N^*+\phi R^*)}{(\mu_2+\varepsilon\rho)^2}$ and $-\frac{\varepsilon\beta(P^*+\alpha N^*+\phi R^*)}{(\mu_2+\varepsilon\rho)^2}$ respectively. Another measure is the elasticity index (normalized sensitivity index) that measures the relative change of R_0 with respect to say ω, denoted by $\gamma_\omega^{R_0}$, and defined by $\gamma_\omega^{R_0} = \frac{\partial R_0}{\partial \omega} \times \frac{\omega}{R_0}$.The sign of the elasticity index tells whether R_0

increases (positive sign) or decreases (negative sign) with the parameter; whereas the magnitude determines the relative importance of the parameter.

5. Conclusion

The study of a model of poverty-crime by modifying [8] has indicated that the CFE is locally and globally asymptotically stable whenever the reproduction number $R_0 < 1$ and unstable whenever $R_0 > 1$ inwhich case CEE is locally and globally asymptotically stable. The parameters that have impact on the model are the proportion of non-impoverished individuals involved in crime paramertes α, the transmission coefficient due to contact (perweek) with criminals β, the probability that a recovered individuals revert back to crime ϕ, the natural death rate μ_2, the proportion of captured criminals that are jailed ε and the rate at which criminals are being apprehended ρ. Therefore, any effort that is aimed at combarting poverty-crime should focus on these parameters.

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