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PROPERTIES OF MEROMORPHIC SOLUTIONS OF NONLINEAR DIFFERENCE EQUATION $w(z+1)w(z-1)=h(z)w^m(z)$

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ABSTRACT. In this paper, we mainly deal with the properties of meromorphic solutions of the nonlinear difference equation of the form

$$w(z+1)w(z-1)=h(z)w^m(z), \tag{$*$}$$

where $h(z)$ is a nonzero rational function and $m=\pm2,\pm1,0$. It is shown that the admissible meromorphic solution $w(z)$ of the equation $(*)$ satisfies $\rho(w)\geq 1$. The relationship of the exponents of convergence of zeros and poles of difference and divided difference to the order of growth of $w(z)$ is also given when $h(z)$ is a non-polynomial rational function.

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1. INTRODUCTION.

Throughout this paper, for a meromorphic function $w(z)$, we use standard notations of the Nevanlinna theory of meromorphic functions such as $T(r,w)$, $m(r,w)$ and $N(r,w)$ (see e.g. [9, 11, 17]) and denote the order of growth of $w(z)$, the hyper order of $w(z)$, the exponent of convergence of the zeros of $w(z)$ and the exponent of convergence of the poles of $w(z)$ by $\rho(w), \rho_2(w), \lambda(w), \lambda(1/w)$, respectively, and define them as follows:

$$\rho(w)=\limsup_{r\to\infty}\frac{\log T(r,w)}{\log r},\ \ \rho_2(w)=\limsup_{r\to\infty}\frac{\log\log T(r,w)}{\log r},$$
$$\lambda(w)=\limsup_{r\to\infty}\frac{\log N(r,1/w)}{\log r},\ \ \lambda(1/f)=\limsup_{r\to\infty}\frac{\log N(r,w)}{\log r}.$$

We say $a(z)$ is a small function with respect to $w(z)$ if $T(r,a)=S(r,w)$, where $S(r,w)=o(T(r,w))$, as $r\to+\infty$ outside of a possible exceptional set of finite logarithmic measure, and say an equation admits an "admissible solution" $w(z)$ if all its coefficients are small functions with respect to $w(z)$.

Furthermore, we need some notations on differences. Let η be a nonzero complex constant and let $w(z)$ be a meromorphic function. We use the notation $\Delta_\eta^n w(z)$ to denote the difference operators of $w(z)$, which are

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defined by

$$\Delta_\eta w(z) = w(z+\eta) - w(z) \quad \text{and} \quad \Delta_c^n w(z) = \Delta_\eta^{n-1}(\Delta_\eta w(z)), \quad n \in \mathbb{N},\ n \geq 2.$$

In particular, if $\eta = 1$, we denote $\Delta_\eta w = \Delta_\eta w(z) = \Delta w(z) = \Delta w$.

Since the application of the classical Nevanlinna theory to difference equations by Ablowitz et al. [1] in 2000, meromorphic solutions of complex difference equations have been a hot topic recently (see e.g. [3–8, 10, 12–15, 18, 19]. Lots of results on the existence, value distribution and growth of meromorphic solutions of kinds of linear and non-linear difference equations are proved.

One of the most important and creative results in this direction can be found in [7] which was given by Halburd and Korhonen. In fact, they considered the following difference equation

$$w(z+1) + w(z-1) = R(z, w), \tag{1.1}$$

where $R(z, w)$ is rational in w and meromorphic in z, and proved that if (1.1) has an admissible meromorphic solution of finite order, then either w satisfies a difference Riccati equation or (1.1) can be transformed into a linear difference equation or difference Painlevé I, II equations.

The other important result is about the difference Painlevé III equations given by Ronkainen in [15]. He showed that if the equation

$$w(z+1)w(z-1) = R(z, w) \tag{1.2}$$

has an admissible meromorphic solution w of hyper order less than one, where $R(z, w)$ is rational and irreducible in w and meromorphic in z, then either w satisfies a difference Riccati equation or (1.2) can be transformed into difference Painlevé III equations. In this paper, we consider the following nonlinear difference equation of the form

$$w(z+1)w(z-1) = h(z)w^m(z), \tag{1.3}$$

where $h(z)$ is a nonzero rational function and $m = \pm 2, \pm 1, 0$. This equation comes from the family of Painlevé III equations by Ronkainen's classification.

Concerning on the value distribution and growth of meromorphic solutions of the equations (1.3), Zhang and Yang [18] and Zhang and Yi [19] gave numbers of results for the case that all coefficients are constants, which were improved into the case that all coefficients are rational functions by Lan and Chen [13, 14]. We recall three results as follows.

Theorem A ([18]). *If $w(z)$ is a nonconstant meromorphic solution with finite order of (1.3), where $m = -2, -1, 0, 1$ and $h(z)$ is a nonzero constant, then*

(i) $w(z)$ cannot be a rational function;

(ii) $\lambda(1/w) = \lambda(w) = \rho(w)$.

Theorem B ([18]). *If $w(z)$ is a nonconstant meromorphic solution with finite order of (1.3), where $m = 2$ and $h(z)$ is a nonzero constant, then*

(i) $w(z)$ has no nonzero Nevanlinna exceptional value;

(*ii*) $w(z)$ *cannot be a rational function.*

Theorem C ([13]). *Suppose that $h(z)$ is a nonconstant rational function. If $w(z)$ is a transcendental meromorphic solution with finite order of (1.3), where $m = -2, -1, 0, 1$, then*

(*i*) $w(z)$ *has no Nevanlinna exceptional value;*

(*ii*) $\lambda(\Delta w) = \lambda(1/\Delta w) = \rho(w), \lambda(\Delta w/w) = \lambda(w/\Delta w) = \rho(w)$.

Remark 1. Examples to show the existences of moromorphic solutions of (1.3), can be found in [13, 18]. In particular, Zhang and Yang [18] gave an example for the case $m = 2$ and $h(z)$ is a nonzero constant such that $w(z)$ has two Picard exceptional values $z = 0, \infty$. From their example, we get the following example in which $h(z)$ is a rational function.

Example 1. $w(z) = ze^z$ satisfies the equation

$$w(z+1)w(z-1) = \frac{z^2-1}{z^2}w^2(z),$$

and $\Delta w = (ez+e-z)e^z, 1/\Delta w = 1/(ez+e-z)e^z, \Delta w/w = (ez+e-z)/z$ and $w/\Delta w = z/(ez+e-z)$ have the same Nevanlinna exceptional values $z = 0, \infty$. This indicates that the conclusion in Theorem 1 does not always hold for the case $m = 2$.

Remark 2. Looking into the proofs of Theorem 1 in [18] and Theorem 1 in [13], we can see that if $w(z)$ is a nonconstant meromorphic solution of (1.3) with finite order, where $m = -2, -1, 0, 1$ and $h(z)$ is a nonzero constant, then $\lambda(1/w) = \lambda(w) = \lambda(1/\Delta w) = \lambda(w/\Delta w) = \rho(w)$. However, we still wonder whether $\lambda(\Delta w) = \lambda(\Delta w/w) = \rho(w)$ holds or not.

Considering Remarks 1 and 2, we ask the following two questions:

Question 1. What can we say about $\rho(w)$ in Theorems B and C?

Question 2. When does $\lambda(\Delta w) = \lambda(1/\Delta w) = \lambda(\Delta w/w) = \lambda(w/\Delta w) = \rho(w)$ hold for the nonconstant finite order meromorphic solution $w(z)$ of (1.3), where $m = -2, -1, 0, 1$ and $h(z)$ is a nonzero constant or where $m = 2$ and $h(z)$ is a nonzero rational function?

We will give some lemmas in the next section, and then consider the Question 1 and Question 2 in the section 3 and section 4 respectively.

2. Lemmas

Lemma 2.1 ([3]). *Let $w(z)$ be a meromorphic function of finite order ρ, ε be a positive constant, η be a nonzero complex constant. Then*

$$N(r, w(z+\eta)) \le N(r, w(z)) + O(r^{\rho-1+\varepsilon}) + O(\log r),$$

$$T(r, w(z+\eta)) \le T(r, w(z)) + O(r^{\rho-1+\varepsilon}) + O(\log r).$$

The lemma below is Hadamard's factorization Theorem of meromorphic function which can be found in [16].

Lemma 2.2 ([16]). *Let $w(z)$ be a meromorphic function of finite order ρ. If*

$$w(z) = c_k z^k + c_{k+1}z^{k+1} + \cdots \quad (c_k \neq 0, k \in \mathbb{Z})$$

near $z=0$. Then we write $w(z)$ as follows

$$w(z)=z^k\frac{P_1(z)}{P_2(z)}e^{Q(z)},$$

where $P_1(z),P_2(z)$ are entire functions and the canonical products of w formed with the non-null zeros and poles of w, respectively, such that $\rho(P_1)=\lambda(P_1)=\lambda(w)$, $\rho(P_2)=\lambda(P_2)=\lambda(1/w)$, and $Q(z)$ is a polynomial such that $\deg Q(z)=q\le\rho$.

Considering the question 2, we need to denote $\hat{E}_1(r,w)=\{z\in\mathbb{C}|w(z+1)=w(z)=0, w(z+1)/w(z)\neq 1\}$ and the corresponding counting function by $\hat{N}_1(r,w)$, and use the following lemma.

Lemma 2.3. *Suppose that $w(z)$ is a nonconstant meromorphic function with finite order such that $\Delta w\not\equiv 0$. Then*
(*i*) $\lambda(1/\Delta w)\le\lambda(1/w), \lambda(w/\Delta w)\le\max\{\lambda(w),\lambda(1/w)\}, \max\{\lambda(\Delta w/w),\lambda(\Delta w)\}\le\rho(w)$;
(*ii*) $\lambda(\Delta w/w)=\rho(w)\Rightarrow\lambda(\Delta w)=\rho(w)$, *and if* $\hat{N}_1(r,w)=S(r,w)$, *then* $\lambda(\Delta w)=\rho(w)\Rightarrow\lambda(\Delta w/w)=\rho(w)$.

Proof. (i) Since $w(z)$ is a nonconstant meromorphic function such that $\rho(w)=\rho<\infty$, we get from lemma 2.1 that

$$\begin{aligned}N(r,\Delta w)&=N(r,w(z+1)-w(z))\\&\le N(r,w(z+1))+N(r,w(z))\le 2N(r,w(z)+O(r^{\rho-1+\varepsilon})+O(\log r).\end{aligned}$$

This means that $\lambda(1/\Delta w)\le\lambda(1/w)$.

$$\begin{aligned}N(r,\Delta w/w)=N(r,w(z+1)/w(z)-1)&\le N(r,w(z+1))+N(r,1/w(z))+O(1)\\&\le N(r,w(z))+N(r,1/w(z))+O(r^{\rho-1+\varepsilon})+O(\log r).\end{aligned}$$

This gives that $\lambda(w/\Delta w)\le\max\{\lambda(w),\lambda(1/w)\}$.

From Lemma 2.1, we have

$$\begin{aligned}T(r,w/\Delta w)&=T(r,\Delta w/w)+O(1)=T(r,w(z+1)/w(z)-1)+O(1)\\&\le T(r,w(z+1))+T(r,w(z))+O(1)\le 2T(r,w(z))+O(r^{\rho-1+\varepsilon})+O(\log r).\end{aligned}$$

which means that $\rho(w/\Delta w)=\rho(\Delta w/w)\le\rho(w)=\rho$. Thus, $\max\{\lambda(\Delta w/w),\lambda(\Delta w)\}\le\rho(w)$.

(ii) Suppose that $\lambda(\Delta w/w)=\rho(w)$. Since $\lambda(\Delta w)\le\rho(\Delta w)\le\rho(w)$, if $\lambda(\Delta w)\neq\rho(w)$, then $0\le\lambda(\Delta w)<\rho(w)$, and there are at most $S(r,w)$ many points such that

$$\Delta w(z)=w(z+1)-w(z)=0. \tag{2.1}$$

Therefore, there are at most $S(r,w)$ many points such that

$$\frac{\Delta w(z)}{w(z)}=\frac{w(z+1)}{w(z)}-1=0. \tag{2.2}$$

which means that $\lambda(\Delta w/w)<\rho(w)$. This contradicts to $\lambda(\Delta w/w)=\rho(w)$. Thus, $\lambda(\Delta w)=\rho(w)$.

Now suppose that $\hat{N}_1(r,w)=S(r,w)$ and $\lambda(\Delta w)=\rho(w)$. Similarly, if $\lambda(\Delta w/w)\neq\rho(w)$, then $0\le\lambda(\Delta w/w)<\rho(w)$, and there are at most $S(r,w)$ many points satisfy (2.2).

Denote

$$E(r,\Delta w)=\{|z|\le r|\Delta w(z)=0\},$$

$$E_0(r,\Delta w)=\{|z|\le r|w(z+1)=w(z)=0\},$$

$$E_c(r, \Delta w) = \{|z| \leq r | w(z+1) = w(z) = c, c \in \mathbb{C} \setminus \{0\}\},$$

$$E_\infty(r, \Delta w) = \{|z| \leq r | \Delta w = 0, w(z+1) = w(z) = \infty\}.$$

Then $E(r, \Delta w) = E_0(r, \Delta w) \cup E_c(r, \Delta w) \cup E_\infty(r, \Delta w)$.

Obviously, all points in $E_c(r, \Delta w) \cup E_\infty(r, \Delta w)$ must satisfy (2.2) and hence $E_c(r, \Delta w) \cup E_\infty(r, \Delta w)$ consists of at most $S(r, w)$ many points. On the other hand, each point of $E_0(r, \Delta w)$ is a point of $E_1(r, w)$ or satisfies (2.2). Since $\hat{N}_1(r, w) = S(r, w)$, there are at most $S(r, w)$ many points in $E_0(r, \Delta w)$. To sum up, there are at most $S(r, w)$ many points in $E(r, \Delta w)$. This indicates that $\lambda(\Delta w) < \rho(w)$. This contradicts to $\lambda(\Delta w) = \rho(w)$. Thus, $\lambda(\Delta w / w) = \rho(w)$.

The following lemma was proved by Chiang and Feng [3] and by Halburd and Korhonen [4] independently, and plays a very important role in studying the difference analogues of Nevanlinna theory and difference equations.

Lemma 2.4 ([3,4]). *Let $w(z)$ be a meromorphic function of finite order $\rho(w) = \rho$, ε be a positive constant, η_1 and η_2 be two distinct nonzero complex constants. Then*

$$m\left(r, \frac{w(z+\eta_1)}{w(z+\eta_2)}\right) = O(r^{\rho-1+\varepsilon}).$$

3. Results on the Question 1

Theorem 3.1. *Suppose that $h(z)$ is a nonzero rational function. If $w(z)$ is a nonconstant meromorphic solution of (1.3), then $\rho(w) \geq 1$.*

Proof. Suppose that $w(z)$ is a nonconstant meromorphic solution of (1.3). If $\rho(w) = \infty$, our conclusion holds. Therefore, we assume that $\rho(w) < \infty$, and discuss case by case in the following.

Case 1: $m = 0$**.** Now (1.3) is of the form

$$w(z+1)w(z-1) = h(z),$$

which gives

$$\frac{w(z+4)}{w(z)} = \frac{h(z+3)}{h(z+1)} := R_1(z). \tag{3.1}$$

Subcase 1.1: $h(z) \equiv c_1 \neq 0$. Then from (3.1), we see that $w(z+4) = w(z)$, which means that $w(z)$ is a nonconstant periodic function of period 4. Thus $\rho(w) = \rho \geq 1$.

Subcase 1.2: $h(z)$ is a nonconstant rational function. Then $\log r = O(T(r, h))$. Since every rational function f satisfies $T(r, w) = O(\log r)$ and $h \in S(w)$, $w(z)$ must be a transcendental meromorphic function. Therefore, from Theorem C, $w(z)$ has infinitely many poles and zeros. Notice that $R_1(z)$ in (3.1) is a rational function. There exists some $r_0 > 0$ such that all zeros and poles of $R_1(z)$ belong to the disk $|z| < r_0$. Choose a zero of $w(z)$, denoted by z_0, such that $|z_0| > r_0 + 4$. Then $|z_0 \pm 4| > |z_0| - 4 > r_0$ and hence $w(z_0 \pm 4) = 0$. If $|z_0 + 4| \geq |z_0| > r_0$, then for all $k \in \mathbb{N}$, $z_k = z_0 + 4k$ is a zero of $w(z)$. If $|z_0 + 4| < |z_0|$, then for all $k \in \mathbb{N}$, $z_k = z_0 - 4k$ is a zero of $w(z)$. As a result, we see that

$$\rho(w) \geq \lambda(w) = \limsup_{r\to\infty} \frac{\log N(r, 1/w)}{\log r} \geq \limsup_{k\to\infty} \frac{\log N(4k, 1/w)}{\log(4k + |z_0|)} = 1.$$

Case 2: $m = -1$**.** Now (1.3) is of the form

$$w(z+1)w(z-1)w(z) = h(z),$$

which gives

$$\frac{w(z+3)}{w(z)} = \frac{h(z+2)}{h(z+1)} := R_2(z).$$

Similarly, we can prove that $\rho(w) = \rho \geq 1$.

Case 3: $m = -2$**.** Now (1.3) is of the form

$$w(z+1)w(z-1)w^2(z) = h(z),$$

which gives

$$\frac{w(z+3)w(z+2)}{w(z+1)w(z)} = \frac{h(z+2)}{h(z+1)}.$$

Set $u(z) = w(z+1)w(z)$, then on one hand, we get from the equation above that

$$\frac{u(z+2)}{u(z)} = \frac{h(z+2)}{h(z+1)} := R_3(z). \tag{3.2}$$

On the other hand, from Lemma 2.1, we obtain

$$T(r, u) = T(r, w(z+1)w(z)) \leq T(r, w(z+1) + T(r, w(z)) \leq 2T(r, w) + S(r, w),$$

which means that $\rho(w) \geq \rho(u)$.

Subcase 3.1: $h(z) \equiv c_2 \neq 0$. Then from (3.2), we see that $u(z+2) = u(z) \neq 0$. This indicates that $u(z)$ is a nonzero periodic function of period 2.

If $u(z) \equiv c_3 \neq 0$, then $w(z+1)w(z) \equiv c_3 \neq 0$. From Theorem A, $w(z)$ has infinitely many zeros and poles. Suppose that $w(z_1) = 0$. Then we can deduce from $w(z+1)w(z) = u(z) \equiv c_3 \neq 0$ that for all $k \in \mathbb{Z}$, $z_k = z_1 + 2k$ is a zero of $w(z)$ while $z_k = z_1 + 2k + 1$ is a pole of $w(z)$. This yields that

$$\rho(w) \geq \lambda(w) = \limsup_{r\to\infty} \frac{\log N(r, 1/w)}{\log r} \geq \limsup_{k\to\infty} \frac{\log N(2k, 1/w)}{\log(2k + |z_1|)} = 1.$$

If $u(z)$ is a nonconstant periodic function of period 2. Then we obtain $\rho(w) = \rho \geq \rho(u) \geq 1$ immediately.

Subcase 3.2: $h(z)$ is a nonconstant rational function. With a similar arguing as in the subcase 1.2, we see that $w(z)$ has infinitely many poles and zeros.

If $u(z)$ has finitely many zeros and poles, with a similar method as in the subcase 1.2, we can choose a zero of $w(z)$, denoted by z_2, such that for $k \in \mathbb{N}$, $u(z_2 + k) \neq 0, \infty$ or $u(z_2 - k) \neq 0, \infty$. Then we can deduce from $u(z) = w(z+1)w(z)$ that for all $k \in \mathbb{N}$, $z_k = z_2 + 2k$ is a zero of $w(z)$ while $z_k = z_2 + 2k + 1$ is a pole of $w(z)$, or $z_k = z_2 - 2k$ is a zero of $w(z)$ while $z_k = z_2 - 2k - 1$ is a pole of $w(z)$. This yields that

$$\rho(w) \geq \lambda(w) = \limsup_{r\to\infty} \frac{\log N(r, 1/w)}{\log r} \geq \limsup_{k\to\infty} \frac{\log N(2k, 1/w)}{\log(2k + |z_2|)} = 1.$$

If $u(z)$ has infinitely many zeros or poles. We may assume that $u(z)$ has infinitely many poles. Then we can choose a pole of $u(z)$, denoted by z_3, such that for $k \in \mathbb{N}$, $R_3(z_3 + 2k) \neq 0, \infty$, or $R_3(z_3 - 2k) \neq 0, \infty$. Then we can deduce from (3.2) that for all $k \in \mathbb{N}$, $z_k = z_3 + 2k$ is a pole of $u(z)$, or $z_k = z_3 - 2k$ is a pole of $u(z)$. This yields that

$$\rho(u) \geq \lambda(1/u) = \limsup_{r\to\infty} \frac{\log N(r,u)}{\log r} \geq \limsup_{k\to\infty} \frac{\log N(2k,u)}{\log(2k + |z_3|)} = 1.$$

And hence $\rho(w) = \rho \geq \rho(u) \geq 1$.

Case 4: $m = 1$**.** Now (1.3) is of the form

$$w(z+1)w(z-1) = h(z)w(z),$$

which gives

$$w(z+3)w(z) = h(z+2)h(z+1).$$

Subcase 4.1: $h(z) \equiv c_4 \neq 0$. With a similar reasoning as in the subcase 3.1, we can get $\rho(w) = \rho \geq 1$.

Subcase 4.2: $h(z)$ is a nonconstant rational function. With a similar reasoning as in the subcase 3.2, we can get $\rho(w) = \rho \geq 1$.

Case 5: $m = 2$**.** Now (1.3) is of the form

$$\frac{w(z+1)w(z-1)}{w^2(z)} = h(z). \tag{3.3}$$

Subcase 5.1: $h(z) \equiv c_5 \neq 0$. From Theorem 1, $w(z)$ is a transcendental meromorphic function. If $w(z)$ has no zeros and poles, then $w(z) = e^{Q_1(z)}$, where $Q_1(z)$ is a nonconstant polynomial. Hence $\rho(w) = \rho(e^{Q_1}) \geq 1$. Otherwise, $w(z)$ has at least one pole or zero.

Suppose that $w(z)$ has a pole, denoted by z_4, with multiplicity k_0, then from (3.3), one can find that either $z_4 + 1$ or $z_4 - 1$ is a pole of $w(z)$. Without loss of generality, assume that $z_4 + 1$ is a pole of $w(z)$ with multiplicity $k_1 \geq 1$. We should discuss two cases as follows: case (i): $k_1 \geq k_0$; case (ii): $k_1 < k_0$.

Case (i): $k_1 \geq k_0$. Then from (3.3), $z_4 + 2$ is a pole of $w(z)$ with multiplicity $k_2 = 2k_1 - k_0 \geq k_1 \geq k_0$. By induction, we can easily prove that for all $l \in \mathbb{N}$, $z_4 + l$ is a pole of $w(z)$ with multiplicity $k_l \geq k_0$, and hence $\rho(w) \geq \lambda(1/w) \geq 1$.

Case (ii): $k_1 < k_0$. Then from (3.3), $z_4 - 1$ is a pole of $w(z)$ with multiplicity $k_{-1} = 2k_0 - k_1 > k_0$. Similarly, we can easily prove that for all $l \in \mathbb{N}$, $z_4 - l$ is a pole of $w(z)$ with multiplicity $k_{-1} \geq k_0$, and hence $\rho(w) \geq \lambda(1/w) \geq 1$.

Suppose that $w(z)$ has a zero. With a similar reasoning above, we can prove that $\rho(w) \geq \lambda(w) \geq 1$.

Subcase 5.2: $h(z)$ is a nonconstant rational function. As reasoning in the subcase 1.2, we see that $w(z)$ is a transcendental meromorphic function. Applying Lemma 2.2, we can write $w(z)$ as the form

$$w(z) = z^k \frac{P_1(z)}{P_2(z)} e^{Q(z)}, \tag{3.4}$$

where $P_1(z), P_2(z)$ are entire functions and the canonical products of w formed with the non-null zeros and poles of w, respectively, such that $\rho(P_1) = \lambda(P_1) = \lambda(w)$, $\rho(P_2) = \lambda(P_2) = \lambda(1/w)$, and $Q(z)$ is a polynomial such that $\deg Q(z) = q \leq \rho$.

Suppose that $w(z)$ has finitely many zeros and poles. Then both $P_1(z)$ and $P_2(z)$ are polynomials, and $Q(z)$ must be a nonconstant polynomial.This yields that $\rho(w) = \rho(e^Q) \geq 1$.

Suppose that $w(z)$ has infinitely many zeros or poles. Note that $h(z)$ has finitely many zeros and poles, and we can prove that either $\rho(w) \geq \lambda(1/w) \geq 1$ or $\rho(w) \geq \lambda(w) \geq 1$ by using the same ideas in the subcase 1.2 and subcase 5.1.

Combining Theorem A and Theorem 3.1, we get

Corollary 3.1. *Suppose that $h(z)$ is a nonconstant rational function. If $w(z)$ is a nonconstant meromorphic solution with finite order of (1.3), where $m = -2, -1, 0, 1$, then $\lambda(w) = \lambda(1/w) = \lambda(\Delta w) = \lambda(1/\Delta w) = \lambda(\Delta w/w) = \lambda(w/\Delta w) = \rho(w) \geq 1$.*

4. Results on the Question 2

Theorem 4.1. *If $w(z)$ is a nonconstant meromorphic solution with finite order of (1.3), where $m = -2, -1, 0, 1$ and $h(z)$ is a nonzero constant, then*

(i) $\lambda(1/w) = \lambda(w) = \lambda(1/\Delta w) = \lambda(w/\Delta w) = \rho(w) \geq 1$;

(ii) $\lambda(\Delta w/w) = \rho(w) \Rightarrow \lambda(\Delta w) = \rho(w)$, and if $\hat{N}_1(r, w) = S(r, w)$, then $\lambda(\Delta w) = \rho(w) \Rightarrow \lambda(\Delta w/w) = \rho(w)$.

Remark 3. We are sorry that we fail to prove that $\lambda(\Delta w/w) = \lambda(\Delta w) = \rho(w)$ or negate it with some counterexamples directly. For the proof of Theorem 4.1, from Theorem A, Theorem 3.1 and Lemma 2.3, we only need to prove that $\lambda(1/\Delta w) = \lambda(w/\Delta w) = \rho(w)$. However, since the idea to prove it is mainly due to [13], all details are omitted here.

Theorem 4.2. *If $w(z)$ is a nonconstant meromorphic solution with finite order of (1.3), where $m = 2$ and $h(z)$ is a nonzero constant, then*

$$\max\{\lambda(1/\Delta w), \lambda(\Delta w/w), \lambda(\Delta w), \lambda(w/\Delta w)\} \geq \rho(w) - 1.$$

More precisely, one or the following case holds:

(i) $w(z)$ has no zeros and poles, $\rho(w) \in \{1, 2\}$ and

$$\lambda(\Delta w) = \lambda(\Delta w/w) = \rho(w) - 1.$$

(ii) $w(z)$ has at least one pole but no zeros, and

$$\max\{\lambda(1/\Delta w), \lambda(\Delta w/w)\} \geq \rho(w) - 1.$$

(iii) $w(z)$ has at least one zero but no poles, and

$$\max\{\lambda(\Delta w), \lambda(w/\Delta w)\} \geq \rho(w) - 1.$$

(iv) $w(z)$ has at least one zero and at least one pole, and

$$\max\{\lambda(1/\Delta w), \lambda(\Delta w/w), \lambda(\Delta w), \lambda(w/\Delta w)\} \geq \rho(w) - 1.$$

Proof. We will use (3.3) and (3.4) by writing $h(z) = h \neq 0$ and denoting

$$Q_1(z) = Q(z+1) + Q(z-1) - 2Q(z) \tag{4.1}$$

and $d = \deg Q(z), d_1 = \deg Q_1(z)$. With a simple calculation, we can see that: (i) $d = 0, 1, 2 \Rightarrow d_1 = 0$; (ii) $d \geq 3 \Rightarrow d_1 = d - 2$. We discuss case by case in the following.

Case 1: $w(z)$ has no zeros and poles. Then $\lambda(w) = \lambda(1/w) = 0$. From (3.4), $w(z) = e^{Q(z)}$, where $Q(z)$ is a nonconstant polynomial. Submitting it and (4.1) into (3.3), we obtain

$$e^{Q_1(z)} = h.$$

Since h is a constant, $Q_1(z)$ must be a constant. This indicates that $Q(z)$ must be of degree $\deg Q(z) \in \{1, 2\}$. Thus $\rho(w) \in \{1, 2\}$.

Notice that

$$\Delta w(z) = (e^{Q(z+1)-Q(z)} - 1)e^{Q(z)}, \;\; \Delta w(z)/w(z) = e^{Q(z+1)-Q(z)} - 1.$$

We can deduce that $\lambda(\Delta w) = \lambda(\Delta w/w) = d - 1 = \rho(w) - 1$.

Case 2: $w(z)$ has at least one pole but no zeros. Then $\lambda(w) = 0$. From (4.4), $w(z) = z^k e^{Q(z)}/P_2(z)$, where $Q(z)$ is a polynomial.Submitting it and (4.1) into (3.3), we get

$$\frac{(z+1)^k(z-1)^k P_2^2(z)}{z^{2k}P_2(z+1)P_2(z-1)} = he^{-Q_1(z)}. \tag{4.2}$$

This implies that $P_2(z) = 0 \Leftrightarrow P_2(z+1) = 0 \Leftrightarrow P_2(z-1) = 0$ except $S(r, P_2)$ many points. Keep in mind that

$$\Delta w(z) = \frac{(z+1)^k P_2(z) e^{Q(z+1)-Q(z)} - z^k P_2(z+1)}{P_2(z)P_2(z+1)} e^{Q(z)},$$

$$\frac{w(z)}{\Delta w(z)} = \frac{z^k P_2(z+1)}{(z+1)^k P_2(z) e^{Q(z+1)-Q(z)} - z^k P_2(z+1)}.$$

Obviously, each pole of $\Delta w(z)$ must be a zero of $P_2(z)$ or $P_2(z+1)$, and $N(r, \Delta w/w) \leq N(r, \Delta w)$.

If $\lambda(1/\Delta w) < \lambda(P_2)$, then $N(r, \Delta w) = S(r, P_2)$ and hence

$$N(r, 1/P_2) = N(r, w) \leq N(r, \Delta w) + N(r, w/\Delta w) = N(r, w/\Delta w) + S(r, P_2),$$

which yields that $\lambda(\Delta w/w) \geq \lambda(P_2)$. Therefore,

$$\max\{\lambda(1/\Delta w), \lambda(\Delta w/w)\} \geq \lambda(P_2) \tag{4.3}$$

always holds.

By lemma 2.4, for any given $\varepsilon > 0$, we have

$$\begin{aligned} T(r, e^{-Q_1(z)}) &= m(r, e^{-Q_1(z)}) \\ &= m\left(r, \frac{(z+1)^k(z-1)^k P_2^2(z)}{z^{2k}P_2(z+1)P_2(z-1)}\right) = O(r^{\rho(P_2)-1+\varepsilon}). \end{aligned} \tag{4.4}$$

Noting now from (4.4), if $d = 0, 1$, then $\rho(P_2) \geq 1$; if $d \geq 2$, then $\rho(P_2) \geq d - 1$. Therefore,

If $d = 0, 1$, then $\rho(P_2) = \rho(w)$. By lemma 2.3, we get from (4.3) that

$$\max\{\lambda(1/\Delta w), \lambda(\Delta w/w)\} = \lambda(P_2) = \rho(P_2) = \rho(w).$$

If $d \geq 2$, then $\rho(w) = \max\{d, \rho(P_2)\}$. Since $\rho(P_2) \geq d - 1$, we get

$$\max\{\lambda(1/\Delta w), \lambda(\Delta w/w)\} \geq \lambda(P_2) \geq \rho(w) - 1. \tag{4.5}$$

Case 3: $w(z)$ has at least one zero but no poles. Then with a similar arguing as in the above case 2, we can get

$$\max\{\lambda(\Delta w), \lambda(w/\Delta w)\} \geq \lambda(P_1) \geq \rho(w) - 1.$$

Case 4: $w(z)$ has at least one zero and one pole. Turn back to the subcase 5.1 in the proof Theorem 3.1, we see that $w(z)$ must have infinitely many zeros and poles such that $\rho(P_1) = \lambda(P_1) = \lambda(w) \geq 1, \rho(P_2) = \lambda(P_2) = \lambda(1/w) \geq 1$. From (3.3) and (4.1), we have

$$\frac{(z+1)^k(z-1)^k P_2^2(z) P_1(z+1) P_1(z-1)}{z^{2k} P_1^2(z) P_2(z+1) P_2(z-1)} = he^{-Q_1(z)}. \tag{4.6}$$

Set $\rho_{1,2} = \max\{\rho(P_1), \rho(P_2)\}$. Applying Lemma 2.4, we see that

$$\begin{aligned} T(r, e^{-Q_1(z)}) &= m(r, e^{-Q_1(z)}) \\ &= m\left(r, \frac{(z+1)^k(z-1)^k P_2^2(z) P_1(z+1) P_1(z-1)}{z^{2k} P_1^2(z) P_2(z+1) P_2(z-1)}\right) = O(r^{\rho_{1,2}-1+\varepsilon}). \end{aligned}$$

Thus, if $d = 0, 1$, then

$$\max\{\lambda(1/\Delta w), \lambda(\Delta w/w), \lambda(\Delta w), \lambda(w/\Delta w)\} = \rho_{1,2} = \rho(w).$$

If $d \geq 2$, then $\rho(w) = \max\{d, \rho_{1,2}\}$. Since $\rho_{1,2} \geq d - 1$, we get

$$\max\{\lambda(1/\Delta w), \lambda(\Delta w/w), \lambda(\Delta w), \lambda(w/\Delta w)\} \geq \rho_{1,2} \geq \rho(w) - 1.$$

Theorem 4.3. *If $w(z)$ is a nonconstant meromorphic solution with finite order of (1.3), where $m = 2$ and $h(z)$ is a nonconstant rational function, then*

$$\max\{\lambda(1/\Delta w), \lambda(\Delta w/w), \lambda(\Delta w), \lambda(w/\Delta w)\} \geq \rho(w) - 1.$$

More precisely, one or the following case holds:

(i) $w(z)$ has finitely many zeros and finitely many poles, $\rho(w) \in \{1, 2\}$ and

$$\lambda(\Delta w) = \lambda(\Delta w/w) = \rho(w) - 1.$$

(ii) $w(z)$ has finitely many zeros and infinitely many poles, and

$$\max\{\lambda(1/\Delta w), \lambda(\Delta w/w)\} \geq \rho(w) - 1.$$

(iii) $w(z)$ has infinitely many zeros and finitely many poles, and

$$\max\{\lambda(\Delta w), \lambda(w/\Delta w)\} \geq \rho(w) - 1.$$

(iv) $w(z)$ has infinitely many zeros and infinitely many poles, and

$$\max\{\lambda(1/\Delta w), \lambda(\Delta w/w), \lambda(\Delta w), \lambda(w/\Delta w)\} \geq \rho(w) - 1.$$

Remark 4. We omit the proof of Theorem 4.3 because it is quite similar as the proof of Theorem 4.2.

Example 2. (1) $w_1(z) = e^{z^2}, w_2(z) = e^{z^2}/\sin(\pi z), w_3(z) = \sin(\pi z)e^{z^2}, w_4(z) = e^{z^2}\tan(\pi z)$ satisfy the equation

$$w(z+1)w(z-1) = e^2 w^2(z)$$

and the conclusion (i)-(iv) in Theorem 4.2 respectively.

(2) $w_1(z) = ze^{z^2}, w_2(z) = ze^{z^2}/\sin(\pi z), w_3(z) = z\sin(\pi z)e^{z^2}, w_4(z) = ze^{z^2}\tan(\pi z)$ satisfy the equation

$$w(z+1)w(z-1) = e^2 \frac{z^2-1}{z^2} w^2(z)$$

and the conclusion (i)-(iv) in Theorem 4.3 respectively.

Remark 5. In the examples above, $h(z)$ is either a nonzero constant or a non-polynomial rational function. We try hard but fail to give corresponding example with a nonconstant polynomial.

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