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SUMMATION FORMULAS FOR THE CONFLUENT HYPERGEOMETRIC FUNCTION $\Psi_2^{(2r)}$ OF SEVERAL VARIABLES

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Abstract. The aim of this paper is to establish a general summation formulas for the confluent hypergeometric function $\Psi_2^{(2r)}$ in several variables by applying the generalized Kummer's summation theorem due to Lavoie *et al*. [5]. Further, a number of generating functions for products of two Laguerre polynomials of two variables are also derived as an applications of our main summation formula.

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1. Introduction

The generalized hypergeometric function ${}_pF_q$ with p numerator parameters and q denominator parameters is defined by (see [10, p.42 (1)])

$$ {}_pF_q\left[\begin{array}{c} a_1,\ldots,a_p\ ; \\ \qquad x \\ b_1,\ldots,b_q\ ; \end{array}\right]=\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\,\frac{x^n}{n!}, \tag{1.1}$$

where $(a)_n$ denotes the Pochhammer's symbol defined by

$$(a)_n=\begin{cases} 1 & , if\ n=0 \\ a(a+1)(a+2)\ldots(a+n-1) & , if\ n=1,2,3,\ldots \end{cases} \tag{1.2}$$

The special case of (1) when $p=2$ and $q=1$ is usually called Gauss's hypergeometric function.

The Kampé de Fériet function of two variables $F^{p:q;k}_{l:m;n}[x,y]$ is defined by (see[10,p.63(16)]

$$F^{p:q;k}_{l:m;n}\left[\begin{array}{c} (a_p):(b_q)\,;(c_k)\,; \\ \qquad x\ ,\ y \\ (\alpha_l):(\beta_m)\,;(\gamma_n)\,; \end{array}\right]=\sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\,\frac{x^r}{r!}\,\frac{y^s}{s!}. \tag{1.3}$$

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The confluent hypergeometric function of several variables $\Psi_2^{(n)}$ is defined as follows (see [10, p.62 (11)]):

$$\Psi_2^{(n)}(a;c_1,\dots,c_n;x_1,\dots,x_n)=\sum_{m_1,\dots,m_n=0}^{\infty}\frac{(a)_{m_1+\dots+m_n}}{(c_1)_{m_1}\dots(c_n)_{m_n}}\frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!} \tag{1.4}$$

Clearly, we have

$$\Psi_2^{(2)}=\Psi_2,$$

where Ψ_2 is the Humbert's confluent hypergeometric function in two variables [10,p.59(42)]

$$\Psi_2[a;b,b';x,y]=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}}{(b)_m(b')_n}\frac{x^m}{m!}\frac{y^n}{n!}. \tag{1.5}$$

Ragab [7] defined Laguerre polynomials of two variables $L_n^{(\alpha,\beta)}(x,y)$ as follows:

$$L_n^{(\alpha,\beta)}(x,y)=\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!}\sum_{r=0}^{n}\frac{(-y)^r L_{n-r}^{(\alpha)}(x)}{r!\Gamma(\alpha+n-r+1)\Gamma(\beta+r+1)}, \tag{1.6}$$

where $L_n^{(\alpha)}(x)$ are the generalized Laguerre polynomials of one variable [8, p.200 (1)].

In particular, we note that

$$L_n^{(\alpha,\beta)}(x,0)=\frac{(\beta+1)_n}{n!}L_n^{(\alpha)}(x)\ and\ L_n^{(\alpha,\beta)}(0,y)=\frac{(\alpha+1)_n}{n!}L_n^{(\beta)}(y). \tag{1.7}$$

Recently, Choi and Rathie [3] obtained certain summation formula for Humbert's hypergeometric function $\Psi_2[a;c,c+i;x,-x]$ by using the following generalizations of the Kummer's summation theorem due to Lavoie *et al*. [5]:

$$\begin{aligned}{}_2F_1\left[\begin{matrix}a\ ,\ b & ; \\ 1+a-b+i & ;\end{matrix}\ -1\right] &= \frac{\Gamma(\frac{1}{2})\Gamma(1+a-b+i)\Gamma(1-b)}{2^a\Gamma(1-b+\frac{1}{2}(i+|i|))} \\ &\times\left\{\frac{A_i}{\Gamma(\frac{1}{2}a+\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])\Gamma(1+\frac{1}{2}a-b+\frac{1}{2}i)}+\frac{B_i}{\Gamma(\frac{1}{2}a+\frac{1}{2}i-[\frac{i}{2}])\Gamma(\frac{1}{2}+\frac{1}{2}a-b+\frac{1}{2}i)}\right\}\end{aligned} \tag{1.8}$$

for $(i=0,\pm1,\pm2,\pm3,\pm4,\pm5)$,

where $[x]$ denotes the greatest integer less than or equal to x and $|x|$ denotes the usual absolute value of x. The coefficients A_i and B_i are given in the following table:

TABLE 1.

i	A_i	B_i
5	$-4(6+a-b)^2+2b(6+a-b)$ $+b^2+22(6+a-b)-13b-20$	$4(6+a-b)^2+2b(6+a-b)$ $-b^2-34(6+a-b)-b+62$
4	$2(a-b+3)(1+a-b)-(b-1)(b-4)$	$-4(a-b+2)$
3	$3b-2a-5$	$2a-b+1$
2	$1+a-b$	-2
1	-1	1
0	1	0
-1	1	1
-2	$a-b-1$	2
-3	$2a-3b-4$	$2a-b-2$
-4	$2(a-b-3)(a-b-1)-b(b+3)$	$4(a-b-2)$
-5	$4(a-b-4)^2-2b(a-b-4)$ $-b^2+8(a-b-4)-7b$	$4(a-b-4)^2+2b(a-b-4)$ $-b^2+16(a-b-4)-b+12$

Here, in this paper we establish further extension formula for the confluent hypergeometric function $\Psi_2^{(2r)}$ in several variables with the help of (1.8). As an application of our main result, we obtain certain generating functions for products of two Laguerre polynomials of one and two variable.

2. MAIN RESULTS

In this section, the following summation formula will be established:

$$
\begin{aligned}
&\Psi_2^{(2r)}(a;c_1,c_1+i,c_2,c_2+i,\ldots,c_r,c_r+i;x_1,-x_1,x_2,-x_2,\ldots,x_r,-x_r)\\
&=\sum_{m_1,\ldots,m_r=0}^{\infty}\frac{(a)_{2m_1+\ldots+2m_r}x_1^{2m_1}\ldots x_r^{2m_r}}{(c_1)_{2m_1}\ldots(c_r)_{2m_r}(2m_1)!\ldots(2m_r)!}\\
&\times(A_i^{(1)}E_1+B_i^{(1)}F_1)(A_i^{(2)}E_2+B_i^{(2)}F_2)\ldots(A_i^{(r)}E_r+B_i^{(r)}F_r)\\
&+\sum_{m_1,\ldots,m_r=0}^{\infty}\frac{(a)_{2m_1+1+2m_2+\ldots+2m_r}x_1^{2m_1+1}x_2^{2m_2}\ldots x_r^{2m_r}}{(c_1)_{2m_1+1}(c_2)_{2m_2}\ldots(c_r)_{2m_r}(2m_1+1)!(2m_2)!\ldots(2m_r)!}\\
&\times(C_i^{(1)}G_1+D_i^{(1)}H_1)(A_i^{(2)}E_2+B_i^{(2)}F_2)\ldots(A_i^{(r)}E_r+B_i^{(r)}F_r)+\ldots
\end{aligned}
$$

$$...+\sum_{m_1,...,m_r=0}^{\infty}\frac{(a)_{2m_1+2m_2+1+...+2m_r+1}x_1^{2m_1}x_2^{2m_2+1}...x_r^{2m_r+1}}{(c_1)_{2m_1}(c_2)_{2m_2+1}...(c_r)_{2m_r+1}(2m_1)!(2m_2+1)!...(2m_r+1)!}$$

$$\times(A_i^{(1)}E_1+B_i^{(1)}F_1)(C_i^{(2)}G_2+D_i^{(2)}H_2)...(C_i^{(r)}G_r+D_i^{(r)}H_r)$$

$$+\sum_{m_1,...,m_r=0}^{\infty}\frac{(a)_{2m_1+1+...+2m_r+1}x_1^{2m_1+1}...x_r^{2m_r+1}}{(c_1)_{2m_1+1}...(c_r)_{2m_r+1}(2m_1+1)!...(2m_r+1)!}$$

(2.1) $$\times(C_i^{(1)}G_1+D_i^{(1)}H_1)(C_i^{(2)}G_2+D_i^{(2)}H_2)...(C_i^{(r)}G_r+D_i^{(r)}H_r),$$

where

$$E_r=\frac{2^{2m_r}\Gamma(\frac{1}{2})\Gamma(c_r+i)\Gamma(c_r+2m_r)}{\Gamma(c_r+2m_r+\frac{1}{2}(i+|i|))\Gamma(-m_r+\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])\Gamma(m_r+c_r+\frac{1}{2}i)}$$

$$F_r=\frac{2^{2m_r}\Gamma(\frac{1}{2})\Gamma(c_r+i)\Gamma(c_r+2m_r)}{\Gamma(c_r+2m_r+\frac{1}{2}(i+|i|))\Gamma(-m_r+\frac{1}{2}i-[\frac{i}{2}])\Gamma(m_r+c_r-\frac{1}{2}+\frac{1}{2}i)}$$

$$G_r=\frac{2^{2m_r+1}\Gamma(\frac{1}{2})\Gamma(c_r+i)\Gamma(c_r+2m_r+1)}{\Gamma(c_r+2m_r+1+\frac{1}{2}(i+|i|))\Gamma(-m_r+\frac{1}{2}i-[\frac{1+i}{2}])\Gamma(m_r+\frac{1}{2}+c_r+\frac{1}{2}i)}$$

$$H_r=\frac{2^{2m_r+1}\Gamma(\frac{1}{2})\Gamma(c_r+i)\Gamma(c_r+2m_r+1)}{\Gamma(c_r+2m_r+1+\frac{1}{2}(i+|i|))\Gamma(-m_r-\frac{1}{2}+\frac{1}{2}i-[\frac{i}{2}])\Gamma(m_r+c_r+\frac{1}{2}i)}$$

for $i=0,\pm1,\pm2,\pm3,\pm4,\pm5$.

The coefficients $A_i^{(1)},...,A_i^{(r)}$ and $B_i^{(1)},...,B_i^{(r)}$ can be obtained from the tables of A_i and B_i by replacing a and b by $-2m_j$ and $1-c_j-2m_j$, $j=1,2,...,r$. The coefficients $C_i^{(1)},...,C_i^{(r)}$ and $D_i^{(1)},...,D_i^{(r)}$ can be obtained from the tables of A_i and B_i by replacing a and b by $-2m_j-1$ and $-c_j-2m_j$, $j=1,2,...,r$.

Proof. Denoting the left hand side of (2.1) by $\Psi_2^{(2r)}$, then from the definition (1.4), we have

$$\Psi_2^{(2r)}=\sum_{n_1,p_1,...,n_r,p_r=0}^{\infty}\frac{(a)_{n_1+p_1+...+n_r+p_r}x_1^{n_1}(-x_1)^{p_1}...x_r^{n_r}(-x_r)^{p_r}}{(c_1)_{n_1}(c_1+i)_{p_1}...(c_r)_{n_r}(c_r+i)_{p_r}n_1!p_1!...n_r!p_r!}$$

Using the well-known result[10,p.22(20)]

(2.2) $$(\alpha)_{m+n}=(\alpha)_m(\alpha+m)_n,$$

we have

$$\Psi_2^{(2r)}=\sum_{n_2,p_2,...,n_r,p_r=0}^{\infty}\frac{(a)_{n_2+p_2+...+n_r+p_r}x_2^{n_2}(-x_2)^{p_2}...x_r^{n_r}(-x_r)^{p_r}}{(c_2)_{n_2}(c_2+i)_{p_2}...(c_r)_{n_r}(c_r+i)_{p_r}n_2!p_2!...n_r!p_r!}$$

(2.3) $$\times\Psi_2[a+n_2+p_2+...+n_r+p_r;c_1,c_1+i;x_1,-x_1]$$

Now, applying the following special case of Manako identity [6]

(2.4) $$\Psi_2(a;b,c;x,-x)=\sum_{m=0}^{\infty}\frac{(a)_m x^m}{(b)_m m!}{}_2F_1\left[\begin{matrix}-m\ ,\ 1-b-m\ ; & \\ & -1\\ c\ ; & \end{matrix}\right],$$

we have

$$\Psi_2^{(2r)}=\sum_{m_1,n_2,p_2,...,n_r,p_r=0}^{\infty}\frac{(a)_{m_1+n_2+p_2+...+n_r+p_r}x_1^{m_1}x_2^{n_2}(-x_2)^{p_2}...x_r^{n_r}(-x_r)^{p_r}}{(c_1)_{m_1}(c_2)_{n_2}(c_2+i)_{p_2}...(c_r)_{n_r}(c_r+i)_{p_r}m_1!n_2!p_2!...n_r!p_r!}$$

$$\times{}_2F_1\left[\begin{matrix}-m_1\ ,\ 1-c_1-m_1\ ; \\ & -1 \\ c_1+i\ ;\end{matrix}\right] \tag{2.5}$$

Then, by repeating the above steps r-times, we have

$$\Psi_2^{(2r)}=\sum_{m_1,...,m_r=0}^{\infty}\frac{(a)_{m_1+...+m_r}x_1^{m_1}...x_r^{m_r}}{(c_1)_{m_1}...(c_r)_{m_r}m_1!...m_r!}f(c_1,i,m_1)...f(c_r,i,m_r), \tag{2.6}$$

where

$$f(c_r,i,m_r)={}_2F_1\left[\begin{matrix}-m_r\ ,\ 1-c_r-m_r\ ; \\ & -1 \\ c_r+i\ ;\end{matrix}\right]$$

Separating (2.6) into its even and odd terms, we have

$$\begin{aligned}\Psi_2^{(2r)}=&\sum_{m_1,...,m_r=0}^{\infty}\frac{(a)_{2m_1+...+2m_r}x_1^{2m_1}...x_r^{2m_r}}{(c_1)_{2m_1}...(c_r)_{2m_r}(2m_1)!...(2m_r)!}\\ &\times f(c_1,i,2m_1)f(c_2,i,2m_2)...f(c_r,i,2m_r)\\ &+\sum_{m_1,...,m_r=0}^{\infty}\frac{(a)_{2m_1+1+2m_2+...+2m_r}x_1^{2m_1+1}x_2^{2m_2}...x_r^{2m_r}}{(c_1)_{2m_1+1}(c_2)_{2m_2}...(c_r)_{2m_r}(2m_1+1)!(2m_2)!...(2m_r)!}\\ &\times f(c_1,i,2m_1+1)f(c_2,i,2m_2)...f(c_r,i,2m_r)+...\\ &...+\sum_{m_1,...,m_r=0}^{\infty}\frac{(a)_{2m_1+2m_2+1+...+2m_r+1}x_1^{2m_1}x_2^{2m_2+1}...x_r^{2m_r+1}}{(c_1)_{2m_1}(c_2)_{2m_2+1}...(c_r)_{2m_r+1}(2m_1)!(2m_2+1)!...(2m_r+1)!}\\ &\times f(c_1,i,2m_1)f(c_2,i,2m_2+1)...f(c_r,i,2m_r+1)\\ &+\sum_{m_1,...,m_r=0}^{\infty}\frac{(a)_{2m_1+1+...+2m_r+1}x_1^{2m_1+1}...x_r^{2m_r+1}}{(c_1)_{2m_1+1}...(c_r)_{2m_r+1}(2m_1+1)!...(2m_r+1)!}\\ &\times f(c_1,i,2m_1+1)f(c_2,i,2m_2+1)...f(c_r,i,2m_r+1)\end{aligned} \tag{2.7}$$

Now, in (2.7) applying the generalized Kummer's theorem (1.8) for each ${}_2F_1(-1)$, we get after some simplification the right hand side of (2.1).This completes the proof of (2.1). □

2.1. **Special Cases of (2.1).** Here we mention some special cases of (2.1) and we will use the following results [10,p.22-23]:

$$\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}=(\alpha)_n,\ \frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}=\frac{(-1)^n}{(1-\alpha)_n} \tag{2.8}$$

$$(\alpha)_{2n}=2^{2n}\left(\frac{1}{2}\alpha\right)_n\left(\frac{1}{2}\alpha+\frac{1}{2}\right)_n \tag{2.9}$$

$$(2n)!=2^{2n}\left(\frac{1}{2}\right)_n n!,\ (2n+1)!=2^{2n}\left(\frac{3}{2}\right)_n n!. \tag{2.10}$$

1.Taking $i=0$ in (2.1), we have

$$\Psi_2^{(2r)}(a;c_1,c_1,\ldots,c_r,c_r;x_1,-x_1,\ldots,x_r,-x_r)=\sum_{m_1,\ldots,m_r=0}^{\infty}\frac{(a)_{2m_1+\ldots+2m_r}x_1^{2m_1}\ldots x_r^{2m_r}}{(c_1)_{2m_1}\ldots(c_r)_{2m_r}(2m_1)!\ldots(2m_r)!}$$
$$\times\frac{2^{2m_1}\Gamma(\frac{1}{2})\Gamma(c_1)}{\Gamma(\frac{1}{2}-m_1)\Gamma(c_1+m_1)}\times\ldots\times\frac{2^{2m_r}\Gamma(\frac{1}{2})\Gamma(c_r)}{\Gamma(\frac{1}{2}-m_r)\Gamma(c_r+m_r)} \tag{2.11}$$

Now using the results (2.8)–(2.10) in (2.11), then after some simplification we obtain the following transformation formula:

$$\Psi_2^{(2r)}(a;c_1,c_1,\ldots,c_r,c_r;x_1,-x_1,\ldots,x_r,-x_r)$$
$$=F\,{}^{2:0;\ldots;0}_{0:3;\ldots;3}\left[\begin{array}{c} \frac{1}{2}a,\frac{1}{2}a+\frac{1}{2}\ :\quad - \quad;\ \ldots\ ;\quad - \quad; \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad -x_1^2,\ldots,-x_r^2 \\ -\quad :\ c_1,\frac{1}{2}c_1,\frac{1}{2}c_1+\frac{1}{2}\ ;\ \ldots\ ;\ c_r,\frac{1}{2}c_r,\frac{1}{2}c_r+\frac{1}{2}\ ; \end{array}\right]. \tag{2.12}$$

Note that, the formula (2.12) is a generalization of the well- known result [9, p.322(188)]

$$\Psi_2(a;c,c;x,-x)={}_2F_3\left[\begin{array}{c} \frac{1}{2}a,\frac{1}{2}a+\frac{1}{2}\ ; \\ \qquad\qquad -x^2 \\ c,\frac{1}{2}c,\frac{1}{2}c+\frac{1}{2}\ ; \end{array}\right]. \tag{2.13}$$

II. Taking $r=1$ in (2.1), we get a known result of Choi and Rathie [3] for Humbert's hypergeometric series $\Psi_2(a;c,c+i;x,-x)$ for $i=0,\pm1,\pm2,\pm3,\pm4,\pm5$.

3. Applications to Generating Functions

The following double generating function for products of two Laguerre polynomials of two variables is given by Atash [1] :

$$\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}m!n!t^m(-t)^n}{(d_1)_m(d_2)_m(c_1)_n(c_2)_n}L_m^{(d_1-1,d_2-1)}(x_1,x_2)L_n^{(c_1-1,c_2-1)}(x_1,x_2)$$
$$=\Psi_2^{(4)}[a;c_1,d_1,c_2,d_2;x_1t,-x_1t,x_2t,-x_2t] \tag{3.1}$$

In (3.1), replacing d_1 by c_1+i and d_2 by c_2+i and applying (2.1) with $r=2$, we get the following generating function:

$$\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}m!n!t^m(-t)^n}{(c_1+i)_m(c_2+i)_m(c_1)_n(c_2)_n}L_m^{(c_1+i-1,c_2+i-1)}(x_1,x_2)L_n^{(c_1-1,c_2-1)}(x_1,x_2)$$
$$=\sum_{m_1,m_2=0}^{\infty}\frac{(a)_{2m_1+2m_2}(x_1t)^{2m_1}(x_2t)^{2m_2}}{(c_1)_{2m_1}(c_2)_{2m_2}(2m_1)!(2m_2)!}(A_i^{(1)}E_1+B_i^{(1)}F_1)(A_i^{(2)}E_2+B_i^{(2)}F_2)$$
$$+\sum_{m_1,m_2=0}^{\infty}\frac{(a)_{2m_1+2m_2+1}(x_1t)^{2m_1+1}(x_2t)^{2m_2}}{(c_1)_{2m_1+1}(c_2)_{2m_2}(2m_1+1)!(2m_2)!}(C_i^{(1)}G_1+D_i^{(1)}H_1)(A_i^{(2)}E_2+B_i^{(2)}F_2)$$
$$+\sum_{m_1,m_2=0}^{\infty}\frac{(a)_{2m_1+2m_2+1}(x_1t)^{2m_1}(x_2t)^{2m_2+1}}{(c_1)_{2m_1}(c_2)_{2m_2+1}(2m_1)!(2m_2+1)!}(A_i^{(1)}E_1+B_i^{(1)}F_1)(C_i^{(2)}G_2+D_i^{(2)}H_2)$$
$$+\sum_{m_1,m_2=0}^{\infty}\frac{(a)_{2m_1+2m_2+2}(x_1t)^{2m_1+1}(x_2t)^{2m_2+1}}{(c_1)_{2m_1+1}(c_2)_{2m_2+1}(2m_1+1)!(2m_2+1)!}(C_i^{(1)}G_1+D_i^{(1)}H_1)(C_i^{(2)}G_2+D_i^{(2)}H_2) \tag{3.2}$$

for $i=0,\pm1,\pm2,\pm3,\pm4,\pm5$.

3.1. **Special cases of (3.2).** Eleven new generating functions for products of two Laguerre polynomials of two variables can be obtained as special cases of (3.2). Here we will mention only the following cases:

I.Taking $i{=}0$ in (3.2) and using the results (2.8)–(2.10), we get the following generating function:

$$\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}m!n!t^m(-t)^n}{(c_1)_m(c_2)_m(c_1)_n(c_2)_n}L_m^{(c_1-1,c_2-1)}(x_1,x_2)L_n^{(c_1-1,c_2-1)}(x_1,x_2)$$

$$=F\,{}^{2:0;0}_{0:3;3}\left[\begin{matrix}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2}: & - & ; & - & ; \\ & & & & & -x_1^2t^2,-x_2^2t^2 \\ -: & c_1,\frac{1}{2}c_1,\frac{1}{2}c_1+\frac{1}{2} & ; & c_2,\frac{1}{2}c_2,\frac{1}{2}c_2+\frac{1}{2} & ; \end{matrix}\right]. \tag{3.3}$$

Further, putting $x_2{=}0$ in (3.3) and using (1.7), we get:

$$\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}t^m(-t)^n}{(c)_m(c)_n}L_m^{(c-1)}(x)L_n^{(c-1)}(x)={}_2F_3\left[\begin{matrix}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2} & ; & \\ & & -x^2t^2 \\ c,\frac{1}{2}c,\frac{1}{2}c+\frac{1}{2} & ; & \end{matrix}\right]. \tag{3.4}$$

Note that, in (3.4) if we take $a{=}c$, we get a known result of Exton [4, p.406 (4.3)].

II. Taking $i{=}1$ in (3.2) and using the results (2.8)–(2.10), we get the following generating function:

$$\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}m!n!t^m(-t)^n}{(c_1+1)_m(c_2+1)_m(c_1)_n(c_2)_n}L_m^{(c_1,c_2)}(x_1,x_2)L_n^{(c_1-1,c_2-1)}(x_1,x_2)$$

$$=F\,{}^{2:0;0}_{0:3;3}\left[\begin{matrix}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2}: & - & ; & - & ; \\ & & & & & -x_1^2t^2,-x_2^2t^2 \\ -: & c_1,\frac{1}{2}c_1+\frac{1}{2},\frac{1}{2}c_1+1 & ; & c_2,\frac{1}{2}c_2+\frac{1}{2},\frac{1}{2}c_2+1 & ; \end{matrix}\right]$$

$$+\frac{ax_1t}{c_1(c_1+1)}F\,{}^{2:0;0}_{0:3;3}\left[\begin{matrix}\frac{1}{2}a+\frac{1}{2},\frac{1}{2}a+1: & - & ; & - & ; \\ & & & & & -x_1^2t^2,-x_2^2t^2 \\ -: & c_1+1,\frac{1}{2}c_1+1,\frac{1}{2}c_1+\frac{3}{2} & ; & c_2,\frac{1}{2}c_2+\frac{1}{2},\frac{1}{2}c_2+1 & ; \end{matrix}\right]$$

$$+\frac{ax_2t}{c_2(c_2+1)}F\,{}^{2:0;0}_{0:3;3}\left[\begin{matrix}\frac{1}{2}a+\frac{1}{2},\frac{1}{2}a+1: & - & ; & - & ; \\ & & & & & -x_1^2t^2,-x_2^2t^2 \\ -: & c_1,\frac{1}{2}c_1+\frac{1}{2},\frac{1}{2}c_1+1 & ; & c_2+1,\frac{1}{2}c_2+1,\frac{1}{2}c_2+\frac{3}{2} & ; \end{matrix}\right]$$

$$+\frac{a(a+1)x_1x_2t^2}{c_1c_2(c_1+1)(c_2+1)}$$

$$\times F\,{}^{2:0;0}_{0:3;3}\left[\begin{matrix}\frac{1}{2}a+1,\frac{1}{2}a+\frac{3}{2}: & - & ; & - & ; \\ & & & & & -x_1^2t^2,-x_2^2t^2 \\ -: & c_1+1,\frac{1}{2}c_1+1,\frac{1}{2}c_1+\frac{3}{2} & ; & c_2+1,\frac{1}{2}c_2+1,\frac{1}{2}c_2+\frac{3}{2} & ; \end{matrix}\right]. \tag{3.5}$$

Further, putting $x_2{=}0$ in (3.5) and using (1.7), we get:

$$\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}t^m(-t)^n}{(c+1)_m(c)_n}L_m^{(c)}(x)L_n^{(c-1)}(x)$$

$$={}_2F_3\left[\begin{matrix}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2} & ; & \\ & & -x^2t^2 \\ c,\frac{1}{2}c+\frac{1}{2},\frac{1}{2}c+1 & ; & \end{matrix}\right]+\frac{axt}{c(c+1)}\,{}_2F_3\left[\begin{matrix}\frac{1}{2}a+\frac{1}{2},\frac{1}{2}a+1 & ; & \\ & & -x^2t^2 \\ c+1,\frac{1}{2}c+1,\frac{1}{2}c+\frac{3}{2} & ; & \end{matrix}\right]. \tag{3.6}$$

Note that, In (3.6), if we take $a{=}c{+}1$, we get a known result of Choi and Rathie [2, p.197 (18)].

III. Taking $i{=}{-}1$ in (3.2) and using the results (2.8)–(2.10), we get the following generating function:

$$\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}m!n!t^m(-t)^n}{(c_1-1)_m(c_2-1)_m(c_1)_n(c_2)_n}L_m^{(c_1-2,c_2-2)}(x_1,x_2)L_n^{(c_1-1,c_2-1)}(x_1,x_2)$$

$$=F\,{}^{2:0;0}_{0:3;3}\left[\begin{array}{l}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2}:\quad -\quad ;\quad -\quad ;\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -x_1^2t^2,-x_2^2t^2\\ \quad -\quad : c_1-1,\frac{1}{2}c_1,\frac{1}{2}c_1+\frac{1}{2}\ ;\ c_2-1,\frac{1}{2}c_2,\frac{1}{2}c_2+\frac{1}{2}\ ;\end{array}\right]$$

$$-\frac{ax_1t}{c_1(c_1-1)}F\,{}^{2:0;0}_{0:3;3}\left[\begin{array}{l}\frac{1}{2}a+\frac{1}{2},\frac{1}{2}a+1:\quad -\quad ;\quad -\quad ;\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -x_1^2t^2,-x_2^2t^2\\ \quad -\quad : c_1,\frac{1}{2}c_1+\frac{1}{2},\frac{1}{2}c_1+1\ ;\ c_2-1,\frac{1}{2}c_2,\frac{1}{2}c_2+\frac{1}{2}\ ;\end{array}\right]$$

$$-\frac{ax_2t}{c_2(c_2-1)}F\,{}^{2:0;0}_{0:3;3}\left[\begin{array}{l}\frac{1}{2}a+\frac{1}{2},\frac{1}{2}a+1:\quad -\quad ;\quad -\quad ;\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -x_1^2t^2,-x_2^2t^2\\ \quad -\quad : c_1-1,\frac{1}{2}c_1,\frac{1}{2}c_1+\frac{1}{2}\ ;\ c_2,\frac{1}{2}c_2+\frac{1}{2},\frac{1}{2}c_2+1\ ;\end{array}\right]$$

$$+\frac{a(a+1)x_1x_2t^2}{c_1c_2(c_1-1)(c_2-1)}$$

$$\times F\,{}^{2:0;0}_{0:3;3}\left[\begin{array}{l}\frac{1}{2}a+1,\frac{1}{2}a+\frac{3}{2}:\quad -\quad ;\quad -\quad ;\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -x_1^2t^2,-x_2^2t^2\\ \quad -\quad : c_1,\frac{1}{2}c_1+\frac{1}{2},\frac{1}{2}c_1+1\ ;\ c_2,\frac{1}{2}c_2+\frac{1}{2},\frac{1}{2}c_2+1\ ;\end{array}\right]. \tag{3.7}$$

Further, putting $x_2{=}0$ in (3.7) and using (1.7), we get:

$$\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}t^m(-t)^n}{(c-1)_m(c)_n}L_m^{(c-2)}(x)L_n^{(c-1)}(x)$$

$$={}_2F_3\left[\begin{array}{l}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2}\quad ;\\ \qquad\qquad\qquad -x^2t^2\\ c-1,\frac{1}{2}c,\frac{1}{2}c+\frac{1}{2}\ ;\end{array}\right]-\frac{axt}{c(c-1)}{}_2F_3\left[\begin{array}{l}\frac{1}{2}a+\frac{1}{2},\frac{1}{2}a+1\ ;\\ \qquad\qquad\qquad -x^2t^2\\ c,\frac{1}{2}c+\frac{1}{2},\frac{1}{2}c+1\ ;\end{array}\right]. \tag{3.8}$$

Note that, in (3.8), if we take $a{=}c$, we get a known result of Choi and Rathie [2, p.198 (21)].

IV. Taking $i{=}2$ in (3.2) and using the results (2.8)–(2.10), we get the following generating function:

$$\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}m!n!t^m(-t)^n}{(c_1+2)_m(c_2+2)_m(c_1)_n(c_2)_n}L_m^{(c_1+1,c_2+1)}(x_1,x_2)L_n^{(c_1-1,c_2-1)}(x_1,x_2)$$

$$=F\,{}^{2:0;0}_{0:3;3}\left[\begin{array}{l}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2}:\quad -\quad ;\quad -\quad ;\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -x_1^2t^2,-x_2^2t^2\\ \quad -\quad : c_1+1,\frac{1}{2}c_1+\frac{3}{2},\frac{1}{2}c_1+1\ ;\ c_2+1,\frac{1}{2}c_2+\frac{3}{2},\frac{1}{2}c_2+1\ ;\end{array}\right]$$

$$+\frac{2ax_1t}{c_1(c_1+2)}F\,{}^{2:0;0}_{0:3;3}\left[\begin{array}{l}\frac{1}{2}a+\frac{1}{2},\frac{1}{2}a+1:\quad -\quad ;\quad -\quad ;\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -x_1^2t^2,-x_2^2t^2\\ \quad -\quad : c_1+1,\frac{1}{2}c_1+\frac{3}{2},\frac{1}{2}c_1+2\ ;\ c_2+1,\frac{1}{2}c_2+\frac{3}{2},\frac{1}{2}c_2+1\ ;\end{array}\right]$$

$$+\frac{2ax_2t}{c_2(c_2+2)}F\,{}^{2:0;0}_{0:3;3}\left[\begin{array}{l}\frac{1}{2}a+\frac{1}{2},\frac{1}{2}a+1\ :\ \ -\ ;\ \ -\ ;\\ \qquad\qquad\qquad\qquad\qquad\qquad -x_1^2t^2,-x_2^2t^2\\ \ -\ :\ c_1+1,\frac{1}{2}c_1+\frac{3}{2},\frac{1}{2}c_1+1\ ;\ c_2+1,\frac{1}{2}c_2+\frac{3}{2},\frac{1}{2}c_2+2\ ;\end{array}\right]$$

$$+\frac{4a(a+1)x_1x_2t^2}{c_1c_2(c_1+2)(c_2+2)}$$

$$\times F\,{}^{2:0;0}_{0:3;3}\left[\begin{array}{l}\frac{1}{2}a+1,\frac{1}{2}a+\frac{3}{2}\ :\ \ -\ ;\ \ -\ ;\\ \qquad\qquad\qquad\qquad\qquad\qquad -x_1^2t^2,-x_2^2t^2\\ \ -\ :\ c_1+1,\frac{1}{2}c_1+\frac{3}{2},\frac{1}{2}c_1+2\ ;\ c_2+1,\frac{1}{2}c_2+\frac{3}{2},\frac{1}{2}c_2+2\ ;\end{array}\right]. \tag{3.9}$$

Further, putting $x_2=0$ in (3.9) and using (1.7), we get:

$$\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}t^m(-t)^n}{(c+2)_m(c)_n}L_m^{(c+1)}(x)L_n^{(c-1)}(x)$$

$$={}_2F_3\left[\begin{array}{l}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2}\ ;\\ \qquad\qquad -x^2t^2\\ c+1,\frac{1}{2}c+\frac{3}{2},\frac{1}{2}c+1\ ;\end{array}\right]+\frac{2axt}{c(c+2)}{}_2F_3\left[\begin{array}{l}\frac{1}{2}a+\frac{1}{2},\frac{1}{2}a+1\ ;\\ \qquad\qquad -x^2t^2\\ c+1,\frac{1}{2}c+\frac{3}{2},\frac{1}{2}c+2\ ;\end{array}\right]. \tag{3.10}$$

The other special cases of (3.2) can also be obtained in the similar manner.

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