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IDEALS OF SEMIRINGS USING MULTISET THEORY

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Abstract. The concept of multi ideal, multi bi-ideal, multi quasi-ideal of a semiring are introduced and some of their related properties are investigated. Some characterizations of regular and intra-regular semiring are also obtained.

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1. Introduction

There are many concepts of universal algebras generalizing an associative ring $(R, +, .)$. Some of them - in particular, semiring have been found very useful for solving problems in different areas of applied mathematics and information sciences, since the structure of a semiring provides an algebraic framework for modelling and studying the key factors in these applied areas. Ideals of semiring play a central role in the structure theory and useful for many purposes.

Multisets is an important generalization of classical set theory which has emerged by violating a basic property of classical sets that an element can belong to a set only once. This set theory has various applications in mathematics and computer science, overview of which can be obtained in [9]. Many authors, like [1–3, 5–7, 10] etc. have enriched the theory of multisets. Nazmul et al [8] applied this to the theory of groups.

As a generalization of this, in this paper we have introduced the concept of multi ideal in the theory of semirings and investigated some of its related properties.

2. Preliminaries

We recall the following preliminaries for subsequent use.

Definition 2.1. [4] *A hemiring [respectively semiring] is a nonempty set S on which operations addition and multiplication have been defined such that the following conditions are satisfied:*

(i) $(S,+)$ *is a commutative monoid with identity 0.*

(ii) $(S,.)$ *is a semigroup [respectively monoid with identity 1_S].*

(iii) *Multiplication distributes over addition from either side.*

(iv) $0.s = 0 = s.0$ *for all* $s \in S$.

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(v) $1_S \neq 0$

Throughout this paper, unless otherwise mentioned S denotes a semiring.

A subset A of a semiring S is called a left(resp. right) ideal of S if A is closed under addition and $SA \subseteq A$ (resp. $AS \subseteq A$). A subset A of a semiring S is called an ideal if it is both left and right ideal of S.

A subset A of a semiring S is called a bi-ideal if A is closed under addition and $ASA \subseteq A$.

A subset A of a semiring S is called a quasi-ideal of S if A is closed under addition and $SA \cap AS \subseteq A$.

Definition 2.2. [3] *A multiset M drawn from the set X is represented by a Count function C_M defined as $C_M : X \to N$, where N represents the set of non-negative integers.*
Here $C_M(x)$ is the number of occurance of the element x in the multiset M.

3. Multi Ideal of Semiring

Definition 3.1. *Let S be a semiring and $x, y \in S$. A multiset M over S called a multi left ideal [resp. multi right ideal] of S if*

(i) $C_M(x+y) \geq \min[C_M(x), C_M(y)]$ *and*

(ii) $C_M(xy) \geq C_M(y)$ [*resp.* $C_M(xy) \geq C_M(x)$]

A multi ideal of a semiring S is a non empty multi subset of S which is a multi left ideal as well as a multi right ideal of S.

Example 3.2. *Let S be the additive commutative semigroup of all integers. Then S is a semiring if ab denotes the usual multiplication of integers a and b where $a, b \in S$. Let C_M be a multi subset of S, defined as follows*

$$C_M(x) = \begin{cases} 7 \text{ if x is even} \\ 5 \text{ if x is odd} \end{cases}$$

The multi subset C_M of S is a multi ideal S.

Definition 3.3. *Let P and Q be two multisets over a semiring S. Define multi-composition of P and Q by*

$$C_P o C_Q(x) = \sup_{x=\sum_{i=1}^{n} a_i b_i}[\min_i\{\min\{C_P(a_i), C_Q(b_i)\}\}]$$

$= 0$, *if x cannot be expressed as above*

where $x, a_i, b_i \in S$.

Definition 3.4. *Let P and Q be two multisets over a semiring S. Define intersection of P and Q by*

$$C_P \cap C_Q(x) = \min\{C_P(x), C_Q(x)\}$$

Lemma 3.5. *Let P and Q be two multi ideal over a semiring S. Then $C_P o C_Q \subseteq C_P \cap C_Q \subseteq C_P, C_Q$*

Proof. P and Q be two multisets over a semiring S with $x \in S$. Then

$$\begin{aligned}(C_P o C_Q)(x) &= \sup_{x=\sum_{i=1}^{n} a_i b_i}\{\min_i\{\min\{C_P(a_i), C_Q(b_i)\}\}\} \\ &\leq \sup\{\min_i\{\min\{C_P(a_i)\}\}\} \\ &\leq C_P(\sum_{i=1}^{n} a_i b_i) = C_P(x)................(1)\end{aligned}$$

Since this is true for every representation of x, $C_P o C_Q \subseteq C_P$.

Similarly we can prove that

$$C_P o C_Q \subseteq C_Q \text{ for all } x \in S...............(2)$$

Combining (1)and (2)we get

$$\begin{aligned}(C_P o C_Q)(x) &\leq \min\{C_P(x), C_Q(x)\} \text{ for all } x \in S \\ &= (C_P \cap C_Q)(x).........(3)\end{aligned}$$

Therefore, combining (1), (2) and (3) and we get that

$$C_P o C_Q \subseteq C_P \cap C_Q \subseteq C_P, C_Q.$$

Hence the lemma. □

Proposition 3.6. *Intersection of a nonempty collection of multi left ideals is a multi left ideal over S.*

Proof. Let $\{M_i : i \in I\}$ be a non-empty family of multi left ideals of S and $x, y \in S$. Then

$$\begin{aligned}(\bigcap_{i\in I} C_{M_i})(x+y) &= \inf_{i\in I}[C_{M_i}(x+y)] \\ &\geq \inf_{i\in I}[\min[C_{M_i}(x), C_{M_i}(y)]] \\ &= \min[\inf_{i\in I} C_{M_i}(x), \inf_{i\in I} C_{M_i}(y)] \\ &= \min[(\bigcap_{i\in I} C_{M_i})(x), (\bigcap_{i\in I} C_{M_i})(y)].\end{aligned}$$

Again

$$\begin{aligned}(\bigcap_{i\in I} C_{M_i})(xy) &= \inf_{i\in I}[C_{M_i}(xy)] \\ &\geq \inf_{i\in I}[C_{M_i}(y)] \\ &= (\bigcap_{i\in I} C_{M_i})(y)\end{aligned}$$

Hence $\bigcap_{i\in I} M_i$ is a multi left ideal of S. □

Definition 3.7. *Let R, S be semirings and $f : R \to S$ be a function and $a, b \in R$. Then f is said to be a homomorphism if*

(i) $f(a+b) = f(a) + f(b)$

(ii) $f(ab) = f(a)f(b)$

(iii) $f(0_R) = 0_S$,

where 0_R and 0_S are the zeroes of R and S respectively.

Definition 3.8. *Let X and Y be two non-empty sets and $f : X \to Y$ be a mapping. Then*

(i) the image of multiset P over X under the mapping f denoted by $f(P)$ where

$$C_{f(P)}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} C_P(x) \\ 0 \text{ otherwise} \end{cases}$$

(ii) the inverse image of a multiset Q over S under the mapping f is denoted by $f^{-1}(Q)$, where $C_{f^{-1}(Q)}(x) = C_Q(f(x))$.

Proposition 3.9. *Let $f : R \to S$ be a morphism of semirings.*

(i) *If Q is a multi left ideal of S, then $f^{-1}(Q)$ is a multi left ideal of R.*

(ii) *If f is surjective morphism and P is a multi left ideal of R, then $f(P)$ is a multi left ideal of S.*

Proof. Let $f : R \to S$ be a morphism of semirings.

(i) Let Q be a multi left ideal of S. Now, for any $r, s \in R$

$$\begin{aligned} C_{f^{-1}(Q)}(r+s) &= C_Q(f(r+s)) = C_Q(f(r)+f(s)) \\ &\geq \min\{C_Q(f(r)), C_Q(f(s))\} \\ &= \min\{(C_{f^{-1}(Q)})(r), (C_{f^{-1}(Q)})(s)\}. \end{aligned}$$

Again

$$\begin{aligned} (C_{f^{-1}(Q)})(rs) &= C_Q(f(rs)) = C_Q(f(r)f(s)) \\ &\geq C_Q(f(s)) = (C_{f^{-1}(Q)})(s). \end{aligned}$$

Thus $f^{-1}(Q)$ is a multi left ideal of R.

(ii) Let P be a multi left ideal of R. Now, for any $x^{'}, y^{'} \in S$

$$\begin{aligned} (C_{f(P)})(x^{'}+y^{'}) &= \sup_{z \in f^{-1}(x^{'}+y^{'})} C_P(z) \\ &\geq \sup_{x \in f^{-1}(x^{'}), y \in f^{-1}(y^{'})} C_P(x+y) \\ &\geq \sup[\min[C_P(x), C_P(y)]] \\ &= \min[\sup_{x \in f^{-1}(x^{'})} C_P(x), \sup_{y \in f^{-1}(y^{'})} C_P(y)] \\ &= \min[(C_{f(P)})(x^{'}), (C_{f(P)})(y^{'})] \end{aligned}$$

Again

$$\begin{aligned} C_{f(P)}(x^{'}y^{'}) &= \sup_{z \in f^{-1}(x^{'}y^{'})} C_P(z) \geq \sup_{x \in f^{-1}(x^{'}), y \in f^{-1}(y^{'})} C_P(xy) \\ &\geq \sup_{y \in f^{-1}(y^{'})} C_P(y) = (C_{f(P)})(y^{'}). \end{aligned}$$

Thus $f(P)$is a multi left ideal of S. □

Let $\{S_i\}_{i \in I}$ be a family of semirings. Now if we define addition $(+)$ and multiplication $(.)$ on the cartesian product $\Pi_{i \in I} S_i$ as follows :

$(x_i)_{i \in I} + (y_i)_{i \in I} = (x_i + y_i)_{i \in I}$ and

$(x_i)_{i \in I}(y_i)_{i \in I} = (x_i y_i)_{i \in I}$ for all $(x_i)_{i \in I}, (y_i)_{i \in I} \in \Pi_{i \in I} S_i$ then $\Pi_{i \in I} S_i$ becomes a semiring.

Definition 3.10. *Let P and Q be multi subsets of X. The cartesian product of P and Q is defined by $(C_P \times C_Q)(x, y) = \min\{C_P(x), C_Q(y)\}$ for all $x, y \in X$.*

Theorem 3.11. *Let P and Q be multi left ideals over a semiring S. Then cartesian product of P and Q is a multi left ideal of $S \times S$.*

Proof. Let $(x_1, x_2), (y_1, y_2) \in S \times S$. Then

$$\begin{aligned}(C_P \times C_Q)((x_1, x_2) + (y_1, y_2)) &= (C_P \times C_Q)(x_1 + y_1, x_2 + y_2)\\ &= \min\{C_P(x_1 + y_1), C_Q(x_2 + y_2)\}\\ &\geq \min\{\min\{C_P(x_1), C_P(y_1)\}, \min\{C_Q(x_2), C_Q(y_2)\}\}\\ &= \min\{\min\{C_P(x_1), C_Q(x_2)\}, \min\{C_P(y_1), C_Q(y_2)\}\}\\ &= \min\{(C_P \times C_Q)(x_1, x_2), (C_P \times C_Q)(y_1, y_2)\}\end{aligned}$$

and

$$\begin{aligned}(C_P \times C_Q)((x_1, x_2)(y_1, y_2)) &= (C_P \times C_Q)(x_1y_1, x_2y_2)\\ &= \min\{C_P(x_1y_1), C_Q(x_2y_2)\}\\ &\geq \min\{C_P(y_1), C_Q(y_2)\}\\ &= (C_P \times C_Q)(y_1, y_2).\end{aligned}$$

Hence cartesian product of P and Q is a multi left ideal of $S \times S$. □

Definition 3.12. *A multi subset P over a semiring S is called multi bi-ideal if for all $x, y, z \in S$ we have*

(i) $C_P(x + y) \geq \min\{C_P(x), C_P(y)\}$

(ii) $C_P(xyz) \geq \min\{C_P(x), C_P(z)\}$

Definition 3.13. *Let S be a semiring and M be a multiset over S. Then the characteristic of the multiset M is defined as*

$$\chi_M(x) = \begin{cases} \infty \text{ if } x \in M \\ 0 \text{ if } x \notin M \end{cases}$$

Definition 3.14. *A multi subset M over a semiring S is called multi quasi-ideal if for all $x, y \in S$ we have*

(i) $C_M(x + y) \geq \min\{C_M(x), C_M(y)\}$

(ii) $(C_M o \chi_M) \cap (\chi_M o C_M) \subseteq C_M$

Lemma 3.15. *A multi subset M over a semiring S is a multi left(resp. right) ideal of S if and only if for all $x, y \in S$, we have*

(i) $C_M(x + y) \geq \min\{C_M(x), C_M(y)\}$

(ii) $\chi_M o C_M \subseteq C_M$(*resp.* $C_M o \chi_M \subseteq C_M$).

Proof. Assume that M is a multi left ideal of S. Then it is sufficient to show that the condition (ii) is satisfied. Let $x \in S$. If $(\chi_M o C_M)(x) = 0$, it is clear that $(\chi_M o C_M)(x) \leq C_M(x)$. Otherwise, there exist elements $a_i, b_i \in S$

and for i=1,...,n such that $x = \sum_{i=1}^{n} a_i b_i$. Then we have

$$\begin{aligned}(\chi_M o C_M)(x) &= \sup_{x=\sum_{i=1}^{n} a_i b_i}[\min_i \{\min\{\chi_M(a_i), C_M(b_i)\}\}] \\ &= \sup_{x=\sum_{i=1}^{n} a_i b_i}[\min_i \{C_M(b_i)\}] \\ &\leq \sup_{x=\sum_{i=1}^{n} a_i b_i}[\min_i \{C_M(a_i b_i)\}] = C_M(x).\end{aligned}$$

This implies that $\chi_M o C_M \subseteq C_M$.

Conversely, assume that the given conditions hold. Then it is sufficient to show the second condition of the definition of ideal. Let $x, y \in S$. Then we have

$C_M(xy) \geq (\chi_M o C_M)(xy) = \sup_{xy=\sum_{i=1}^{n} a_i b_i}[\min_i \{\min\{\chi_M(a_i), C_M(b_i)\}\}] \geq C_M(y)$. Hence C_M is a multi left ideal of S.

The case for multi right ideal can be proved similarly. □

Lemma 3.16. *Any multi quasi-ideal of S is a multi bi-ideal of S.*

Proof. Let M be any multi quasi-ideal of S. It is sufficient to show that $C_M(xyz) \geq \min\{C_M(x), C_M(z)\}$ and $C_M(xy) \geq \min\{C_M(x), C_M(y)\}$ for all $x, y, z \in S$.

In fact, by the assumption, we have

$$\begin{aligned}C_M(xyz) &\geq ((C_M o \chi_M) \cap (\chi_M o C_M))(xyz) \\ &= \min\{(C_M o \chi_M)(xyz), (\chi_M o C_M)(xyz)\} \\ &= \min\{\sup_{xyz=\sum_{i=1}^{n} a_i b_i}(\min(C_M(a_i), \chi_M(b_i))), \sup_{xyz=\sum_{i=1}^{n} a_i b_i}(\min(\chi_M(a_i), C_M(b_i)))\} \\ &\geq \min\{C_M(x), C_M(z)\}\end{aligned}$$

Similarly, we can show that $C_M(xy) \geq \min\{C_M(x), C_M(y)\}$. □

4. Multi Ideal of Regular and Intra-regular Semiring

Definition 4.1. *A semiring S is said to be regular if for each $x \in S$, there exists $a \in S$ such that $x = xax$.*

Lemma 4.2. *A semiring S is regular if and only if for any right ideal R and any left ideal L of S we have $RL = R \cap L$.*

Theorem 4.3. *If semiring S is regular then for any multi right ideal M and any multi left ideal N of S we have $C_M o C_N = C_M \cap C_N$.*

Proof. Let S be a regular semiring. By Lemma 3.5, we have $C_M o C_N \subseteq C_M \cap C_N$.

For any $x \in S$, there exist $a \in S$ such that $x = xax$.

Then

$$\begin{aligned}(C_M o C_N)(x) &= \sup_{x=\sum_{i=1}^{n} a_i b_i}\{\min \ \{C_M(a_i), C_N(b_i)\}\} \\ &\geq \min\{C_M(xa), C_N(x)\} \\ &\geq \min\{C_M(x), C_N(x)\} \\ &= (C_M \cap C_N)(x).\end{aligned}$$

Therefore $(C_M \cap C_N) \subseteq (C_M o C_N)$.

Hence $(C_M o C_N) = (C_M \cap C_N)$. □

Theorem 4.4. *Let S be a regular semiring. Then*

(i) $C_M \subseteq C_M o \chi_M o C_M$ *for every multi bi-ideal M of S.*

(ii) $C_M \subseteq C_M o \chi_M o C_M$ *for every multi quasi-ideal M of S.*

Proof. (i) Let M be any multi bi-ideal of S and x be any element of S. Since S is regular there exists $a \in S$ such that $x = xax$.

$$\begin{aligned}(C_M o \chi_M o C_M)(x) &= \sup_{x=\sum_{i=1}^{n} a_i b_i}(\min \ \{(C_M o \chi_M)(a_i), C_M(b_i))\}) \\ &\geq \min\{(C_M o \chi_M)(xa), C_M(x)\} \\ &= \min\{\min\{C_M(x), \chi_M(a)\}, C_M(x)\} = C_M(x)\end{aligned}$$

This implies that $C_M \subseteq C_M o \chi_M C_M$.

(i)⇒(ii) This is straight forward from Lemma 3.16 □

Theorem 4.5. *Let S is a regular semiring. Then*

(i) $C_M \cap C_N \subseteq C_M o C_N o C_M$ *for every multi bi-ideal M and every multi ideal N of S.*

(ii) $C_M \cap C_N \subseteq C_M o C_N o C_M$ *for every multi quasi-ideal M and every multi ideal N of S.*

Proof. (i) Let M and N be any multi bi-ideal and multi ideal of S, respectively and x be any element of S. Since S is regular, there exists $a \in S$ such that $x = xax$.

$$\begin{aligned}(C_M o C_N o C_M)(x) &= \sup_{x=\sum_{i=1}^{n} a_i b_i}\{\min \ \{(C_M o C_N)(a_i), C_M(b_i)\}\} \\ &\geq \min\{(C_M o C_N)(xa), C_M(x)\} \\ &= \min\{ \ \sup_{xa=\sum_{i=1}^{n} a_i b_i}\{\min \ \{C_M(a_i), C_N(b_i)\}\}, C_M(x)\} \\ &\geq \min\{\min\{C_M(x), C_N(axa), C_M(x)\}(\text{since } \ xa = xaxa) \\ &\geq \min\{C_M(x), C_N(x)\} = (C_M \cap C_N)(x)\end{aligned}$$

(i)⇒(ii) This is straight forward from Lemma 3.16. □

Definition 4.6. *A semiring S is said to be intra-regular if for each $x \in S$, there exist $a, b \in S$, such that $x = axxb$.*

Theorem 4.7. *Let S be a intra-regular semiring. Then $C_M \cap C_N \subseteq C_M o C_N$ for every multi left ideal C_M and every multi right ideal C_N of S.*

Proof. Suppose S is regular semiring. Let M and N be any multi left ideal and multi right ideal of S respectively. Now let $x \in S$. Then by hypothesis there exist $a, b \in S$, such that $x = axxb$. Therefore

$$\begin{aligned}(C_M o C_N)(x) &= \sup_{x=\sum_{i=1}^{n} a_i b_i}[\min_i \{\min\{C_M(a_i), C_N(b_i)\}\}] \\ &\geq \min_i[\min\{C_M(ax), C_N(xb)\}] \\ &\geq \min\{C_M(x), C_N(x)\} = (C_M \cap C_N)(x)\end{aligned}$$

□

Theorem 4.8. *Let S be both regular and intra-regular semiring. Then*

(ii) *$C_M = C_M o C_M$ for every multi bi-ideal M of S.*

(ii) *$C_M = C_M o C_M$ for every multi quasi-ideal M of S.*

Proof. (i) Let $x \in S$ and M be any multi bi-ideal of S. Since S is both regular and intra-regular there exist $a, b \in S$, such that $x = xaxxbx$. Therefore

$$\begin{aligned}(C_M o C_M)(x) &= \sup_{x=\sum_{i=1}^{n} a_i b_i}[\min_i \{\min\{C_M(a_i), C_M(b_i)\}\}] \\ &\geq \min_{x=xaxxbx}\{C_M(xax), C_M(xbx)\} \\ &\geq C_M(x).\end{aligned}$$

Now $C_M o C_M \subseteq C_M \cap C_M = C_M$. Hence $C_M o C_M = C_M$ for every multi bi-ideal M of S.
(i)⇒(ii) This is straightforward from the Lemma 3.16. □

Theorem 4.9. *Let S be a regular and intra-regular semiring. Then*

(i) *$C_M \cap C_N \subseteq C_M o C_N$ for all multi bi-ideals M and N of S.*

(ii) *$C_M \cap C_N \subseteq C_M o C_N$ for every multi bi-ideals M and every multi quasi-ideal N of S.*

(iii) *$C_M \cap C_N \subseteq C_M o C_N$ for every multi quasi-ideals M and every multi bi-ideal N of S.*

(iv) *$C_M \cap C_N \subseteq C_M o C_N$ for all multi quasi-ideals M and N of S.*

Proof. (i) Let $x \in S$ and M be any multi bi-ideal of S. Since S is both regular and intra-regular there exist $a, b \in S$, such that $x = xaxxbx$. Therefore

$$\begin{aligned}(C_M C_N)(x) &= \sup_{x=\sum_{i=1}^{n} a_i b_i}[\min_i \{\min\{C_M(a_i), C_N(b_i)\}\}] \\ &\geq \min\{C_M(xax), C_N(xbx)\} \\ &\geq \min\{C_M(x), C_N(x)\} = (C_M \cap C_N)(x)\end{aligned}$$

which implies $C_M \cap C_N \subseteq C_M o C_N$.

$(i) \Rightarrow (ii) \Rightarrow (iv)$ and $(i) \Rightarrow (iii) \Rightarrow (iv)$ are obvious from Lemma 3.16. □

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