Acia Pac. J. Math. 2019 **6**:21

A FIXED POINT THEOREMS FOR MULTIVALUED SEMI-QUASI CONTRACTION MAPS ON METRIC SPACE

ABDUL-MAJEED AL-IZERI

Department of Mathematics (LMBP), CNRS, Clermont Auvergne University, France

Corresponding author: Abdul_Majeed.Al_izeri@math.univ-bpclermont.fr

Received Jul. 14, 2019

Abstract. The aim of this paper is to obtain a fixed point theorems for multivalued mappings of semi-quasi contraction type in the framework of complete metric spaces and prove the existence and uniqueness of some fixed point theorems. These results generalize other well known fixed point theorems including Ćirić quasi contraction theorem.

2010 Mathematics Subject Classification. 47H10; 54H25.

Key words and phrases. fixed point; quasi-contraction; semi-quasi contraction; generalized semi-quasi contraction.

1. Introduction and Preliminaries

In 1992, the Banach contraction principle introduced by Stefan Banach [2] is one of the most useful and important theorems of classical functional analysis. These theorems extend other well know fundamental metrical fixed point theorems in the literature (see, for exemple, [3,5,7,11] and the references therein). This principle states that, if (X, d) is a complete metric space and $T : \mathsf{X} \to \mathsf{X}$ is a contraction map (i.e., $d(Tx, Ty) \leq \lambda d(x, y)$ for all $x, y \in \mathsf{X}$ where $0 \leq \lambda < 1$ is a constant), then T has a unique fixed point.

The Banach contraction principle has been generalized in many ways over the years. In some generalization, Ćirić's [4] introduced the notion of a quasi-contraction and obtained a generalisation of the classical Banach contraction principale (BCP), we recall that a self-mapping T of a matric space X is called a quasi-contraction if there exists a number $q \in [0, 1)$, such that for all $x, y \in \mathsf{X}$

$$d(Tx, Ty) \leq qM(x, y) \tag{1.1}$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$.

It is well known that $H : CB(\mathsf{X}) \times CB(\mathsf{X}) \to \mathbb{R}$ defined by

$$H(A, B) = \max\{\sup_{a\in A} D(a, B), \sup_{b\in B} D(A, b)\}$$

DOI: 10.28924/APJM/6-21

is a metric on $CB(\mathsf{X})$, which is called the Pompeiu Hausdorff metric, where $CB(\mathsf{X})$ is the class of all nonempty, closed, and bounded subsets of X and $D(a,B) = \inf\{d(a,b) : b \in B\}$. A fixed point of a multivalued mapping $F : \mathsf{X} \to \mathcal{P}(\mathsf{X})$, which is the class of all nonempty subsets of X, is an element $x \in \mathsf{X}$ such that $x \in Fx$.

The defintion of F-orbitally complete is modeled by the following definition (see, [4,8]).

Definition 1.1. *Let $F : \mathsf{X} \to CB(\mathsf{X})$ be a multi-valued mapping. Let $u_0 \in \mathsf{X}$, an orbit of F at u_0 is a sequence*

$$\{u_n : u_n \in Fu_{n-1},\ n \in \mathbb{N}\}.$$

A space X is said to be F-orbitally complete if every Cauchy sequence which is a subsequence of an orbit of F at $u \in \mathsf{X}$ converges in X.

Let n be any positive integer, $\mathcal{O}(x_k, n)$ denotes the set of points $\{x_k, x_{k+1}, \cdots, x_{k+n}\}$ and $\delta[\mathcal{O}(x_k, n)]$ the diametre of $\mathcal{O}(x_k, n)$. Notice that if $\delta[\mathcal{O}(x_k, n)] > 0$ for $k, n \in \mathbb{N}$ then

$$\delta[\mathcal{O}(x_k, n)] = d(x_k, x_j), \quad \text{where} \quad k < j < k+n.$$

In 1969, Nadler [9] extended the Banach contraction principle to multivalued mappings and first initiated the study of fixed point results for multivalued linear contraction.

Theorem 1.1. *(Nadler [9]) Let (X, d) be a complete metric space and $F : \mathsf{X} \to CB(\mathsf{X})$ a multivalued contraction; that is, there exists $r \in [0,1)$ such that*

$$H(Fx, Fy) \le r d(x,y)$$

for all $x, y \in \mathsf{X}$. Then F has a fixed point.

Later on, several studies were conducted on a variety of generalizations, extensions, and applications of this result of Nadler (see [1,6,10]). Recently, Ćirić's [4] has extended the Banach quasi-contraction to multivalued quasi-contraction mappings. The Ćirić's fixed point theorem of multivalued map is given by

Theorem 1.2. *[4, Theorem 1] Let $F : \mathsf{X} \to CB(\mathsf{X})$ be a multi-valued mapping on a metric space X and let X be F-orbitally complete. If F satisfies*

$$H(Fx, Fx^*) \le q \max\{d(x,x^*), D(x,Fx), D(x^*,Fx^*), D(x,Fx^*), D(x^*,Fx)\})$$

for some $q < 1$ and all $x, x^* \in \mathsf{X}$, then

(i) F has a unique fixed point $x^* \in \mathsf{X}$ and $Fx^* = \{x^*\}$.
(ii) For each $x_0 \in \mathsf{X}$, there exists an orbit $\{x_n\}_{n\in\mathbb{N}}$ of F at x_0 such that $\lim_{n\to\infty} x_n = x^*$ for all $x \in \mathsf{X}$, and
(iii) $d(x_n, x^*) \le \dfrac{(q^{a-1})^n}{(1-q)^{a-1}} d(x_0, x_1)$ for all $n \in \mathbb{N}$ where $a < 1$ is any fixed positive number.

Kumam et al. [8] introduced the notion of a generalized quasi-contraction, they were added four new values, for all $x, y \in \mathsf{X}$, $D(F^2x, x), D(F^2x, Fx), D(F^2x, y), D(F^2x, Fy)$ to a quasi-contraction condition and they have extended these results of fixed point theorems for Multi-valued generalized quasi-contraction.

Theorem 1.3. *[8, Theorem 3.1] Let $F : \mathsf{X} \to CB(\mathsf{X})$ be a multi-valued mapping on a metric space X and let X be F-orbitally complete. Suppose that F is a generalized quasi-contraction. Then we have*

(i) F has a unique fixed point $x^* \in \mathsf{X}$ and $Fx^* = \{x^*\}$.

(ii) For each $x_0 \in \mathsf{X}$, there exists an orbit $\{x_n\}_{n\in\mathbb{N}}$ of F at x_0 such that $\lim_{n\to\infty} x_n = x^*$ for all $x \in \mathsf{X}$, and

(iii) $d(x_n, x^*) \leq \dfrac{(q^{a-1})^n}{(1-q)^{a-1}} d(x_0, x_1)$ for all $n \in \mathbb{N}$ where $a < 1$ is any fixed positive number.

Our goal in this note is to extend and generalize the notion of quasi contractions to multivalued semi-quasi contractions and obtain the existence and uniqueness of a fixed point theorems for such contractions.

2. Multi-valued semi-quasi contractions

In the framework of complete metric spaces, we establish some approximate fixed point results for multi-valued mappings satisfying a semi-quasi contraction.

Now we need the following definition.

Definition 2.1. *A multivalued mapping $F : \mathsf{X} \to CB(\mathsf{X})$ on a metric space X will be called a semi-quasi contraction if there exists a number $q \in [0, 1)$, such that*

$$(1-q)D(u, Fu) \leq d(u, u^*) \quad \textit{implies} \quad H(Fu, Fu^*) \leq qM_D(u, u^*) \tag{2.1}$$

for all $u, u^* \in \mathsf{X}$ where

$$M_D(u, u^*) = \max\{d(u, u^*), D(u, Fu), D(u^*, Fu^*), D(u, Fu^*), D(u^*, Fu)\}.$$

Before going further we shall prove the following lemmas which is required below.

Lemma 2.1. *Let $F : \mathsf{X} \to CB(\mathsf{X})$ be a multivalued semi-quasi contraction mapping and we suppose that $\delta[\mathcal{O}(u_{n+1}; 1)] > 0$ for $n \in \mathbb{N}$, then*

$$\delta[\mathcal{O}(u_n; 1)] \leq q^n \delta[\mathcal{O}(u_0; n+1)].$$

Proof. Let $u_0 \in \mathsf{X}$ and we define a sequence $\{u_n\}$ in X by

$$u_n = F^n u_0 \quad \text{for} \quad n \in \mathbb{N}.$$

We assume that there exists a positive integer $n \in \mathbb{N}$ such that

$$(1-q)D(u_n, Fu_n) \leq D(u_n, Fu_n) = d(u_n, u_{n+1}).$$

By Eq. (2.1), we have

$$\begin{aligned} d(u_{n+1}, u_{n+2}) &= D(Fu_n, Fu_{n+1}) \leq H(Fu_n, Fu_{n+1}) \\ &\leq q \max\{d(u_n, u_{n+1}), D(u_n, Fu_n), D(u_{n+1}, Fu_{n+1}), \\ &\quad D(u_n, Fu_{n+1}), D(u_{n+1}, Fu_n)\} \\ &= q \max\{d(u_n, u_{n+1}), d(u_{n+1}, u_{n+2}), d(u_n, u_{n+2})\}. \end{aligned}$$

Let n be any positive integer, we deduce that

$$d(u_{n+1}, u_{n+2}) = \delta[\mathcal{O}(u_{n+1}; 1)] \leq q\delta[\mathcal{O}(u_n; 2)] \leq \cdots \leq q^n \delta[\mathcal{O}(u_0; n+1)].$$

□

Lemma 2.2. *Let $F : \mathsf{X} \to CB(\mathsf{X})$ be a multivalued semi-quasi contraction mapping and we suppose that $\delta[\mathcal{O}(u_{n+1}; 1)] > 0$ for $n \in \mathbb{N}$, then the following statements holds:*

(1) $\delta[\mathcal{O}(u_0; n+1)] \leq \dfrac{1}{1-q} d(u_0, x_1)$,

(2) $D(Fu_n, Fu_{n+1}) \leq \dfrac{q}{1-q} d(u_n, u_{n+1})$.

Proof. The proof of (1). It is obvious that $\delta[\mathcal{O}(u_0; n+1)] = d(u_0, u_k)$ for positive integer $k \leq n+1$. So, we get

$$\begin{aligned} \delta[\mathcal{O}(u_0; n+1)] &= d(u_0, u_k) \leq d(u_0, u_1) + d(u_1, u_k) \\ &= d(u_0, u_1) + \delta[\mathcal{O}(u_1; n)] \\ &\leq d(u_0, u_1) + q\delta[\mathcal{O}(u_0; n+1)]. \end{aligned}$$

Thus

$$\delta[\mathcal{O}(u_0; n+1)] \leq \frac{1}{1-q} d(u_0, u_1).$$

Now, we will prove the second estimate, let $u_0 \in \mathsf{X}$, as the previous Lemma, for each $n \in \mathbb{N} \cap \{0\}$ we define a sequence $\{u_n\}$ by the following inequality

$$(1-q)D(u_n, Fu_n) \leq D(u_n, Fu_n) = d(u_n, u_{n+1}).$$

Thus

$$\begin{aligned} D(Fu_n, Fu_{n+1}) &\leq H(Fu_n, Fu_{n+1}) \\ &\leq q \max\{d(u_n, u_{n+1}), D(u_n, Fu_n), D(u_{n+1}, Fu_{n+1}), D(u_n, Fu_{n+1}), D(u_{n+1}, Fu_n)\} \\ &= q \max\{d(u_n, u_{n+1}), d(u_{n+1}, u_{n+2}), d(u_n, u_{n+2})\} \\ &\leq q \max\{d(u_n, u_{n+1}), d(u_{n+1}, u_{n+2}), d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2})\} \\ &= \frac{q}{1-q} D(u_n, Fu_n). \end{aligned}$$

□

Now we are ready to state our first result:

Theorem 2.1. *Let $F : \mathsf{X} \to CB(\mathsf{X})$ be a multivalued semi-quasi contraction mapping on a F-orbitally complete metric space X. Then we have:*

(i) *F has a unique fixed point $u^* \in \mathsf{X}$ and $Fu^* = \{u^*\}$.*

(ii) *For each $u_0 \in \mathsf{X}$, there exists an orbit $\{u_n\}_{n\in\mathbb{N}}$ of F at u_0 such that $\lim_{n\to\infty} u_n = u^*$ for all $u \in \mathsf{X}$, and*

(iii) $D(F^n u, u^*) \leq \dfrac{q^{n-1}}{(1-q)^2} D(u, Fu)$ *for all $n \in \mathbb{N}$.*

Proof. Let $u_0 \in \mathsf{X}$, for each $n \in \mathbb{N} \cap \{0\}$ and $\delta[\mathcal{O}(u_n;1)] = 0$, then

$$u_n = u_{n+1},\ i.e., u_n = Fu_n \quad \text{and } u_n \text{ is a fixed point of } \ F.$$

Since $q < 1$, the sequence $\{u_n\}$ is Cauchy. Moreover by the defintions 2.1 and since X is F-orbitally complete, $\{u_n\}$ has a limite u^* in X. We claim that

$$(1-q)D(u_n, Fu_n) \le d(u_n, u^*).$$

Suppose that for some $n \in \mathbb{N} \cup \{0\}$, either

$$d(u_n, u^*) \le (1-q)D(u_n, Fu_n)$$

and

$$D(Fu_n, u^*) < (1-q)D(Fu_n, Fu_{n+1}).$$

Then

$$\begin{aligned} D(u_n, Fu_n) &\le d(u_n, u^*) + D(Fu_n, u^*) \\ &< (1-q)D(u_n, Fu_n) + (1-q)D(Fu_n, Fu_{n+1}). \end{aligned}$$

Thus, by using Lemma 2.2(2), we get

$$\begin{aligned} D(u_n, Fu_n) &< (1-q)\Big[D(u_n, Fu_n) + \frac{q}{1-q}D(u_n, Fu_n)\Big] \\ &= (1-q)\Big[1 + \frac{q}{1-q}\Big]D(u_n, Fu_n) = D(u_n, Fu_n), \end{aligned}$$

a contradiction. Therefore $(1-q)D(u_n, Fu_n) \le d(u_n, u^*)$ holds for every $n \in \mathbb{N} \cup \{0\}$.

Now, by Eq. (2.1), we have for every $n \in \mathbb{N} \cup \{0\}$

$$\begin{aligned} D(Fu_n, Fu^*) &\le H(Fu_n, Fu^*) \\ &\le q \max\{d(u_n, u^*), D(u_n, Fu_n), D(u^*, Fu^*), D(u_n, Fu^*), D(u^*, Fu_n)\}. \end{aligned}$$

Hence, if we put $n \to \infty$, then

$$D(u^*, Fu^*) \le qD(u^*, Fu^*),$$

a contradiction, so $u^* = Fu^*$, i.e. u^* is a fixed point of F. Now, we will prove the unicity of a fixed point of F, we assume that u^* and w^* are tow fixed points of F. Then $(1-q)D(u^*, Fu^*) \le d(u^*, w^*)$. Hence, by Eq. (2.1), we get

$$\begin{aligned} D(w^*, u^*) = D(Fw^*, Fu^*) &\le H(Fw^*, Fu^*) \\ &\le q \max\{d(w^*, u^*), D(w^*, Fw^*), D(u^*, Fu^*), D(w^*, Fu^*), D(u^*, Fw^*)\} \\ &= qD(w^*, u^*) \end{aligned}$$

a contradiction, so $u^* = w^*$. Moreover, F has a unique fixed point.

Let $u_0 \in \mathsf{X}$ is arbitrary, then

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} F^n u_0 = u^*.$$

In the other hand, if $\delta[\mathcal{O}(u_n;1)] > 0$ then by Lemmas 2.1 and 2.2(2), we obtain for $u_0 \in \mathsf{X}$,

$$d(u_n, u_{n+1}) \leq \frac{q^{n-1}}{1-q} d(u_0, u_1).$$

For every $m > n$, we have

$$\begin{aligned} d(u_n, u_m) &\leq (r^{n-1} + r^n + \cdots + r^{m-1}) \frac{1}{1-q} d(u_0, u_1) \\ &\leq \frac{q^{n-1}}{(1-q)^2} d(u_0, u_1), \end{aligned} \tag{2.2}$$

we conclude that

$$D(F^n u, F^m u) = d(u_n, u_m) \leq \frac{q^{n-1}}{(1-q)^2} D(u, Fu),$$

for all $u \in \mathsf{X}$. So, if we put $m \to \infty$, then

$$D(F^n u, u^*) \leq \frac{q^{n-1}}{(1-q)^2} D(u, Fu),$$

for all $n \in \mathbb{N}$. This proves of Theorem 2.1. □

The next result readily follows from the above theorem.

Theorem 2.2. *Let $F : \mathsf{X} \to CB(\mathsf{X})$ be a multi-valued mapping on a metric space X satisfying the following:*

1) X is F-orbitally complete.

2) There exists $k \in \mathbb{N}$ and $q \in [0,1)$ such that for all $x, y \in \mathsf{X}$,

$$(1-q)D(x, F^k x) \leq d(x,y) \quad \textit{implies} \quad H(F^k x, F^k y) \leq q M_K(x,y)$$

where

$$M_K(x,y) = \max\{d(x,y), D(x, F^k x), D(y, F^k y), D(x, F^k y), D(y, F^k x)\}$$

Then we have

(i). F has a unique fixed point $x^* \in \mathsf{X}$.

(ii). $D(F^n x, x^*) \leq \dfrac{q^{m-1}}{(1-q)^2} \max\{D(F^i x, F^{i+k} x) : i = 0, 1, \cdots, k-1\}$ for all $x \in \mathsf{X}$ and $n \in \mathbb{N}$ where m is the greatest integer not exceeding $\dfrac{n}{k}$.

(iii). $\lim_{n\to\infty} F^n x = x^*$ for all $x \in \mathsf{X}$.

Proof. (i) By the conclusion of Theorem 2.1, F^k has a unique fixed point x^* and $F^k(Fx^*) = F(F^k x^*) = Fx^*$, it follows that $Fx^* = x^*$, that is, F has a fixed point x^*. The uniqueness of the fixed point of F is obvious.

(ii) Let n be any positive integer. Then $n = (m-1)k + j$, $0 \leq j < k$ and for every $x \in \mathsf{X}$, $F^n x = (F^k)^{m-1} F^j x$. It follows from Theorem 2.1(iii) that

$$\begin{aligned} D(F^n x, x^*) &\leq \frac{q^{(m-1)}}{(1-q)^2} D(F^j x, F^k F^j x) \\ &\leq \frac{q^{(m-1)}}{(1-q)^2} \max\{D(F^i x, F^{i+k} x) : i = 0, 1, \cdots, k-1\}. \end{aligned}$$

(iii). It is a direct consequence of (ii). □

Now, we define a new generalized semi-quasi contraction by adding four new values,
for all $x, y \in \mathsf{X}$, $D(F^2x, x)$,$D(F^2x, Fx)$, $D(F^2x, y)$, $D(F^2x, Fy)$ to a semi-quasi contraction condition and obtain a generalization of Theorem 2.1.

Theorem 2.3. *Let $F : \mathsf{X} \to CB(\mathsf{X})$ be a multi-valued mapping on a metric space X and let X be F-orbitally complete. If F satisfies*

$$(1-q)D(u, Fu) \leq d(u, u^*) \quad \textit{implies} \quad H(Fu, Fu^*) \leq qM_{G_D}(u, u^*)$$

where

$$M_{G_D}(x, y) = \max\{d(u, u^*), D(u, Fu), D(u^*), Fu^*)), D(u, Fu^*)), D(u^*), Fu),$$
$$D(F^2u, u), D(F^2u, Fu), D(F^2u, u^*)), D(F^2u, Fu^*))\}.$$

for some $q < 1$ and all $u, u^*)^* \in \mathsf{X}$, then we have

(i) F has a unique fixed point $u^* \in \mathsf{X}$ and $Fu^* = \{u^*\}$.

(ii) For each $u_0 \in \mathsf{X}$, there exists an orbit $\{u_n\}_{n\in\mathbb{N}}$ of F at u_0 such that

$\lim_{n\to\infty} u_n = u^*$ for all $u \in \mathsf{X}$, and

(iii) $D(F^n u, u^*) \leq \dfrac{q^{n-1}}{(1-q)^2} D(u, Fu)$ for all $n \in \mathbb{N}$.

Proof. The proof is similar of the proof of Theorem 2.1 , it is enough to replace $M_D(u, u^*)$ by $M_{G_D}(u, u^*)$ and so it is omitted. □

Corollary 2.1. *Let $F : \mathsf{X} \to CB(\mathsf{X})$ be a multi-valued mapping on a metric space X satisfying the following:*

1) X is F-orbitally complete.

2) There exists $k \in \mathbb{N}$ and $q \in [0, 1)$ such that for all $x, y \in \mathsf{X}$,

$$(1-q)D(x, F^k x) \leq d(x, y) \quad \textit{implies} \quad H(F^k x, F^k y) \leq qM_{G_K}(x, y)$$

where

$$M_{G_K}(x, y) = \max\{d(x, y), D(x, F^k x), D(y, F^k y), D(x, F^k y), D(y, F^k x),$$
$$D(F^{2k}x, x), D(F^{2k}x, F^k x), D(F^{2k}x, y), D(F^{2k}x, F^k y)\}$$

Then we have

(i). F has a unique fixed point $x^* \in \mathsf{X}$.

(ii). $D(F^n x, x^*) \leq \dfrac{q^{m-1}}{(1-q)^2} \max\{D(F^i x, F^{i+k} x) : i = 0, 1, \cdots, k-1\}$ for all $x \in \mathsf{X}$ and $n \in \mathbb{N}$ where m is the greatest integer not exceeding $\dfrac{n}{k}$.

(iii). $\lim_{n\to\infty} F^n x = x^*$ for all $x \in \mathsf{X}$.

Proof. It comes from Theorem 2.2 when $M_K(x, y) = M_{G_K}(x, y)$. □

Now we give a significant example showing that F is multi-valued semi-quasi contractions:

Example 2.1. *Let* $X = \{(0,0),(4,0),(0,4),(4,5),(5,4)\}$ *be endowed with the metric* d *defined by*

$$d((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2 - y_2|$$

Then, (X,d) *is a complete metric space. Define a mapping* $F : X \to CB(X)$ *by*

$$F(x_1,x_2) = \begin{cases} (x_1,0), & \text{if} \quad x_1 \le x_2 \\ (0,x_2), & \text{if} \quad x_2 \le x_1. \end{cases}$$

Then, we note that

$$D(Fx,Fy) \le H(Fx,Fy) \le \frac{4}{5} M_D(x,y)$$

if $(x,y) \neq ((4,5),(5,4))$ *and* $(y,x) \neq ((4,5),(5,4))$.

Since

$$(1-q)D((4,5),F(4,5)) > \frac{5}{2} > 2 = d((4,5),(5,4))$$

and

$$(1-q)D((5,4),F(5,4)) > d((5,4),(4,5)) \text{ for every } q \in [0,1)$$

Therefore, Theorem 2.1 holds.

References

[1] RP. Agarwal, DO. Regan, and DR. Sahu, Fixed Point Theory for Lipschitzian-Type Mappings with Applications, Topological Fixed Point Theory and Its Applications. **6** . New York, Springer, 2009.

[2] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math. **3** (1922), 133-181.

[3] S. K. Chatterjee, Fixed point theorems, Rend. Acad. Bulgare Sc. **25** (1972), 727-730.

[4] Lj. B. Ćirić, A generalization of Banach's contraction principle, Proc. Amer. Math. Soc., **45** (1974), 267-273.

[5] G. S. Jeong and B. E. Rhoades, More maps for which $F(T) = F(Tn)$, Demonstratio Math. **40** (2007), 671-680.

[6] T. Kamran and Q. Kiran, Fixed point theorems for multi-valued mappings obtained by altering distances, Math Comput Model. **54** (2011), 2772-2777.

[7] R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc., **60** (1968), 71-76.

[8] P. Kumam, V. D. Nguyen and K. Sitthithakerngkiet, A generalization of Ćirić fixed point theorems, Filomat. **29(7)** (2015), 1549-1556.

[9] S. B. Jr. Nadler, Multi-valued contraction mappings, Pacic J. Math. **30** (1969), 475-488.

[10] S. Reich, Fixed points of contractive functions, Boll. Un. Mat. Ital. **5(4)** (1972), 26-42.

[11] T. Zamfirescu, Fixed point theorems in metric spaces, Arch. Math. (Basel). **23** (1972), 292-298.