Acia Pac. J. Math. 2019 **6**:3

THE NEW SUB-EQUATION METHOD FOR GENERAL KAUP-KUPERSHMIDT EQUATION FROM CAPILLARY GRAVITY WAVES WITH CONFORMABLE DERIVATIVE

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Received Feb. 14, 2019

Abstract. In this present paper, authors obtained the exact solution of time fractional general Kaup-Kupershmidt equation where the fracitonal derivative operator is in conformable sense by using the new sub-equation method. This method implemented firstly in the literature to a fractional partial differential equation.

2010 Mathematics Subject Classification. 35R11; 35A20; 35C05.

Key words and phrases. conformable fractional derivative; the new sub-equation method; fractional general Kaup-Kupershmidt; exact solutions.

1. Introduction

The generalizations of differentiation and integration with integer orders are called fractional calculus. In the last decades, interest to fractional calculus has been increasing considerably because of its huge application area in various fields such as physics, engineering, dynamical systems, control systems [1-4]. As a result of this interest, many powerful methods to solve fractional differential equations were presented by many authors. For example Kurt et. al. [5] obtained approximate anaytical solutions of Whitham-Broer-Kaup Equation by using homotopy analysis with Caputo derivative.Çelik et. al. [6] used Crank-Nicolson method for solvingfractional diffusion equation with the Riesz fractional derivative. Tasbozan et. al. [7] employed finite element method for obtaining the approximate solutions of diffusion equation with Riemann-Liouville fractional derivative. As it is seen from the the references given above, authors applied numerical methods to obtain the approximate solutionsof considered equations. Because scientists can not obtain the anaytical solutions for Caputo, Riemann-Liouville and the Riesz fractional derivatives. But newly defined conformable fractional derivative let us to obtain the

DOI: 10.28924/APJM/6-3

analytical solutions of the considered equations by using wave transform [8] and chain rule [9]. For instance Korkmaz et. al. [10] used two different methods to obtain the analytical solution of time-fractional parabolic equation with exponential nonlinearity. Çenesiz et. al. [11] employed first integral method to get the exact solutions of conformable fractional Burgers' type equations. Kurt et. al. [12] considered method to get the analytical solutions of conformable fractional Nizhnik-Novikov-Veselov system. For further details see papers [13-22]

In this study we regard the conformable time-fractional general Kaup-Kupershmidt equation (For $\alpha = 1$ see [23])

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{1}{5}\gamma^2 u^2 \frac{\partial u}{\partial x} + \frac{5}{2}\gamma \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + \gamma u \frac{\partial^3 u}{\partial x^3} + \frac{\partial^5 u}{\partial x^5} = 0 \tag{1.1}$$

where $\alpha \in (0, 1]$. Also, in the case $\gamma = 3$, Eq. (1.1) reduces to the standard Kaup-Kupershmidt equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + 20u^2 \frac{\partial u}{\partial x} + 25 \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + 10u \frac{\partial^3 u}{\partial x^3} + \frac{\partial^5 u}{\partial x^5} = 0. \tag{1.2}$$

The fifth order Kaup-Kupershmidt equation (1.1) is one of the solitonic equations related to the integrable cases of the Henon-Heiles system and belongs to the completely integrable hierarchy of higher order KdV equations [24]. These Eq (1.1) and Eq (1.2) for $\alpha = 1$ studied using various techniques, for example the extended tanh method [23], the Fan sub-equation method [24], the Projective Riccati equation method [25], the simplified Hirota's method [26].

The rest of the work is arranged as follows: In Section 2, the definition of conformable fractional derivative is introduced, and the basic properties of fractional derivative are investigated. In Section 3, we give a description of the the new sub-equation method. In Section 4, we employ the new sub-equation method to conformable time-fractional general Kaup-Kupershmidt equation. Finally, we give a concluding remarks in section 5.

2. Conformable Fractional Calculus

Conformable fractional calculus firstly mentioned by R. Khalil et. al. [27] is well behaved, applicable and obeys many rules that known derivative and integral satisfies.

Definition 2.1. *Let $f : [0, \infty) \to \mathbb{R}$ be a function. α^{th} order "conformable fractional derivative" of f is defined by*

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0$, $\alpha \in (0, 1)$. If f is α differentiable in some $(0, a)$, $a > 0$ and

$$\lim_{t \to 0} f(t)$$

exists, then define

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t).$$

But other fractional derivatives such as Caputo, Riemann-Liouville, Grünwald so not satisfy basic rules. For instance

(1) Let λ be a constant and $\alpha \in R$. So $D_a^\alpha(\lambda) \neq 0$ for Riemann-Liouville derivative.

(2) All fractional derivatives do not satisfy the known formula of the derivative of the product of two functions.

$$D_a^\alpha (fg) \neq f D_a^\alpha (g) + g D_a^\alpha (f)$$

(3) All fractional derivatives do not satisfy the known formula of the derivative of the quotient of two functions.

$$D_a^\alpha \left(\frac{f}{g}\right) \neq \frac{g D_a^\alpha (f) - f D_a^\alpha (g)}{g^2}$$

(4) All fractional derivatives do not satisfy the chain rule.

$$D_a^\alpha (fog) \neq f^{(\alpha)} (g(t))\, g^{(\alpha)}(t)$$

(5) All fractional derivatives do not satisfy

$$D^\alpha D^\beta f = D^{\alpha+\beta} f$$

in general.

(6) The Caputo derivative assumes that the function f is differentiable.

This new definition satisfies the properties which are given in the following theorem.

Theorem 2.1. *Let $\alpha \in (0,1)$ and f, g be $\alpha-$ differentiable at point $t > 0$. Then*

(1) $T_\alpha(cf + dg) = cT_\alpha(f) + dT_\alpha(g)$,*for all* $a, b \in \mathbb{R}$.
(2) $T_\alpha(t^p) = pt^{p-\alpha}$*for all*$p \in \mathbb{R}$.
(3) $T_\alpha(\lambda) = 0$ *for all constant functions* $f(t) = \lambda$.
(4) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
(5) $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(g) - fT_\alpha(f).}{g^2}$
(6) *If, in addition to f is differentiable, then* $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}$.

3. Description of the new sub-equation method

A general fractional nonlinear wave equation can be written as

$$F\left(u, \frac{\partial^\alpha u}{\partial t^\alpha}, \frac{\partial u}{\partial x}, u\frac{\partial u}{\partial x}, u^2\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots\right) = 0. \quad (3.1)$$

We seek its travelling wave solution $u(\xi)$ by letting

$$\xi = \lambda x - \eta\frac{t^\mu}{\mu} \quad (3.2)$$

where λ and η are parameters to be determined later.Now we briefly illustrate the new sub-equation method.

Step 1. Uniting the independent variables x and t into one variable ξ as usual, then from Eq.(3) we obtain

$$G(u, u', u'', \dots) = 0. \quad (3.3)$$

Step 2. The solution of Eq. (5) can be expressed by a polynomial in $f(\xi)$ as

$$u(\xi) = \sum_{j=0}^{\vartheta} b_j\, f^j(\xi), \quad (3.4)$$

where b_j $(0 \leq j \leq \vartheta)$ are constant coefficients to be determined later and $f(\xi)$ satisfies the first order linear ODE of the form

$$f'(\xi) = \alpha + \beta f'(\xi) + \sigma f''(\xi) \tag{3.5}$$

where α, β, σ are constant. On the other hand, Eq. (7) has the following travelling wave solutions

Family1. If $\beta^2 - 4\alpha\sigma > 0$ and $\sigma \neq 0$, then we have

$$f(\xi) = \frac{-\beta}{2\sigma} - \frac{\sqrt{\beta^2 - 4\alpha\sigma}}{2\sigma}\tanh_{pq}\left(\frac{\sqrt{\beta^2 - 4\alpha\sigma}}{2}\xi\right),$$
$$f(\xi) = \frac{-\beta}{2\sigma} - \frac{\sqrt{\beta^2 - 4\alpha\sigma}}{2\sigma}\coth_{pq}\left(\frac{\sqrt{\beta^2 - 4\alpha\sigma}}{2}\xi\right).$$

Family 2. If $\beta^2 - 4\alpha\sigma < 0$ and $\sigma \neq 0$, then we have

$$f(\xi) = \frac{-\beta}{2\sigma} + \frac{\sqrt{-\left(\beta^2 - 4\alpha\sigma\right)}}{2\sigma}\tan_{pq}\left(\frac{\sqrt{-\left(\beta^2 - 4\alpha\sigma\right)}}{2}\xi\right),$$
$$f(\xi) = \frac{-\beta}{2\sigma} - \frac{\sqrt{-\left(\beta^2 - 4\alpha\sigma\right)}}{2\sigma}\cot_{pq}\left(\frac{\sqrt{-\left(\beta^2 - 4\alpha\sigma\right)}}{2}\xi\right).$$

Family 3. If $\alpha\sigma < 0,\ \sigma \neq 0$ and $\beta = 0$, then we have

$$f(\xi) = -\sqrt{\frac{-\alpha}{\sigma}}\tanh_{pq}\left(\sqrt{-\alpha\sigma}\xi\right),$$
$$f(\xi) = -\sqrt{\frac{-\alpha}{\sigma}}\coth_{pq}\left(\sqrt{-\alpha\sigma}\xi\right).$$

Family 4. If $\alpha\sigma > 0,\ \sigma \neq 0$ and $\beta = 0$, then we have

$$f(\xi) = \sqrt{\frac{\alpha}{\sigma}}\tan_{pq}\left(\sqrt{\alpha\sigma}\xi\right),$$
$$f(\xi) = -\sqrt{\frac{\alpha}{\sigma}}\cot_{pq}\left(\sqrt{\alpha\sigma}\xi\right).$$

Family 5. If $\beta = 0$ and $\alpha = -\sigma$, then we have

$$f(\xi) = -\tanh_{pq}\left(\alpha\xi\right),$$
$$f(\xi) = -\coth_{pq}\left(\alpha\xi\right).$$

Case 6. If $\beta = 0$ and $\alpha = \sigma$, then we have

$$f(\xi) = \tan_{pq}\left(\alpha\xi\right),$$
$$f(\xi) = -\cot_{pq}\left(\alpha\xi\right).$$

Family 7. If $\beta^2 = 4\alpha\sigma$, then we have

$$f(\xi) = \frac{-2\alpha(\beta\xi + 2)}{\beta^2\xi}.$$

Family8. If $\alpha = 0$ and $\beta \neq 0$, then we have

$$f(\xi) = -\frac{p\beta}{\sigma\left(\cosh_{pq}(\beta\xi) - \sinh_{pq}(\beta\xi) + p\right)},$$
$$f(\xi) = -\frac{\beta\left(\sinh_{pq}(\beta\xi) + \cosh_{pq}(\beta\xi)\right)}{\sigma\left(\sinh_{pq}(\beta\xi) + \cosh_{pq}(\beta\xi) + q\right)}.$$

Family 9. If $\beta = \alpha = 0$, then we have

$$f(\xi) = \frac{-1}{\sigma\xi}.$$

Family 10. If $\beta = \upsilon, \sigma = m\upsilon$ and $\alpha = 0$, then we have

$$f(\xi) = \frac{pe^{\upsilon\xi}}{q - pme^{\upsilon\xi}}.$$

Family11. If $\beta = \upsilon$, $\alpha = m\upsilon$ $(m \neq 0)$ and $\sigma = 0$, then we have

$$f(\xi) = e^{\upsilon\xi} - m.$$

Family 12. If $\sigma = 0$, then

$$f(\xi) = e^{\beta\xi} - \frac{\alpha}{\beta}.$$

Family 13. If $\beta = \sigma = 0$, then

$$f(\xi) = \alpha\xi.$$

The generalized hypergeometric functions and the generalized trigonometric functions are defined as [28]

$$\begin{aligned}
\sinh_A(\xi) &= \frac{pe^{\xi} - qe^{-\xi}}{2}, \cosh_A(\xi) = \frac{pe^{\xi} + qe^{-\xi}}{2},\\
\tanh_A(\xi) &= \frac{pe^{\xi} - qe^{-\xi}}{pe^{\xi} + qe^{-\xi}}, \coth_A(\xi) = \frac{pe^{\xi} + qe^{-\xi}}{pe^{\xi} - qe^{-\xi}},\\
\mathrm{sech}_A(\xi) &= \frac{2}{pe^{\xi} + qe^{-\xi}}, \mathrm{csch}_A(\xi) = \frac{2}{pe^{\xi} - qe^{-\xi}},\\
\sin_A(\xi) &= \frac{pe^{i\xi} - qe^{-i\xi}}{2i}, \cos_A(\xi) = \frac{pe^{i\xi} + qe^{-i\xi}}{2},\\
\tan_A(\xi) &= -i\frac{pe^{i\xi} - qe^{-i\xi}}{pe^{i\xi} + qe^{-i\xi}}, \cot_A(\xi) = i\frac{pe^{i\xi} + qe^{-i\xi}}{pe^{i\xi} - qe^{-i\xi}},\\
\sec_A(\xi) &= \frac{2}{pe^{i\xi} + qe^{-i\xi}}, \csc_A(\xi) = \frac{2i}{pe^{i\xi} - qe^{-i\xi}}
\end{aligned}$$

where ξ is an independent variable, $p, q > 0$ are constants. Also,the positive integer ϑ can be determined by considering the homogeneous balance between the highest order derivative linear term and nonlinear terms appearing in ODE (4).

Step 3. Substituting Eq.(6) into ODE (5), making use of Eq.(7) and setting the coefficients of all powers of $f(\xi)$ to zeros, we will get a system of algebraic equations, from which λ, η and b_j $(0 \leq j \leq \vartheta)$ can be found explicitly.

Step 4. Substituting the values b_j $(0 \leq j \leq \vartheta)$ obtained in Step 3 into Eq.(6), we may get all possible solutions.

4. Exact Solutions for the conformable time-fractional general Kaup-Kupershmidt Equation

Upon using the transformation

$$u(x,t) = u(\xi), \qquad \xi = \lambda x - \eta\frac{t^{\mu}}{\mu} \tag{4.1}$$

Eq. (1) is transferred to

$$-\eta u' + \frac{1}{5}\gamma^2\lambda u^2 u' + \frac{5}{2}\gamma\lambda^3 u'u'' + \gamma\lambda^3 uu''' + \lambda^5 u^{(5)} = 0, \tag{4.2}$$

where the prime symbolizes the derivation with respect to ξ. Considering the homogeneous balance between u^2u' and $u^{(5)}$ in Eq. (9) we required that $\vartheta+5=3\vartheta+1$; then $\vartheta=2$; so we can write (6) as

$$u(\xi)=b_0+b_1\,f(\xi)+b_2\,f^2(\xi). \tag{4.3}$$

Substituting Eq. (10) along with (7) into Eq. (9) and collecting all terms with the same power of $a^{f(\xi)}$ together, the left hand side of Eq. (9) is converted into a polynomial in $f(\xi)$. Equating each coefficient to be zero yields a set of simultaneous algebraic equations. Solving the this algebraic system with respect to the unknowns variables $b_0,\ b_1\ ,\ \ b_2$and η we find the following sets of solutions

Set 1:

$$b_0=-\frac{5}{4}\frac{\lambda^2\left(8\sigma\alpha+\beta^2\right)}{\gamma},\qquad b_1=-15\frac{\lambda^2\beta\sigma}{\gamma},\quad b_2=-15\frac{\lambda^2\sigma^2}{\gamma},\qquad \eta=\frac{\lambda^5}{16}(\beta^2-4\sigma\alpha)^2.$$

Set 2:

$$b_0=-\frac{10\lambda^2\left(8\sigma\alpha+\beta^2\right)}{\gamma},\qquad b_1=-120\frac{\lambda^2\beta\sigma}{\gamma},\quad b_2=-120\frac{\lambda^2\sigma^2}{\gamma},\qquad \eta=11\lambda^5(\beta^2-4\sigma\alpha)^2.$$

Substituting Set 1 along with (8) into (10) and solutions of ODE (7), we have following travelling wave solutions of Eq. (1).

Case1. When $\beta^2-4\alpha\sigma>0$ and $\sigma\neq 0$, then

$$u(x,t)=\frac{5\lambda^2\Delta}{4\gamma}\left(2-3\tanh^2_{pq}\left(\frac{\sqrt{\Delta}}{2}\left(\lambda x-\frac{\lambda^5}{16}\Delta^2\frac{t^\mu}{\mu}\right)\right)\right),$$
$$u(x,t)=\frac{5\lambda^2\Delta}{4\gamma}\left(2-3\coth^2_{pq}\left(\frac{\sqrt{\Delta}}{2}\left(\lambda x-\frac{\lambda^5}{16}\Delta^2\frac{t^\mu}{\mu}\right)\right)\right),$$

where $\Delta=\beta^2-4\alpha\sigma$.

Case 2. When $\beta^2-4\alpha\sigma<0$ and $\sigma\neq 0$, then

$$u(x,t)=\frac{5\lambda^2\Delta}{4\gamma}\left(2+3\tan^2_{pq}\left(\frac{\sqrt{-\Delta}}{2}\left(\lambda x-\frac{\lambda^5}{16}\Delta^2\frac{t^\mu}{\mu}\right)\right)\right),$$
$$u(x,t)=\frac{5\lambda^2\Delta}{4\gamma}\left(2-3\cot^2_{pq}\left(\frac{\sqrt{-\Delta}}{2}\left(\lambda x-\frac{\lambda^5}{16}\Delta^2\frac{t^\mu}{\mu}\right)\right)\right),$$

where $\Delta=\beta^2-4\alpha\sigma$.

Case 3. When $\alpha\sigma<0,\ \sigma\neq 0$ and $\beta=0$, then

$$u(x,t)=\frac{5\lambda^2\sigma\alpha}{\gamma}\left(-2+3\tanh^2_{pq}\left(\sqrt{-\sigma\alpha}\left(\lambda x-\lambda^5(\sigma\alpha)^2\frac{t^\mu}{\mu}\right)\right)\right),$$
$$u(x,t)=\frac{5\lambda^2\sigma\alpha}{\gamma}\left(-2+3\coth^2_{pq}\left(\sqrt{-\sigma\alpha}\left(\lambda x-\lambda^5(\sigma\alpha)^2\frac{t^\mu}{\mu}\right)\right)\right).$$

Case 4. When $\alpha\sigma>0,\ \sigma\neq 0$ and $\beta=0$, then

$$u(x,t)=-\frac{5\lambda^2\sigma\alpha}{\gamma}\left(2+3\tan^2_{pq}\left(\sqrt{\sigma\alpha}\left(\lambda x-\lambda^5(\sigma\alpha)^2\frac{t^\mu}{\mu}\right)\right)\right),$$
$$u(x,t)=\frac{5\lambda^2\sigma\alpha}{\gamma}\left(-2+3\cot^2_{pq}\left(\sqrt{\sigma\alpha}\left(\lambda x-\lambda^5(\sigma\alpha)^2\frac{t^\mu}{\mu}\right)\right)\right).$$

Case 5. When $\beta=0$ and $\alpha=-\sigma$, then

$$u(x,t)=-\frac{5\lambda^2\alpha^2}{\gamma}\left(-2+3\tanh_{pq}^2\left(\alpha\left(\lambda x-\lambda^5\alpha^4\frac{t^\mu}{\mu}\right)\right)\right),$$
$$u(x,t)=-\frac{5\lambda^2\alpha^2}{\gamma}\left(-2+3\coth_{pq}^2\left(\alpha\left(\lambda x-\lambda^5\alpha^4\frac{t^\mu}{\mu}\right)\right)\right),$$

Case 6. When $\beta=0$ and $\alpha=\sigma$, then

$$u(x,t)=-\frac{5\lambda^2\alpha^2}{\gamma}\left(2+3\tan_{pq}^2\left(\alpha\left(\lambda x-\lambda^5\alpha^4\frac{t^\mu}{\mu}\right)\right)\right),$$
$$u(x,t)=-\frac{5\lambda^2\alpha^2}{\gamma}\left(-2+3\cot_{pq}^2\left(\alpha\left(\lambda x-\lambda^5\alpha^4\frac{t^\mu}{\mu}\right)\right)\right),$$

Case 8. When $\alpha=0$ and $\beta\neq 0$, then

$$u(x,t)=\frac{10\lambda^2\beta^2}{\gamma}\left(-\frac{1}{4}+\frac{12p}{\left(\cosh_{pq}\left(\beta\left(\lambda x-\frac{\lambda^5\beta^4}{16}\frac{t^\mu}{\mu}\right)\right)-\sinh_{pq}\left(\beta\left(\lambda x-\frac{\lambda^5\beta^4}{16}\frac{t^\mu}{\mu}\right)\right)+p\right)}\right.$$
$$\left.-12\left(-\frac{p}{\left(\cosh_{pq}\left(\beta\left(\lambda x-\frac{\lambda^5\beta^4}{16}\frac{t^\mu}{\mu}\right)\right)-\sinh_{pq}\left(\beta\left(\lambda x-\frac{\lambda^5\beta^4}{16}\frac{t^\mu}{\mu}\right)\right)+p\right)}\right)^2\right),$$

$$u(x,t)=\frac{10\lambda^2\beta^2}{\gamma}\left(-\frac{1}{4}+\frac{12\left(\sinh_{pq}\left(\beta\left(\lambda x-\frac{\lambda^5\beta^4}{16}\frac{t^\mu}{\mu}\right)\right)+\cosh_{pq}\left(\beta\left(\lambda x-\frac{\lambda^5\beta^4}{16}\frac{t^\mu}{\mu}\right)\right)\right)}{\left(\sinh_{pq}\left(\beta\left(\lambda x-\frac{\lambda^5\beta^4}{16}\frac{t^\mu}{\mu}\right)\right)+\cosh_{pq}\left(\beta\left(\lambda x-\frac{\lambda^5\beta^4}{16}\frac{t^\mu}{\mu}\right)\right)+q\right)}\right.$$
$$\left.-12\left(\frac{\left(\sinh_{pq}\left(\beta\left(\lambda x-\frac{\lambda^5\beta^4}{16}\frac{t^\mu}{\mu}\right)\right)+\cosh_{pq}\left(\beta\left(\lambda x-\frac{\lambda^5\beta^4}{16}\frac{t^\mu}{\mu}\right)\right)\right)}{\left(\sinh_{pq}\left(\beta\left(\lambda x-\frac{\lambda^5\beta^4}{16}\frac{t^\mu}{\mu}\right)\right)+\cosh_{pq}\left(\beta\left(\lambda x-\frac{\lambda^5\beta^4}{16}\frac{t^\mu}{\mu}\right)\right)+q\right)}\right)^2\right).$$

Case 10. When $beta=\upsilon,\sigma=m\upsilon$ and $\alpha=0$ then

$$u(x,t)=-\frac{10\lambda^2\upsilon^2}{\gamma}\left(\frac{1}{4}+12m\,\frac{pe^{\upsilon\left(\lambda x-\frac{\lambda^5\upsilon^4}{16}\frac{t^\mu}{\mu}\right)}}{q-pme^{\upsilon\left(\lambda x-\frac{\lambda^5\upsilon^4}{16}\frac{t^\mu}{\mu}\right)}}\;+12m\left(\frac{pe^{\upsilon\left(\lambda x-\frac{\lambda^5\upsilon^4}{16}\frac{t^\mu}{\mu}\right)}}{q-pme^{\upsilon\left(\lambda x-\frac{\lambda^5\upsilon^4}{16}\frac{t^\mu}{\mu}\right)}}\right)^2\right).$$

Substituting Set 2 along with (8) into (10) and solutions of ODE (7), we have following travelling wave solutions of Eq. (1).

Case1. When$\beta^2-4\alpha\sigma>0$ and $\sigma\neq 0$, then

$$u(x,t)=\frac{10\lambda^2\Delta}{\gamma}\left(2-3\tanh_{pq}^2\left(\frac{\sqrt{\Delta}}{2}\left(\lambda x-11\lambda^5\Delta^2\frac{t^\mu}{\mu}\right)\right)\right),$$

$$u(x,t)=\frac{10\lambda^2\Delta}{\gamma}\left(2-3\coth_{pq}^2\left(\frac{\sqrt{\Delta}}{2}\left(\lambda x-11\lambda^5\Delta^2\frac{t^\mu}{\mu}\right)\right)\right),$$

where $\Delta=\beta^2-4\alpha\sigma$.

Case 2. When $\beta^2-4\alpha\sigma<0$ and $\sigma\neq 0$, then

$$u(x,t)=\frac{10\lambda^2\Delta}{\gamma}\left(2+3\tan_{pq}^2\left(\frac{\sqrt{-\Delta}}{2}\left(\lambda x-11\lambda^5\Delta^2\frac{t^\mu}{\mu}\right)\right)\right),$$

$$u(x,t)=\frac{10\lambda^2\Delta}{\gamma}\left(2-3\cot_{pq}^2\left(\frac{\sqrt{-\Delta}}{2}\left(\lambda x-11\lambda^5\Delta^2\frac{t^\mu}{\mu}\right)\right)\right),$$

where $\Delta=\beta^2-4\alpha\sigma$.

Case 3. When $\alpha\sigma<0,\ \sigma\neq 0$ and $\beta=0$, then

$$u(x,t)=\frac{40\lambda^2\sigma\alpha}{\gamma}\left(-2+3\tanh_{pq}^2\left(\sqrt{-\sigma\alpha}\left(\lambda x-176\lambda^5(\sigma\alpha)^2\frac{t^\mu}{\mu}\right)\right)\right),$$

$$u(x,t)=\frac{40\lambda^2\sigma\alpha}{\gamma}\left(-2+3\coth^2_{pq}\left(\sqrt{-\sigma\alpha}\left(\lambda x-176\lambda^5(\sigma\alpha)^2\frac{t^\mu}{\mu}\right)\right)\right).$$

Case 4. When $\alpha\sigma>0,\ \sigma\neq 0$ and $\beta=0$, then

$$u(x,t)=-\frac{40\lambda^2\sigma\alpha}{\gamma}\left(2+3\tan^2_{pq}\left(\sqrt{\sigma\alpha}\left(\lambda x-176\lambda^5(\sigma\alpha)^2\frac{t^\mu}{\mu}\right)\right)\right),$$

$$u(x,t)=\frac{40\lambda^2\sigma\alpha}{\gamma}\left(-2+3\cot^2_{pq}\left(\sqrt{\sigma\alpha}\left(\lambda x-176\lambda^5(\sigma\alpha)^2\frac{t^\mu}{\mu}\right)\right)\right).$$

Case 5. When$\beta=0$ and $\alpha=-\sigma$, then

$$u(x,t)=-\frac{40\lambda^2\alpha^2}{\gamma}\left(-2+3\tanh^2_{pq}\left(\alpha\left(\lambda x-176\lambda^5\alpha^4\frac{t^\mu}{\mu}\right)\right)\right),$$

$$u(x,t)=-\frac{40\lambda^2\alpha^2}{\gamma}\left(-2+3\coth^2_{pq}\left(\alpha\left(\lambda x-176\lambda^5\alpha^4\frac{t^\mu}{\mu}\right)\right)\right),$$

Case 6. When $\beta=0$ and $\alpha=\sigma$, then

$$u(x,t)=-\frac{40\lambda^2\alpha^2}{\gamma}\left(2+3\tan^2_{pq}\left(\alpha\left(\lambda x-176\lambda^5\alpha^4\frac{t^\mu}{\mu}\right)\right)\right),$$

$$u(x,t)=-\frac{40\lambda^2\alpha^2}{\gamma}\left(-2+3\cot^2_{pq}\left(\alpha\left(\lambda x-176\lambda^5\alpha^4\frac{t^\mu}{\mu}\right)\right)\right),$$

Case 8. When $\alpha=0$ and $\beta\neq 0$, then

$$u(x,t)=\frac{10\lambda^2\beta^2}{\gamma}\left(-\frac{1}{4}+\frac{12p}{\left(\cosh_{pq}\left(\beta\left(\lambda x-11\lambda^5\beta^4\frac{t^\mu}{\mu}\right)\right)-\sinh_{pq}\left(\beta\left(\lambda x-11\lambda^5\beta^4\frac{t^\mu}{\mu}\right)\right)+p\right)}\right.$$
$$\left.-12\left(-\frac{p}{\left(\cosh_{pq}\left(\beta\left(\lambda x-11\lambda^5\beta^4\frac{t^\mu}{\mu}\right)\right)-\sinh_{pq}\left(\beta\left(\lambda x-11\lambda^5\beta^4\frac{t^\mu}{\mu}\right)\right)+p\right)}\right)^2\right),$$

$$u(x,t)=\frac{10\lambda^2\beta^2}{\gamma}\left(-\frac{1}{4}+\frac{12\left(\sinh_{pq}\left(\beta\left(\lambda x-11\lambda^5\beta^4\frac{t^\mu}{\mu}\right)\right)+\cosh_{pq}\left(\beta\left(\lambda x-11\lambda^5\beta^4\frac{t^\mu}{\mu}\right)\right)\right)}{\left(\sinh_{pq}\left(\beta\left(\lambda x-11\lambda^5\beta^4\frac{t^\mu}{\mu}\right)\right)+\cosh_{pq}\left(\beta\left(\lambda x-11\lambda^5\beta^4\frac{t^\mu}{\mu}\right)\right)+q\right)}\right.$$
$$\left.-12\left(\frac{\left(\sinh_{pq}\left(\beta\left(\lambda x-11\lambda^5\beta^4\frac{t^\mu}{\mu}\right)\right)+\cosh_{pq}\left(\beta\left(\lambda x-11\lambda^5\beta^4\frac{t^\mu}{\mu}\right)\right)\right)}{\left(\sinh_{pq}\left(\beta\left(\lambda x-11\lambda^5\beta^4\frac{t^\mu}{\mu}\right)\right)+\cosh_{pq}\left(\beta\left(\lambda x-11\lambda^5\beta^4\frac{t^\mu}{\mu}\right)\right)+q\right)}\right)^2\right).$$

Case 10. When $\beta=\upsilon,\sigma=m\upsilon$ and $\alpha=0$ then

$$u(x,t)=-\frac{10\lambda^2\upsilon^2}{\gamma}\left(\frac{1}{4}+12m\,\frac{pe^{\upsilon\left(\lambda x-11\lambda^5\upsilon^4\frac{t^\mu}{\mu}\right)}}{q-pme^{\upsilon\left(\lambda x-11\lambda^5\upsilon^4\frac{t^\mu}{\mu}\right)}}+12m\left(\frac{pe^{\upsilon\left(\lambda x-11\lambda^5\upsilon^4\frac{t^\mu}{\mu}\right)}}{q-pme^{\upsilon\left(\lambda x-11\lambda^5\upsilon^4\frac{t^\mu}{\mu}\right)}}\right)^2\right).$$

5. Conclusions

The new sub-equation method has been successfully employed to time fractional general Kaup-Kuperschmidt equation. Authors fistly used chain rule and wave transform so the nonlinear conformable fractional differential equation turns into differential equation with integer order derivative. All the results show that both arguments(chain rule, wave transform, new sub-equation method) are applicable, reliable are efficient tools for obtaining the exact solutions of nonlinear partial differential equations with conformable fractional derivative.

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