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A SUBCLASS OF HARMONIC UNIVALENT FUNCTION INVOLVING ERROR FUNCTION

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Abstract. In this investigation we introduce and study a subclass of harmonic univalent functions associated with error function and salagean operator using convolutional approach. Coefficient estimate, distortion bounds, growth theorem, extreme points and convex combination were established for a new class $\overline{T}_{H,n}(\lambda)$. Consequently, our results are new approaches to those corresponding to previously known results.

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1. Introduction

The theory of special functions are significantly inportant to scientist and engineers with mathematically calculations [Oladipo, 10].The special function does not have specific definition but its applications extends to physics, computer e.t.c. There are various special functions but we shall concern ourselves with error function which has application in probability, statistics,materials science and partial differential equations. Abramowitz el al[1] defined error function as

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{k=2}^{\infty} \frac{(-1)^{k-1} z^{2k+1}}{(2k+1)k!}$$

For detailed about properties and inequalities error function and properties of complementary error function see [3,5,7].

We shall denote by A the class of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z|, z \in U\}$nt functions with normalization $f(0) = f'(0) - 1 = 0$. Also, let S be the subclass of A consisting of univalent functions in U.

A function $f \in A$ is said to be in the class S^* and class C of starlike and convex in A, if it satisfies the following inequalities respectifully

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in U \tag{1.2}$$

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$$Re\left(1+\frac{zf''(z)}{f'(z)}\right)>0,\quad z\in U \tag{1.3}$$

Let E be the class of modified error function which was intrduced and studied by Ramchandran et al [11]as

$$E_rf(z)=\frac{\sqrt{\pi z}}{2}erf(\sqrt{z})=z+\sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{(2k-1)(k-1)!}z^k \tag{1.4}$$

where E_rf is anormalized analytic function which is obtained from (1) A continous complex-valued function $f=u+iv$ is said to be harmonic in a simply-connected domain D if both u and v are real harmonic in D. Various classes of harmonic functions have been extensively investigated in the literature of the subject for example, see [2,6,8,13] and other references there in.The function $f=h+\overline{g}$ is said to be harmonic univalent in D if the mapping $z\to f(z)$ is orietation preserving, harmonic and one-to-one in D. We call h the analytic part and g the co-analytic off. For detailed see Clunie and Sheil-Small [4].

Denote by H the class of the class of the functions that are harmonic univalent and orientation preserving in the open unit disk $U=\{z:|z|<1\}$ for which $f(0)=f_z(0)-1=0$. Then for $f=h+\overline{g}\in H$, we may express the analytic functions f and g as

A function $f=h+\overline{g}$ with h and g be given by (1) is said to be harmonic starlike of order β, $0\le\beta<1$ for $|z|=r<1$, if

$$\frac{\partial}{\partial\theta}(argf(re^{i\theta}))=R\left\{\frac{zh'(z)-\overline{zzg'(z)}}{h(z)+\overline{g}(z)}\right\}\geq\beta \tag{1.5}$$

The class of all harmonic starlike of order β is denoted by $S_H^*(\beta)$ and vigoriously study by jahangiri [9].

The convolution of two power series

$$D^n(f)(z)=z-\sum_{k=2}^{\infty}k^na_kz^k$$

and

$$E_rf(z)=z-\sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{(2k-1)(k-1)!}z^k$$

is defined by

$$(D^n*E_r)(z)=z-\sum_{k=2}^{\infty}\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}a_kz^k$$

where D^n isderivative operator defined by Salagean [12]

Definition 1: Let $D^nE_rf=D^nE_rh+\overline{D^nE_rg}$ with h and g be given by (1) where

$$D^nE_rh(z)=z-\sum_{k=2}^{\infty}\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}a_kz^k,$$

$$D^nE_rg(z)=\sum_{k=1}^{\infty}\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}\overline{b_k}z^k,\quad |b_1|<1.$$

Then $f\in\overline{T}_{H,n}(\lambda)$ if and only if , for $\beta, 0\le\beta<1$ and $[-1,1]$

$$Re\left(\frac{zD^nErf'(z)-\overline{zD^nErg'(z)}}{D^nErf(z)+\overline{D^nErg(z)}}-\lambda\right)\leq\left|\frac{zD^nErf'(z)-\overline{zD^nErg'(z)}}{D^nErf(z)+\overline{D^nErg(z)}}-1\right| \tag{1.6}$$

Theorem 1:Let $f \in \overline{T}S_{H,n}(\lambda)$ then

$$\sum_{k=2}^{\infty} \frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}\left[2k-\lambda-1\right]|a_k| + \sum_{k=1}^{\infty} \frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}\left[2k+\lambda+1\right]\left|b_{\overline{k}}\right| \leq 1-\lambda \tag{1.7}$$

Proof: Let $f \in \overline{T}_{H,n}(\lambda)$ with $\lambda \in [-1,1]$. We have

$$Re\left(\frac{zD^nErf'(z) - \overline{zD^nErg'(z)}}{D^nErf(z) + \overline{D^nErg(z)}} - \lambda\right) \leq \left|\frac{zD^nErf'(z) - \overline{zD^nErg'(z)}}{D^nErf(z) + \overline{D^nErg(z)}} - 1\right| \tag{1.8}$$

If we take $z \in [0,1]$, we have

$$\frac{1 - \sum_{k=2}^{\infty} \frac{k^{n+1}(-1)^{k-1}}{(2k-1)(k-1)!} a_k z^{k-1} - \sum_{k=1}^{\infty} \frac{k^{n+1}(-1)^{k-1}}{(2k-1)(k-1)!} \overline{b_k} z^{k-1}}{1 - \sum_{k=2}^{\infty} \frac{k^{n}(-1)^{k-1}}{(2k-1)(k-1)!} a_k z^{k-1} + \sum_{k=1}^{\infty} \frac{k^{n}(-1)^{k-1}}{(2k-1)(k-1)!} \overline{b_k} z^{k-1}} - \lambda$$

$$\geq 1 - \frac{1 - \sum_{k=2}^{\infty} \frac{k^{n+1}(-1)^{k-1}}{(2k-1)(k-1)!} a_k z^{k-1} - \sum_{k=1}^{\infty} \frac{k^{n+1}(-1)^{k-1}}{(2k-1)(k-1)!} \overline{b_k} z^{k-1}}{1 - \sum_{k=2}^{\infty} \frac{k^{n}(-1)^{k-1}}{(2k-1)(k-1)!} a_k z^{k-1} + \sum_{k=1}^{\infty} \frac{k^{n}(-1)^{k-1}}{(2k-1)(k-1)!} \overline{b_k} z^{k-1}} \tag{1.9}$$

This yields

$$\sum_{k=2}^{\infty} \frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}\left[2k-\lambda-1\right]|a_k| + \sum_{k=1}^{\infty} \frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}\left[2k+\lambda+1\right]\left|b_{\overline{k}}\right| \leq 1-\lambda \tag{1.10}$$

Letting $z \to 1^{-1}$ along the real axis we have

$$\sum_{k=2}^{\infty} \frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}\left[2k-\lambda-1\right]|a_k| + \sum_{k=1}^{\infty} \frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}\left[2k+\lambda+1\right]\left|b_{\overline{k}}\right| < 1-\lambda \tag{1.11}$$

Conversely, let us take $f(z) \in T$ for which the relation () hold. It suffices to show that

$$\left|\frac{zD^nErf'(z) - \overline{zD^nErg'(z)}}{D^nErf(z) + \overline{D^nErg(z)}} - 1\right| - Re\left\{\frac{zD^nErf'(z) - \overline{zD^nErg'(z)}}{D^nErf(z) + \overline{D^nErg(z)}} - 1\right\} \leq 1-\lambda \tag{1.12}$$

We have

$$\left|\frac{zD^nErf'(z) - \overline{zD^nErg'(z)}}{D^nErf(z) + \overline{D^nErg(z)}} - 1\right| - Re\left\{\frac{zD^nErf'(z) - \overline{zD^nErg'(z)}}{D^nErf(z) + \overline{D^nErg(z)}} - 1\right\} \leq$$

$$2\left|\frac{zD^nErf'(z) - \overline{zD^nErg'(z)}}{D^nErf(z) + \overline{D^nErg(z)}} - 1\right| \tag{1.13}$$

$$= 2\left|\frac{zD^nErf'(z) - \overline{zD^nErg'(z)} - D^nErf(z) - \overline{D^nErg(z)}}{D^nErf(z) + \overline{D^nErg(z)}}\right| \tag{1.14}$$

$$\leq \frac{1 - \sum_{k=2}^{\infty} \frac{2k^{n}(-1)^{k-1}}{(2k-1)(k-1)!}[k-1]a_k + \sum_{k=1}^{\infty} \frac{2k^{n}(-1)^{k-1}}{(2k-1)(k-1)!}[k+1]a_k}{1 - \sum_{k=2}^{\infty} \frac{k^{n+1}(-1)^{k-1}}{(2k-1)(k-1)!} a_k z^{k-1} - \sum_{k=1}^{\infty} \frac{k^{n+1}(-1)^{k-1}}{(2k-1)(k-1)!} \overline{b_k} z^{k-1}} \tag{1.15}$$

The last expression is bounded above by $1-\lambda$ if

$$\sum_{k=2}^{\infty} \frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}\left[2k-\lambda-1\right]|a_k| + \sum_{k=1}^{\infty} \frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}\left[2k+\lambda+1\right]\left|b_{\overline{k}}\right| \leq 1-\lambda \tag{1.16}$$

If we choose $\lambda = 0$ in the theorem above,yields Corrolary 1 below:

Corrolary 1:Let $f \in \overline{T}S_{H,n}(0)$ then

$$\sum_{k=2}^{\infty} \frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}\left[2k-1\right]|a_k| + \sum_{k=1}^{\infty} \frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}\left[2k+1\right]\left|b_{\overline{k}}\right| \leq 1 \tag{1.17}$$

Theorem 2: Let $f \in \overline{T}_{H,n}(\lambda)$ with $\lambda \in [-1,1]$. We have

$$|f(z)| \leq (1+|b_1|)r - \frac{3}{2^n}\left[\frac{1-\lambda}{3-\lambda} - \frac{3+\lambda}{3-\lambda}\right] r^2 \tag{1.18}$$

$$|f(z)| \geq (1+|b_1|)r + \frac{3}{2^n}\left[\frac{1-\lambda}{3-\lambda} - \frac{3+\lambda}{3-\lambda}\right] r^2 \quad where\, |b_1| \leq \frac{1-\lambda}{3+\lambda} \tag{1.19}$$

The result is sharp

Proof: We shall prove the first inequality. Let $f \in \overline{T}_{H,n}(\lambda)$. Then we have

$$|f(z)| \geq (1+|b_1|)r + \sum_{k=2}^{\infty}\left(|a_k|+|b_k|\right) r^k \leq (1+|b_1|)r + r^2\sum_{k=2}^{\infty}\left(|a_k|+|b_k|\right) \tag{1.20}$$

$$= (1+|b_1|)r - \frac{3(1-\lambda)}{2^n(3-\lambda)}\sum_{k=2}^{\infty}\frac{2^n(3-\lambda)}{3(1-\lambda)}\left(|a_k|+|b_k|\right) r^2 \tag{1.21}$$

and so

$$|f(z)| \leq (1+|b_1|)r - \frac{3(1-\lambda)}{2^n(3-\lambda)}\sum_{k=2}^{\infty} A\left[\frac{2k-\lambda-1}{1-\lambda}|a_k| + \frac{2k+\lambda+1}{1-\lambda}|b_k|\right] r^2 \tag{1.22}$$

$$\leq (1+|b_1|)r - \frac{3(1-\lambda)}{2^n(3-\lambda)}\left[1 - \frac{3+\lambda}{1-\lambda}|b_1|\right] r^2 \tag{1.23}$$

$$=\leq (1+|b_1|)r - \frac{3}{2^n}\left[\frac{1-\lambda}{3-\lambda} - \frac{3+\lambda}{3-\lambda}\right] r^2 \tag{1.24}$$

$$=\leq (1+|b_1|)r - \frac{3}{2^n}\left[\frac{1-\lambda}{3-\lambda} - \frac{3+\lambda}{3-\lambda}\right] r^2 \tag{1.25}$$

where $A = \frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}$

On the other hand

$$|f(z)| \geq (1+|b_1|)r - \sum_{k=2}^{\infty}\left(|a_k|+|b_k|\right) r^k \leq (1+|b_1|)r + r^2\sum_{k=2}^{\infty}\left(|a_k|+|b_k|\right) \tag{1.26}$$

$$= (1+|b_1|)r + \frac{3(1-\lambda)}{2^n(3-\lambda)}\sum_{k=2}^{\infty}\frac{2^n(3-\lambda)}{3(1-\lambda)}\left(|a_k|+|b_k|\right) r^2 \tag{1.27}$$

and so

$$|f(z)| \geq (1+|b_1|)r + \frac{3(1-\lambda)}{2^n(3-\lambda)}\sum_{k=2}^{\infty} A\left[\frac{2k-\lambda-1}{1-\lambda}|a_k| + \frac{2k+\lambda+1}{1-\lambda}|b_k|\right] r^2 \tag{1.28}$$

$$\geq (1+|b_1|)r + \frac{3(1-\lambda)}{2^n(3-\lambda)}\left[1 - \frac{3+\lambda}{1-\lambda}|b_1|\right] r^2 \tag{1.29}$$

$$= (1+|b_1|)r + \frac{3}{2^n}\left[\frac{1-\lambda}{3-\lambda} - \frac{3+\lambda}{3-\lambda}\right] r^2 \tag{1.30}$$

$$= (1+|b_1|)r + \frac{3}{2^n}\left[\frac{1-\lambda}{3-\lambda} - \frac{3+\lambda}{3-\lambda}\right] r^2 \tag{1.31}$$

which complete the proof.

Theorem 3: Let $f \in \overline{T}_{H,n}(\lambda)$ with $\lambda \in [-1,1]$. We have

$$|f'(z)| \leq 1 + |b_1| - \frac{6}{2^n}\left[\frac{1-\lambda}{3-\lambda} - \frac{3+\lambda}{3-\lambda}\right] r \tag{1.32}$$

$$|f'(z)| \geq (1+|b_1|)r + \frac{6}{2^n}\left[\frac{1-\lambda}{3-\lambda} - \frac{3+\lambda}{3-\lambda}\right] r \quad where\, |b_1| \leq \frac{1-\lambda}{3+\lambda} \tag{1.33}$$

The result is sharp

Proof: We shall prove the first inequality. Let $f \in \overline{T}_{H,n}(\lambda)$. Then we have

$$|f'(z)| \geq 1 + |b_1| + \sum_{k=2}^{\infty} k\left(|a_k| + |b_k|\right) r^k \geq 1 + |b_1| + r\sum_{k=2}^{\infty} k\left(|a_k| + |b_k|\right) \tag{1.34}$$

$$= 1 + |b_1| - \frac{6(1-\lambda)}{2^n(3-\lambda)} \sum_{k=2}^{\infty} \frac{2^n(3-\lambda)}{6(1-\lambda)}\left(|a_k| + |b_k|\right) r \tag{1.35}$$

and so

$$|f(z)| \geq 1 + |b_1| - \frac{6(1-\lambda)}{2^n(3-\lambda)} \sum_{k=2}^{\infty} A\left[\frac{2k-\lambda-1}{1-\lambda}|a_k| + \frac{2k+\lambda+1}{1-\lambda}|b_k|\right] r \tag{1.36}$$

$$\leq 1 + |b_1| - \frac{3(1-\lambda)}{2^n(3-\lambda)}\left[1 - \frac{3+\lambda}{1-\lambda}|b_1|\right] r \tag{1.37}$$

$$= 1 + |b_1| - \frac{6}{2^n}\left[\frac{1-\lambda}{3-\lambda} - \frac{3+\lambda}{3-\lambda}\right] r \tag{1.38}$$

$$= (1 + |b_1| - \frac{6}{2^n}\left[\frac{1-\lambda}{3-\lambda} - \frac{3+\lambda}{3-\lambda}\right] r \tag{1.39}$$

where $A = \frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}$

On the other hand

$$|f'(z)| \geq 1 + |b_1| + \sum_{k=2}^{\infty} k\left(|a_k| + |b_k|\right) r^k \geq (1 + |b_1|) + r\sum_{k=2}^{\infty} k\left(|a_k| + |b_k|\right) \tag{1.40}$$

$$= 1 + |b_1| - \frac{6(1-\lambda)}{2^n(3-\lambda)} \sum_{k=2}^{\infty} \frac{2^n(3-\lambda)}{6(1-\lambda)}\left(|a_k| + |b_k|\right) r \tag{1.41}$$

and so

$$|f(z)| \geq 1 + |b_1| - \frac{6(1-\lambda)}{2^n(3-\lambda)} \sum_{k=2}^{\infty} A\left[\frac{2k-\lambda-1}{1-\lambda}|a_k| + \frac{2k+\lambda+1}{1-\lambda}|b_k|\right] r \tag{1.42}$$

$$\leq 1 + |b_1| - \frac{3(1-\lambda)}{2^n(3-\lambda)}\left[1 - \frac{3+\lambda}{1-\lambda}|b_1|\right] r \tag{1.43}$$

$$= 1 + |b_1| - \frac{6}{2^n}\left[\frac{1-\lambda}{3-\lambda} - \frac{3+\lambda}{3-\lambda}\right] r \tag{1.44}$$

$$= (1 + |b_1| - \frac{6}{2^n}\left[\frac{1-\lambda}{3-\lambda} - \frac{3+\lambda}{3-\lambda}\right] r \tag{1.45}$$

where $A = \frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}$

On the other hand which complete the proof.

Theorem 4: Let $f = h + \overline{g}$, where h and g are given by (1). Then $f \in clcoH\overline{T}_{H,n}(\lambda)$ if and only if

$$f(z) = \sum_{k=1}^{\infty} (X_k h_k + Y_k g_k) \tag{1.46}$$

where

$$h_1(z) = z$$
$$h_k(z) = z + \frac{1-\lambda}{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k-\lambda-1]} z^k \quad (k = 2, 3, ..),$$
$$g_k(z) = z + \frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k+\lambda+1]\,\overline{z}^k \quad (k = 1, 2, 3, ..),$$

$\sum_{k=1}^{\infty}(X_k + Y_k), X_k \geq 0$ and $Y_k \geq 0$. In particular, the extrem points of the class $\overline{T}_{H,n}$ are $\{h_k\}$ and $\{g_k\}$ respectively.

Proof. For a function of the form (9), we have

$$f(z) = \sum_{k=1}^{\infty} (X_k h_k + Y_k g_k)$$

$$= \sum_{k=1}^{\infty}(X_k + Y_k)(z) + \sum_{k=2}^{\infty}\frac{1-\lambda}{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k-\lambda-1]} X_k z^k + \sum_{k=1}^{\infty}\frac{1-\lambda}{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k+\lambda+1]} Y_k \overline{z}^k$$

$$= z + \sum_{k=2}^{\infty}\frac{1-\lambda}{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k-\lambda-1]} X_k z^k + \sum_{k=1}^{\infty}\frac{1-\lambda}{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k+\lambda+1]} Y_k \overline{z}^k$$

But

$$\sum_{k=2}^{\infty}\frac{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k-\lambda-1]}{1-\lambda}\left[\frac{1-\lambda}{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k-\lambda-1]} X_k\right] +$$
$$\sum_{k=2}^{\infty}\frac{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k+\lambda+1]}{1-\lambda}\left[\frac{1-\lambda}{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k+\lambda+1]} X_k\right]$$
$$\sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \leq 1$$

Thus $f \in clcoH\overline{T}_{H,n}(\lambda)$.

Conversely, suppose that $f \in clcoH\overline{T}_{H,n}(\lambda)$. Set

$$X_k = \frac{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k-\lambda-1]}{1-\lambda}|a_k| \quad (k = 2, 3, ..),$$

and

$$Y_k = \frac{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k+\lambda+1]}{1-\lambda}|b_k| \quad (k = 1, 2, 3, ..),$$

Then by the inequality Theorem 1,we have $0 \leq X_k \leq 1 (k = 2, 3, ..)$ and $0 \leq Y_k \leq 1 (k = 1, 2, 3, ..)$. Define $X_1 = 1 - \sum_{k=2}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k$ and note that $X_1 \geq 0$. Thus we obain $f(z) = \sum_{k=2}^{\infty} X_k h_k + Y_k g_k$. This completes the proof of Theorem 4.

Convolution and Convex Combinations

For two harmonic functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b}_k \overline{z}^k$$

$$F(z) = z + \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} \overline{B}_k \overline{z}^k$$

we define their convolution

$$(f * F)(z) = z + \sum_{k=2}^{\infty} a_k A_k z^k + \sum_{k=1}^{\infty} \overline{b_k B_k} \overline{z}^k,$$

using this definition, we show that the class $\overline{T}_{H,n}(\lambda)$ is close under convolution.

Theorem 5. Let $f, F \in \overline{T}_{H,n}(\lambda)$.Then $f * F \in \overline{T}_{H,n}(\lambda)$.

Proof: We note that $|A_k| \leq 1$ and $|B_k| \leq 1$.Now for the convolution $f * F$ we have

$$\sum_{k=2}^{\infty} \frac{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k-\lambda-1]}{1-\lambda}|A_k a_k| + \sum_{k=1}^{\infty} \frac{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k+\lambda+1]}{1-\lambda}|B_k b_k|$$

$$\leq \sum_{k=2}^{\infty} \frac{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k-\lambda-1]}{1-\lambda}|a_k| + \sum_{k=1}^{\infty} \frac{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k+\lambda+1]}{1-\lambda}|b_k| \leq 1$$

Therefore $f * F \in \overline{T}_{H,n}(\lambda)$, which complete the proof.

We show that the class $\overline{T}_{H,n}(\lambda)$ is closed under convex combination of its members.

Theorem 6: The class $\overline{T}_{H,n}(\lambda)$ is closed under convex combination.

Proof: For $(i = 1, 2, 3, ...)$let $f_i \in \overline{T}_{H,n}(\lambda)$ where $f_i(z)$ is given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b}_k \overline{z}^k$$

Then by Theorem 1, we have

$$\sum_{k=2}^{\infty} \frac{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k-\lambda-1]}{1-\lambda}|a_{k_i}| + \sum_{k=1}^{\infty} \frac{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k+\lambda+1]}{1-\lambda}|b_{k_i}| \leq 1$$

For $\sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i = z + \sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i a_{k_i}\right) z^k + \sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i \overline{b_{k_i}}\right) z^k$$

Then by Theorem 1, we have

$$\sum_{k=2}^{\infty} \frac{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k-\lambda-1]}{1-\lambda}\left|\sum_{i=1}^{\infty} t_i a_{k_i}\right| + \sum_{k=1}^{\infty} \frac{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k+\lambda+1]}{1-\lambda}\left|\overline{\sum_{i=1}^{\infty} t_i b_{k_i}}\right|$$

$$\leq \sum_{t=1}^{\infty} t_1 \left(\sum_{k=2}^{\infty} \frac{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k-\lambda-1]}{1-\lambda}|a_{k_i}| + \sum_{k=1}^{\infty} \frac{\frac{k^n(-1)^{k-1}}{(2k-1)(k-1)!}[2k+\lambda+1]}{1-\lambda}|b_{k_i}|\right)$$

$$\leq \sum_{t=1}^{\infty} t_1 = 1$$

Therefore $\sum_{i=1}^{\infty} t_i f_i \in \overline{T}_{H,n}(\lambda)$

2. Conclusions

Varying various parameter involved in class $f \in \overline{T}S_{H,n}(\lambda)$,known or new result could be arrived at.

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