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ON SOME PROPERTIES OF GENERALIZED INVERSE HYPERBOLIC FUNCTIONS

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Abstract. In this paper, we provide inverses for some generalized hyperbolic functions. We also study some properties of these inverse functions. Furthermore, by using basic techniques, we establish some inequalities (or bounds) for the inverse functions. As a by-product of the established results, we obtain some inequalities (or bounds) for the logarithmic function.

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1. Introduction

Inverse trigonometric and inverse hyperbolic functions are very useful in several areas of applied mathematics. They have important applications in engineering in particular. In recent years, inequalities concerning these functions have been studied extensively by many researchers and as a result, there exist a rich literature on this subject. See for example [1], [2], [3], [4], [5], [6], [7], [8], [9], [11], [13], [17], [18], [19], [20], [21], [23], [24] and the related references therein.

In a recent work, the authors [15] considered the following generalizations of the hyperbolic cosine, hyperbolic sine and hyperbolic tangent functions.

$$\cosh_a(z) = \frac{a^z + a^{-z}}{2}, \tag{1.1}$$

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(1.2) $$\sinh_a(z) = \frac{a^z - a^{-z}}{2},$$

(1.3) $$\tanh_a(z) = \frac{\sinh_a(z)}{\cosh_a(z)} = \frac{a^z - a^{-z}}{a^z + a^{-z}} = 1 - \frac{2}{1 + a^{2z}},$$

where $a > 1$ and $z \in (-\infty, \infty)$. Cosequently, the generalized hyperbolic secant, hyperbolic cosecant and hyperbolic cotangent functions are respectively defined as

(1.4) $$\operatorname{sech}_a(z) = \frac{1}{\cosh_a(z)}, \quad \operatorname{cosech}_a(z) = \frac{1}{\sinh_a(z)}, \quad \coth_a(z) = \frac{1}{\tanh_a(z)},$$

and in particular, if $a = e$, where $e = 2.71828...$ is the Euler's number, then the above definitions reduce to their ordinary counterparts. For more information on these generalized functions, one could refer to the works [12], [16] and [14].

Motivated by the above works, the objective of this work is to provide inverses of the functions (1.1)-(1.4) and to further study some properties of the inverse functions. We also provide some inequalities (or bounds) for the inverse functions.

2. Results and Discussion

Proposition 2.1. *Let* $\operatorname{arsinh}_a(z)$, $\operatorname{arcosh}_a(z)$, $\operatorname{artanh}_a(z)$, $\operatorname{arcosech}_a(z)$, $\operatorname{arsech}_a(z)$ *and* $\operatorname{arcoth}_a(z)$ *respectively be the inverses of the functions* $\sinh_a(z)$, $\cosh_a(z)$, $\tanh_a(z)$, $\operatorname{cosech}_a(z)$, $\operatorname{sech}_a(z)$ *and* $\coth_a(z)$. *Then*

(2.1) $$\operatorname{arsinh}_a(z) = \frac{1}{\ln a} \ln\left(z + \sqrt{z^2 + 1}\right), \quad z \in (-\infty, \infty),$$

(2.2) $$\operatorname{arcosh}_a(z) = \frac{1}{\ln a} \ln\left(z + \sqrt{z^2 - 1}\right), \quad z \in [1, \infty),$$

(2.3) $$\operatorname{artanh}_a(z) = \frac{1}{2\ln a} \ln\left(\frac{1+z}{1-z}\right), \quad z \in (-1, 1),$$

(2.4) $$\operatorname{arcosech}_a(z) = \frac{1}{\ln a} \ln\left(\frac{1}{z} + \sqrt{\frac{1}{z^2} + 1}\right) = \frac{1}{\ln a} \ln\left(\frac{1 + \sqrt{1 + z^2}}{z}\right), \quad z \neq 0,$$

(2.5) $$\operatorname{arsech}_a(z) = \frac{1}{\ln a} \ln\left(\frac{1}{z} + \sqrt{\frac{1}{z^2} - 1}\right) = \frac{1}{\ln a} \ln\left(\frac{1 + \sqrt{1 - z^2}}{z}\right), \quad z \in (0, 1],$$

$$\text{(2.6)} \qquad \operatorname{arcoth}_a(z) = \frac{1}{2\ln a} \ln\left(\frac{z+1}{z-1}\right), \quad z \in (-\infty,-1)\cup(1,\infty).$$

Proof. Let $u = \operatorname{arsinh}_a(z)$ so that $z = \sinh_a(u)$. This implies that $a^{2u} - 2za^u - 1 = 0$ which is a quadratic equation in a^u. Thus $a^u = z \pm \sqrt{z^2+1}$. Since $a^u > 0$ for all u, then we consider the root $z + \sqrt{z^2+1}$ which is positive for all $z \in (-\infty,\infty)$. Hence $u = \frac{1}{\ln a} \ln\left(z + \sqrt{z^2+1}\right)$ which gives (2.1).

Next, let $v = \operatorname{arcosh}_a(z)$ so that $z = \cosh_a(v)$. Since $\cosh_a(v)$ is not one-to-one, we have to restrict its domain to $[0,\infty)$ in order for the inverse to exist. Now $z = \cosh_a(v)$ implies that $a^{2v} - 2za^v + 1 = 0$ and so $a^v = z \pm \sqrt{z^2-1}$. Here, both roots are positive for all $z \in [1,\infty)$. Since $\ln(z - \sqrt{z^2-1}) = -\ln(z + \sqrt{z^2-1})$, then $\ln(z \pm \sqrt{z^2-1}) = \pm\ln(z + \sqrt{z^2-1})$ and because of the restriction, we consider the positive case. Hence $v = \frac{1}{\ln a} \ln\left(z + \sqrt{z^2-1}\right)$ which gives (2.2).

Next, let $w = \operatorname{artanh}_a(z)$ so that $z = \tanh_a(w)$. This implies that $a^{2w} = \frac{1+z}{1-z}$ which is positive for all $z \in (-1,1)$. Hence $w = \frac{1}{2\ln a} \ln\left(\frac{1+z}{1-z}\right)$ which gives (2.3).

Next, let $r = \operatorname{arcosech}_a(z)$ so that $z = \operatorname{cosech}_a(r)$. This implies that $a^r = \frac{1\pm\sqrt{1+z^2}}{z} = \frac{1}{z} \pm \sqrt{\frac{1}{z^2}+1}$. Since $\frac{1}{z}+\sqrt{\frac{1}{z^2}+1}$ is positive for all $z \neq 0$, then we conclude that $r = \frac{1}{\ln a} \ln\left(\frac{1}{z} + \sqrt{\frac{1}{z^2}+1}\right)$ which gives (2.4).

Next, let $k = \operatorname{arsech}_a(z)$ so that $z = \operatorname{sech}_a(k)$. Similarly, since $\operatorname{sech}_a(k)$ is not one-to-one, we have to restrict its domain to $[0,\infty)$ in order for the inverse to exist. Now $z = \operatorname{sech}_a(k)$ implies that $za^{2k} - 2a^k + z = 0$ and so $a^k = \frac{1\pm\sqrt{1-z^2}}{z}$. Here too, both roots are positive for all $z \in (0,1]$. Since $\ln(\frac{1-\sqrt{1-z^2}}{z}) = -\ln(\frac{1+\sqrt{1-z^2}}{z})$, then $\ln(\frac{1\pm\sqrt{1-z^2}}{z}) = \pm\ln(\frac{1+\sqrt{1-z^2}}{z}) = \pm\ln(\frac{1}{z} + \sqrt{\frac{1}{z^2}-1})$. Because of the restriction, we consider the positive case. Hence $k = \frac{1}{\ln a} \ln\left(\frac{1}{z} + \sqrt{\frac{1}{z^2}-1}\right)$ which gives (2.5).

Finally, let $\delta = \operatorname{arcoth}_a(z)$ so that $z = \coth_a(\delta)$. This implies that $a^{2\delta} = \frac{z+1}{z-1}$ which is positive for all $z \in (-\infty,-1)\cup(1,\infty)$. Hence $\delta = \frac{1}{2\ln a} \ln\left(\frac{z+1}{z-1}\right)$ which gives (2.6). □

Proposition 2.2. *The following identities hold.*

$$\text{(2.7)} \qquad \operatorname{arsinh}_a\left(\frac{z}{r}\right) = \frac{1}{\ln a} \int \frac{dz}{\sqrt{z^2+r^2}} + \beta, \quad z \in (-\infty,\infty),\ r > 0,$$

$$\text{(2.8)} \qquad \operatorname{arcosh}_a\left(\frac{z}{r}\right) = \frac{1}{\ln a} \int \frac{dz}{\sqrt{z^2-r^2}} + \beta, \quad z > r > 0,$$

$$\operatorname{artanh}_a\left(\frac{z}{r}\right) = \frac{r}{\ln a}\int \frac{dz}{r^2 - z^2} + \beta, \quad |z| < r, \tag{2.9}$$

$$\operatorname{arcoth}_a\left(\frac{z}{r}\right) = \frac{r}{\ln a}\int \frac{dz}{r^2 - z^2} + \beta, \quad |z| > r > 0, \tag{2.10}$$

where β is a constant.

Proof. Let $z \in (-\infty, \infty)$ and $r > 0$. Then by (2.1), we have

$$\operatorname{arsinh}_a\left(\frac{z}{r}\right) = \frac{1}{\ln a}\ln\left(\frac{z + \sqrt{z^2 + r^2}}{r}\right),$$

which implies that

$$\frac{d}{dz}\operatorname{arsinh}_a\left(\frac{z}{r}\right) = \frac{1}{\ln a}\frac{1}{\sqrt{z^2 + r^2}},$$

and this is equivalent to (2.7). Next, let $z > r > 0$. Then by using (2.2), we arrive at

$$\frac{d}{dz}\operatorname{arcosh}_a\left(\frac{z}{r}\right) = \frac{1}{\ln a}\frac{1}{\sqrt{z^2 - r^2}},$$

which is equivalent to (2.8). Next let $|z| < r$. Then by using (2.3), we have

$$\operatorname{artanh}_a\left(\frac{z}{r}\right) = \frac{1}{\ln a}\left[\ln(r + z) - \ln(r - z)\right],$$

which implies that

$$\frac{d}{dz}\operatorname{artanh}_a\left(\frac{z}{r}\right) = \frac{1}{\ln a}\frac{r}{r^2 - z^2},$$

and this is equivalent to (2.9). Finally, let $|z| > r > 0$. Then by using (2.6), we arrive at

$$\frac{d}{dz}\operatorname{arcoth}_a\left(\frac{z}{r}\right) = -\frac{1}{\ln a}\frac{r}{z^2 - r^2},$$

which is equivalent to (2.10). □

Proposition 2.3. *The generalized inverse hyperbolic functions satisfy the following properties*

$$\sinh_a(\operatorname{arcosh}_a(z)) = \sqrt{z^2 - 1}, \quad |z| > 1, \tag{2.11}$$

$$\sinh_a(\operatorname{artanh}_a(z)) = \frac{z}{\sqrt{1 - z^2}}, \quad |z| < 1, \tag{2.12}$$

$$\cosh_a(\operatorname{arsinh}_a(z)) = \sqrt{z^2 + 1}, \quad z \in (-\infty, \infty), \tag{2.13}$$

$$\cosh_a(\operatorname{artanh}_a(z)) = \frac{1}{\sqrt{1-z^2}}, \quad |z|<1, \tag{2.14}$$

$$\tanh_a(\operatorname{arsinh}_a(z)) = \frac{z}{\sqrt{z^2+1}}, \quad z \in (-\infty,\infty), \tag{2.15}$$

$$\tanh_a(\operatorname{arcosh}_a(z)) = \frac{\sqrt{z^2-1}}{z}, \quad |z|>1. \tag{2.16}$$

Proof. Let $\sinh_a(\operatorname{arcosh}_a(z)) = y$ and $\operatorname{arcosh}_a(z) = \theta$. This implies that $z = \cosh_a(\theta)$ and $y = \sinh_a(\theta)$. Then by applying identity $\cosh_a^2(z) - \sinh_a^2(z) = 1$, we obtain $z^2 - y^2 = 1$ which implies that $y = \sqrt{z^2-1}$ where $|z| > 1$. This gives (2.11).

Similarly, let $\sinh_a(\operatorname{artanh}_a(z)) = y$ and $\operatorname{artanh}_a(z) = \phi$. Thus, $z = \tanh_a(\phi)$ and $y = \sinh_a(\phi)$. Then by the identity, we obtain $\frac{1}{z^2} - 1 = \frac{1}{y^2}$ which implies that $y = \frac{z}{\sqrt{1-z^2}}$ where $|z| < 1$. This gives (2.12).

The proofs of (2.13)-(2.16) follow the same procedure. As a result, we omit the details. □

Theorem 2.1. *The following inequalities are valid.*

$$\frac{1}{\ln a}\frac{v-u}{\sqrt{v^2+1}} < \operatorname{arsinh}_a(v) - \operatorname{arsinh}_a(u) < \frac{1}{\ln a}\frac{v-u}{\sqrt{u^2+1}}, \quad 0 \le u < v, \tag{2.17}$$

$$\frac{1}{\ln a}\frac{v-u}{\sqrt{v^2-1}} < \operatorname{arcosh}_a(v) - \operatorname{arcosh}_a(u) < \frac{1}{\ln a}\frac{v-u}{\sqrt{u^2-1}}, \quad 1 < u < v, \tag{2.18}$$

$$\frac{1}{\ln a}\frac{v-u}{1-u^2} < \operatorname{artanh}_a(v) - \operatorname{artanh}_a(u) < \frac{1}{\ln a}\frac{v-u}{1-v^2}, \quad 0 \le u < v < 1, \tag{2.19}$$

$$\frac{1}{\ln a}\frac{v-u}{1-u^2} < \operatorname{arcoth}_a(v) - \operatorname{arcoth}_a(u) < \frac{1}{\ln a}\frac{v-u}{1-v^2}, \quad 1 < u < v. \tag{2.20}$$

Proof. Consider the function $\operatorname{arsinh}_a(t)$ on the interval $0 \le u < v$. Then by the classical mean value theorem, there exist $c \in (u,v)$ such that

$$\frac{\operatorname{arsinh}_a(v) - \operatorname{arsinh}_a(u)}{v-u} = \frac{1}{\ln a}\frac{1}{\sqrt{c^2+1}} = \psi(c).$$

Since $\psi(t) = \frac{1}{\ln a}\frac{1}{\sqrt{t^2+1}}$ is decreasing for $t \ge 0$, then for $c \in (u,v)$, we have $\psi(v) < \psi(c) < \psi(u)$ which yields (2.17).

Next, consider the function $\operatorname{arcosh}_a(t)$ on the interval $1 < u < v$. Then by the mean value theorem, there exist $c \in (u, v)$ such that

$$\frac{\operatorname{arcosh}_a(v) - \operatorname{arcosh}_a(u)}{v - u} = \frac{1}{\ln a}\frac{1}{\sqrt{c^2 - 1}} = \theta(c).$$

Since $\theta(t) = \frac{1}{\ln a}\frac{1}{\sqrt{t^2-1}}$ is decreasing for $t > 1$, then for $c \in (u, v)$, we have $\theta(v) < \theta(c) < \theta(u)$ which yields (2.18).

Next, consider the function $\operatorname{artanh}_a(t)$ on the interval $0 \leq u < v < 1$. Then by the mean value theorem, there exist $c \in (u, v)$ such that

$$\frac{\operatorname{artanh}_a(v) - \operatorname{artanh}_a(u)}{v - u} = \frac{1}{\ln a}\frac{1}{1 - c^2} = \phi(c).$$

Since $\phi(t) = \frac{1}{\ln a}\frac{1}{1-t^2}$ is increasing for $t \geq 0$, then for $c \in (u, v)$, we have $\phi(u) < \phi(c) < \phi(v)$ which yields (2.19).

Finally, consider the function $\operatorname{arcoth}_a(t)$ on the interval $1 < u < v$. Then by the mean value theorem, there exist $c \in (u, v)$ such that

$$\frac{\operatorname{arcoth}_a(v) - \operatorname{arcoth}_a(u)}{v - u} = \frac{1}{\ln a}\frac{1}{1 - c^2} = \beta(c).$$

Since $\beta(t) = \frac{1}{\ln a}\frac{1}{1-t^2}$ is increasing for $t > 1$, then for $c \in (u, v)$, we have $\beta(u) < \beta(c) < \beta(v)$ which yields (2.20). □

Corollary 2.1. *The following inequalities are valid.*

(2.21)
$$\frac{1}{\ln a}\frac{z}{\sqrt{z^2+1}} < \operatorname{arsinh}_a(z) < \frac{1}{\ln a}z, \quad z > 0,$$

(2.22)
$$\frac{1}{\ln a}\left(\ln(2+\sqrt{3}) + \frac{z-2}{\sqrt{z^2-1}}\right) < \operatorname{arcosh}_a(z) < \frac{1}{\ln a}\left(\ln(2+\sqrt{3}) + \frac{z-2}{\sqrt{3}}\right), \; z > 2,$$

(2.23)
$$\frac{1}{\ln a}z < \operatorname{artanh}_a(z) < \frac{1}{\ln a}\frac{z}{1-z^2}, \quad 0 < z < 1,$$

(2.24)
$$\frac{1}{\ln a}\left(\frac{\ln 3}{2} - \frac{z-2}{3}\right) < \operatorname{arcoth}_a(z) < \frac{1}{\ln a}\left(\frac{\ln 3}{2} + \frac{z-2}{1-z^2}\right), \quad z > 2.$$

Proof. By letting $u = 0$ and $v = z$ in (2.17), we obtain (2.21). By letting $u = 2$ and $v = z$ in (2.18), we obtain (2.22). By letting $u = 0$ and $v = z$ in (2.19), we obtain (2.23). By letting $u = 2$ and $v = z$ in (2.20), we obtain (2.24). □

Remark 2.1. *Inequalities (2.21), (2.22), (2.23) and (2.24) respectively imply the following results concerning the logarithmic function.*

$$\frac{z}{\sqrt{z^2+1}} < \ln(z+\sqrt{z^2+1}) < z, \quad z>0, \tag{2.25}$$

$$\ln(2+\sqrt{3}) + \frac{z-2}{\sqrt{z^2-1}} < \ln(z+\sqrt{z^2-1}) < \ln(2+\sqrt{3}) + \frac{z-2}{\sqrt{3}}, \quad z>2, \tag{2.26}$$

$$2z < \ln\left(\frac{1+z}{1-z}\right) < \frac{2z}{1-z^2}, \quad 0<z<1, \tag{2.27}$$

$$\ln 3 - \frac{2}{3}(z-2) < \ln\left(\frac{z+1}{z-1}\right) < \ln 3 + 2\left(\frac{z-2}{1-z^2}\right), \quad z>2. \tag{2.28}$$

Remark 2.2. *By letting $s=\frac{1+z}{1-z}$ in (2.27) and $s=\frac{z+1}{z-1}$ in (2.28), we respectively obtain*

$$2\left(\frac{s-1}{s+1}\right) < \ln s < \frac{s^2-1}{2s}, \quad s>1, \tag{2.29}$$

and

$$\ln 3 + 2\left(\frac{s-3}{s-1}\right) < \ln s < \ln 3 + \frac{(s-3)(s-1)}{2s}, \quad 1<s<3. \tag{2.30}$$

By letting $s=x+1$ in (2.29), we recover inequality (3) of [22]. Furthermore, by letting $s=1+\frac{1}{x}$ in [10, Problem 3.6.18, p.273], we obtain

$$2\left(\frac{s-1}{s+1}\right) < \ln s < \frac{s-1}{\sqrt{s}}, \quad s>1. \tag{2.31}$$

It is observed that the upper part of (2.31) is stronger than the upper part of (2.29).

Theorem 2.2. *The inequality*

$$\frac{1}{\ln a}\left(\frac{4(v-u)}{4-(u+v)^2}\right) < \operatorname{artanh}_a(v) - \operatorname{artanh}_a(u) < \frac{1}{2\ln a}\left(\frac{v-u}{1-u^2} + \frac{v-u}{1-v^2}\right), \tag{2.32}$$

holds for $-1<u<v<1$ and consequently, the inequality

$$\frac{1}{\ln a}\left(\frac{4z}{4-z^2}\right) < \operatorname{artanh}_a(z) < \frac{1}{2\ln a}\left(z + \frac{z}{1-z^2}\right), \quad 0<z<1, \tag{2.33}$$

also holds.

Proof. We employ the Hermite-Hadamard inequality

$$p\left(\frac{k_1+k_2}{2}\right) \leq \frac{1}{k_2-k_1}\int_{k_1}^{k_2} p(z)dz \leq \frac{p(k_1)+p(k_2)}{2} \tag{2.34}$$

for a convex functin p on the interval $[k_1, k_2]$. Let $p(z) = \frac{1}{\ln a}\frac{1}{1-z^2}$ where $-1 < z < 1$. Then p is convex. Next, let $-1 < u < v < 1$. Then by (2.34), we obtain

$$\frac{1}{\ln a}\left(\frac{4}{4-(u+v)^2}\right) < \frac{\operatorname{artanh}_a(v) - \operatorname{artanh}_a(u)}{v-u} < \frac{1}{2\ln a}\left(\frac{1}{1-u^2} + \frac{1}{1-v^2}\right),$$

which gives (2.32). Inequality (2.33) is obtained by letting $u = 0$ and $v = z$ in (2.32). □

Remark 2.3. *Since $z < \frac{4z}{1-z^2}$ and $\frac{1}{2}\left(z + \frac{z}{1-z^2}\right) < \frac{z}{1-z^2}$ for all $0 < z < 1$, then inequality (2.33) is sharper than inequality (2.23).*

Remark 2.4. *Inequality (2.33) implies that*

$$\frac{8z}{4-z^2} < \ln\left(\frac{1+z}{1-z}\right) < z + \frac{z}{1-z^2}, \quad 0 < z < 1, \tag{2.35}$$

and by letting $s = \frac{1+z}{1-z}$ in (2.35), we obtain

$$\frac{8s^2-8}{3s^2+10s+3} < \ln s < \frac{s^3+5s^2-5s-1}{4s^2+4s}, \quad s > 1, \tag{2.36}$$

which is a better estimate than (2.29). Thus, (2.36) is a refinement of inequality (3) in [22]. The upper part of (2.36) is however weaker than inequality (22) in [22]

Theorem 2.3. *The inequality*

$$a^{-\operatorname{arcosh}_a(y)} < y < a^{\operatorname{arcosh}_a(y)}, \quad y > 1, \tag{2.37}$$

also holds.

Proof. This follows directly from the inequality [16, (3.3)]

$$a^{-z} < \cosh_a(z) < a^z, \quad z > 0,$$

by letting $\cosh_a(z) = y$ so that $z = \operatorname{arcosh}_a(y)$. □

Theorem 2.4. *The inequalities*

$$\frac{1}{\ln a}\frac{1}{\sqrt{y^2+1}} < \frac{\operatorname{arsinh}_a(y)}{y} < \frac{1}{\ln a}\sqrt{y^2+1}, \quad y \neq 0, \tag{2.38}$$

$$\frac{1}{\ln a}\frac{\sqrt{y^2-1}}{y^2} < \frac{\operatorname{arcosh}_a(y)}{y} < \frac{1}{\ln a}\sqrt{y^2-1}, \quad y > 1, \tag{2.39}$$

are valid.

Proof. We make use of the inequality [16, (3.17)]

$$\frac{\ln a}{\cosh_a(z)} < \frac{\sinh_a(z)}{z} < (\ln a)\cosh_a(z), \quad z \neq 0. \tag{2.40}$$

Let $\sinh_a(z) = y$ so that $z = \operatorname{arsinh}_a(y)$. The condition that $z \neq 0$ implies that $y \neq 0$. Moreover, $\cosh_a(z) = \cosh_a(\operatorname{arsinh}_a(y)) = \sqrt{y^2+1}$. Substituting these into (2.40) yields (2.38).

Next, let $\cosh_a(z) = y$ so that $z = \operatorname{arcosh}_a(y)$. The condition that $z \neq 0$ implies that $y > 1$. Additionally, $\sinh_a(z) = \sinh_a(\operatorname{arcosh}_a(y)) = \sqrt{y^2-1}$. Substituting these into (2.40) yields (2.39). □

Theorem 2.5. *For $p \geq 3$, the inequalities*

$$\left(\frac{\operatorname{arsinh}_a(y)}{y}\right)^p < \frac{1}{\sqrt{y^2+1}}, \quad y \neq 0, \tag{2.41}$$

$$\frac{\operatorname{arcosh}_a(y)}{y} < \frac{\sqrt{y^2-1}}{y^{1+1/p}}, \quad y > 1, \tag{2.42}$$

are valid.

Proof. We use the inequality [12]

$$\cosh_a(z) < \left(\frac{\sinh_a(z)}{z}\right)^p, \quad z \neq 0, \quad p \geq 3. \tag{2.43}$$

By letting $\sinh_a(z) = y$ in (2.43), and adopting the technique of the proof of Theorem 2.4, we obtain (2.41). Likewise, by letting $\cosh_a(z) = y$ in (2.43), we obtain (2.42). □

Theorem 2.6. *The inequality*

$$\operatorname{arcosh}_a(y) > \frac{3\sqrt{y^2-1}}{(\ln a)(2+y)}, \quad y > 1, \tag{2.44}$$

is valid.

Proof. This follows directly from the inequality [12]

$$\frac{\sinh_a(z)}{z} < \frac{2\ln a + (\ln a)\cosh_a(z)}{3}, \quad z \neq 0,$$

by letting $\cosh_a(z) = y$. □

Theorem 2.7. *The inequalities*

$$\left(\frac{y}{\operatorname{arsinh}_a(y)}\right)^2 + \frac{1}{\sqrt{y^2+1}}\left(\frac{y}{\operatorname{arsinh}_a(y)}\right) > 2, \quad y \neq 0, \tag{2.45}$$

$$\frac{1}{1-y^2}\left(\frac{y}{\operatorname{artanh}_a(y)}\right)^2+\frac{y}{\operatorname{artanh}_a(y)}>2, \quad y\in(-1,0)\cup(0,1), \tag{2.46}$$

are valid.

Proof. We use the inequality [12]

$$\left(\frac{\sinh_a(z)}{z}\right)^2+\frac{\tanh_a(z)}{z}>2, \quad z\neq 0. \tag{2.47}$$

Let $\sinh_a(z)=y$ so that $z=\operatorname{arsinh}_a(y)$ and $\tanh_a(z)=\tanh_a(\operatorname{arsinh}_a(y))=\frac{y}{\sqrt{y^2+1}}$. The condition that $z\neq 0$ implies $y\neq 0$. Substituting these into (2.47) yields (2.45). Also, let $\tanh_a(z)=y$ so that $z=\operatorname{artanh}_a(y)$ and $\sinh_a(z)=\sinh_a(\operatorname{artanh}_a(y))=\frac{y}{\sqrt{1-y^2}}$. The condition that $z\neq 0$ implies that $y\in(-1,0)\cup(0,1)$. Substituting these into (2.47) yields (2.46). □

Theorem 2.8. *The inequalities*

$$\frac{3}{(\ln a)(2+\sqrt{y^2+1})}<\frac{\operatorname{arsinh}_a(y)}{y}<\frac{1}{3\ln a}\left(2+\frac{1}{\sqrt{y^2+1}}\right), \quad y\neq 0, \tag{2.48}$$

$$\frac{3}{(\ln a)(1+2\sqrt{1-y^2})}<\frac{\operatorname{artanh}_a(y)}{y}<\frac{1}{3\ln a}\left(1+\frac{2}{\sqrt{1-y^2}}\right), \; y\in(-1,0)\cup(0,1), \tag{2.49}$$

are valid.

Proof. Here we employ the Huygen's type inequalities [12]

$$2\frac{\sinh_a(z)}{z}+\frac{\tanh_a(z)}{z}>3\ln a, \quad z\neq 0, \tag{2.50}$$

$$2\frac{z}{\sinh_a(z)}+\frac{z}{\tanh_a(z)}>\frac{3}{\ln a}, \quad z\neq 0. \tag{2.51}$$

As in the above, let $\sinh_a(z)=y$ so that $z=\operatorname{arsinh}_a(y)$ and $\tanh_a(z)=\tanh_a(\operatorname{arsinh}_a(y))=\frac{y}{\sqrt{y^2+1}}$. Then substituting these into (2.50) and (2.51) respectively yields the upper and lower bounds of (2.48). In the same way, let $\tanh_a(z)=y$ so that $z=\operatorname{artanh}_a(y)$ and $\sinh_a(z)=\sinh_a(\operatorname{artanh}_a(y))=\frac{y}{\sqrt{1-y^2}}$. Then substituting these into (2.50) and (2.51) respectively yields the upper and lower bounds of (2.49). □

3. Concluding Remarks

We have provided inverses for the generalized hyperbolic functions $\cosh_a(z)$, $\sinh_a(z)$ and $\tanh_a(z)$, where $a > 1$ and $z \in (-\infty, \infty)$. We have also considered some properties satisfied by these functions. Furthermore, we have established some inequalities (or bounds) for the inverse functions and as a by-product, we obtained some inequalities (or bounds) for the logarithmic function. For the particular case where $a = e$, we obtain the corresponding results for the ordinary inverse hyperbolic functions.

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