Asia Pac. J. Math. 2020 **7**:23

EXISTENCE RESULTS FOR BVP OF A CLASS OF GENERALIZED FRACTIONAL-ORDER IMPLICIT DIFFERENTIAL EQUATIONS

KADDA MAAZOUZ[1,*], DEVARAJ VIVEK[2]

[1]Department of Mathematics, University of Tiaret, Algeria

[2]Department of Mathematics, PSG College of Arts & Science, Coimbatore, India-641 014

*Corresponding author: kadda.maazouz@univ-tiaret.dz

Received Jun. 3, 2020

Abstract. In this paper, we study the existence of solutions to boundary value problem for implicit differential equations involving generalized fractional derivative via fixed point methods.

2010 Mathematics Subject Classification. 26A33; 34A08; 34B15.

Key words and phrases. generalized fractional derivative; boundary value problem; integral equation; fixed point.

1. Introduction

Because of its wide applicability in biology, medicine and in more and more fields, the theory of fractional differential equations has recently been attracting increasing interest. Especially, many research papers had devoted to generalized fractional differential operator, this concept of generalized integral and derivative was given through Katugampola [9,10]. The use of Katugampola fractional derivative (KFD) is to generalize the Hadamard and Riemann-Liouville integrals and derivatives which widely discussed by many researchers, one can refer to [6,9,10,18]. Anderson et al. [1] studied some properties of KFD with potential application in quantum mechanics. In [6], Janaki et al. established existence and uniqueness of solutions to the impulsive differential equations with inclusions, and the authors also established some conditions for the uniqueness and existence of solutions for a class of fractional implicit differential equations with KFD [7]. Recently, Vivek et al. [18] investigated existence and stability of solutions for impulsive type integro-differential equations. Followed by the work,

DOI: 10.28924/APJM/7-23

the existence and Ulam stability of solutions for impulsive type pantograph equations was considered in [19].

As a result of unifying different techniques for initial or boundary conditions, nonlinear boundary conditions received more and more attention, see [2–4, 8, 11–16, 20].

In this paper, we consider the following boundary value problem for implicit differential equations with KFD of the form

$$(1.1)\qquad \begin{cases} {}^{\rho}D^{\alpha}u(t) = \Psi(t, u(t), {}^{\rho}D^{\alpha}u(t)), \quad t \in J := [a, b], \quad 1 < \alpha < 2, \quad \rho > 0, \\ c_1 u(a) - d_1 u'(a) = u_1, \\ c_2 u(b) - d_2 u'(b) = u_2, \end{cases}$$

where ${}^{\rho}D^{\alpha}$ is the generalized fractional derivative of order α, $\Psi : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, is given function, $c_1, c_2, d_1, d_2, u_1, u_2 \in \mathbb{R}$ and $0 \leq a < b < \infty$.

The paper is organized as follows: In Section 2, we present definitions, lemmas, and some results. Section 3 is devoted to establish our main results. Finally, two illustrative examples are given to illustrate the theoritical results.

2. Fundamental results

We now introduce some definitions, preliminary facts about the fractional calculus, notations, and some auxiliary results, which will be used later.

Definition 2.1. [10] *The generalized left-sided fractional integral of order $\alpha \in \mathbb{C}$, $(Re(\alpha) > 0)$ is defined for $t > a$ by*

$$(2.1)\qquad {}^{\rho}I^{\alpha}h(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \left(t^{\rho} - s^{\rho}\right)^{\alpha-1} s^{\rho-1} h(s) ds$$

if the integral exists, where $\Gamma(.)$ is the Gamma function.

Definition 2.2. [10] *The generalized left-sided fractional derivative, corresponding to the generalized fractional integral (2.1) is defined for $t > a$ by*

$${}^{\rho}D^{\alpha}h(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right) \int_a^t \left(t^{\rho} - s^{\rho}\right)^{n-\alpha-1} s^{\rho-1} h(s) ds$$

where $n = [\alpha] + 1$, if the integral exists.

Lemma 2.1. *Let $\alpha > 0$ and $\rho > 0$, then the differential equation*

$${}^{\rho}D^{\alpha}f(t) = 0$$

has solutions

$$f(t) = a_0 + \sum_{k=1}^{n-1} a_k \Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\alpha-k}, \quad a_k \in \mathbb{R},\ k = 0, 1, 2, \ldots, n-1, \quad n = [\alpha] + 1.$$

Lemma 2.2. *Let* $\alpha > 0$ *and* $\rho > 0$, *then*

$${}^\rho I^\alpha \Big({}^\rho D^\alpha f(t)\Big) = f(t) + a_0 + \sum_{k=1}^{n-1} a_k \Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\alpha-k},$$

for some

$$a_k \in \mathbb{R},\ k = 0, 1, 2, \ldots, n-1, \quad n = [\alpha] + 1.$$

Theorem 2.1. [5](*Nonlinear alternative*)
Let X *be a Banach space with* $C \subset X$ *closed and convex. Assume* U *is a relatively open subset of* C *with* $0 \in U$ *and* $T : \overline{U} \longrightarrow C$ *is a compact. Then either,*

(1) T *has a fixed point in* $\overline{U}$*; or*
(2) *there is a point* $u \in \partial U$ *and* $\lambda \in (0, 1)$ *with* $u = \lambda T u$.

Theorem 2.2. [17] (*Krasnoselskii's fixed point theorem*)
Let E *be a bounded closed convex and nonempty subset of a Banach space* X. *Let* A, B *two operators such that* $Ax + By \in E$ *for every pair* $x, y \in E$. *If* A *is a contraction and* B *is completely continuous. Then there exists* $z \in E$ *such that* $Az + Bz = z$.

3. Main results

The following lemma is essential to state and prove our main result

Lemma 3.1. *Let* $1 < \alpha < 2$, $\rho > 0$ *and* $\psi \in C(J, \mathbb{R})$ *be a continuous function. Then the following boundary value problem*

$$\begin{cases} {}^\rho D^\alpha u(t) = \psi(t), \quad t \in J, \\ c_1 u(a) - d_1 u'(a) = u_1, \\ c_2 u(b) - d_2 u'(b) = u_2, \end{cases} \tag{3.1}$$

has a unique solution given by

$$u(t) = \frac{c_2 u_1}{c_1} - u_2 + \phi_{a,b}\sigma_t + \int_a^t K_t^\alpha(s)\psi(s)ds,$$

where

$$K_t^\alpha(s) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}(t^\rho - s^\rho)^{\alpha-1} s^{\rho-1}, \qquad \sigma_t = \Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\alpha-1},$$

$$\phi_{a,b} = \frac{1}{\delta}\Big(\frac{c_2 u_1}{c_1} - u_2 + \int_a^b K(s)\psi(s)ds\Big), \quad K(s) = c_2 K_b^{\alpha}(s) - d_2 b^{\rho-1} K_b^{\alpha-1}(s),$$

and

$$\delta = d_2(\alpha-1)b^{\rho-1}\Big(\frac{b^\rho - a^\rho}{\rho}\Big)^{\alpha-2} - c_2\Big(\frac{b^\rho - a^\rho}{\rho}\Big)^{\alpha-1}.$$

Proof. Let u satisfies (3.1) then, by Lemmas 2.1 and 2.1 we have

$$\begin{aligned} u(t) &= a_0 + a_1\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\alpha-1} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1}\psi(s)ds \\ &= a_0 + a_1\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\alpha-1} + \int_a^t K_t^{\alpha}(s)\psi(s)ds, \end{aligned}$$

Then

$$u'(t) = a_1(\alpha-1)t^{\rho-1}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\alpha-2} + t^{\rho-1}\int_a^t K_t^{\alpha-1}(s)\psi(s)ds.$$

Therefore

$$u(a) = a_0, \quad \text{and} \quad u'(a) = 0$$

so we have

$$c_1 u(a) - d_1 u'(a) = c_1 a_0 = u_1$$

it follows that

$$a_0 = \frac{u_1}{c_1}.$$

On the other hand we have

$$c_2 u(b) = c_2 a_0 + c_2 a_1\Big(\frac{b^\rho - a^\rho}{\rho}\Big)^{\alpha-1} + c_2\int_a^b K_b^{\alpha}(s)\psi(s)ds,$$

and

$$d_2 u'(b) = d_2 a_1(\alpha-1)b^{\rho-1}\Big(\frac{b^\rho - a^\rho}{\rho}\Big)^{\alpha-2} + d_2 b^{\rho-1}\int_a^b K_b^{\alpha-1}(s)\psi(s)ds.$$

Then we obtain

$$\begin{aligned}
c_2u(b) - d_2u'(b) &= c_2a_0 + c_2a_1\Big(\frac{b^\rho - a^\rho}{\rho}\Big)^{\alpha-1} - d_2a_1(\alpha-1)b^{\rho-1}\Big(\frac{b^\rho - a^\rho}{\rho}\Big)^{\alpha-2} \\
&\quad + \int_a^b \Big[c_2K_b^\alpha(s) - d_2b^{\rho-1}K_b^{\alpha-1}(s)\Big]\psi(s)ds = u_2 \\
&= \frac{c_2u_1}{c_1} + c_2a_1\Big(\frac{b^\rho - a^\rho}{\rho}\Big)^{\alpha-1} - d_2a_1(\alpha-1)b^{\rho-1}\Big(\frac{b^\rho - a^\rho}{\rho}\Big)^{\alpha-2} \\
&\quad + \int_a^b \Big[c_2K_b^\alpha(s) - d_2b^{\rho-1}K_b^{\alpha-1}(s)\Big]\psi(s)ds = u_2 \\
&= \frac{c_2u_1}{c_1} - a_1\Bigg(d_2(\alpha-1)b^{\rho-1}\Big(\frac{b^\rho - a^\rho}{\rho}\Big)^{\alpha-2} - c_2\Big(\frac{b^\rho - a^\rho}{\rho}\Big)^{\alpha-1}\Bigg) \\
&\quad + \int_a^b \Big[c_2K_b^\alpha(s) - d_2b^{\rho-1}K_b^{\alpha-1}(s)\Big]\psi(s)ds.
\end{aligned}$$

From $c_2u(b) - d_2u'(b) = u_2$ we deduce that

$$\begin{aligned}
a_1 &= \frac{1}{\delta}\Big(\frac{c_2u_1}{c_1} - u_2 + \int_a^b \Big[c_2K_b^\alpha(s) - d_2b^{\rho-1}K_b^{\alpha-1}(s)\Big]\psi(s)ds\Big) \\
&= \frac{1}{\delta}\Big[\frac{c_2u_1}{c_1} - u_2 + \int_a^b \Big(c_2K_b^\alpha(s) - d_2b^{\rho-1}K_b^{\alpha-1}(s)\Big)\psi(s)ds\Big] \\
&= \frac{1}{\delta}\Big(\frac{c_2u_1}{c_1} - u_2 + \int_a^b K(s)\psi(s)ds\Big) = \phi_{a,b}.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
u(t) &= \frac{c_2u_1}{c_1} - u_2 + \phi_{a,b}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\alpha-1} + \int_a^t K_t^\alpha(s)\psi(s)ds \\
&= \frac{c_2u_1}{c_1} - u_2 + \phi_{a,b}\sigma_t + \int_a^t K_t^\alpha(s)\psi(s)ds.
\end{aligned}$$

Then we can accomplish the purpose desired, which complete the proof.

For sake of brevity, we need the following proposition which is very useful in what follows.

Proposition 3.1. *For* $1 < \alpha < 2, \quad \rho > 0,$ *and* $t, s \in J$ *we have*

(i) $\displaystyle\int_a^t K_t^\alpha(s)ds \le \int_a^b K_b^\alpha(s)ds = \frac{(b^\rho - a^\rho)^\alpha\rho^{-\alpha}}{\Gamma(\alpha+1)}$

(ii) $\displaystyle\int_a^b K_b^{\alpha-1}(s)ds = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}(b^\rho - a^\rho)^{\alpha-1}$

(iii) $\displaystyle\int_a^b |K(s)|ds \le \frac{(b^\rho - a^\rho)^{\alpha-1}\rho^{-\alpha}}{\Gamma(\alpha)}\Big(|c_2|(b^\rho - a^\rho) + |d_2|\rho b^{\rho-1}\Big) := K^*.$

Proof. The proof of (i) and (ii) is immediate, it remains to prove (iii). Indeed we have

$$\begin{aligned}\int_a^b |K(s)|ds &= \int_a^b \Big|c_2 K_b^{\alpha}(s) - d_2 b^{\rho-1} K_b^{\alpha-1}(s)\Big|ds \\ &\leq \frac{|c_2|\rho^{-\alpha}}{\Gamma(\alpha+1)}(b^\rho - a^\rho)^{\alpha} + \frac{|d_2|b^{\rho-1}\rho^{1-\alpha}}{\Gamma(\alpha)}(b^\rho - a^\rho)^{\alpha-1} \\ &\leq \frac{(b^\rho - a^\rho)^{\alpha-1}\rho^{-\alpha}}{\Gamma(\alpha)}\Big(|c_2|\frac{b^\rho - a^\rho}{\alpha} + |d_2|\rho b^{\rho-1}\Big) \\ &\leq \frac{(b^\rho - a^\rho)^{\alpha-1}\rho^{-\alpha}}{\Gamma(\alpha)}\Big(|c_2|(b^\rho - a^\rho) + |d_2|\rho b^{\rho-1}\Big) = K^*.\end{aligned}$$

Now, we are in position to first result which is based on Theorem 2.1.

Theorem 3.1. *Assume that*

$(\mathcal{A}_1)$ Ψ *is continuous.*

$(\mathcal{A}_2)$ *There exist constants* $k > 0$ *and* $0 < l < 1$ *such that*

$$|\Psi(t, u_2, v_2) - \Psi(t, u_1, v_1)| \leq k|u_2 - u_1| + l|v_2 - v_1|$$

for any $u_1, v_1, u_2, v_2 \in \mathbb{R}$, *and* $t \in J$.

Then the problem (1.1) *has at least one solution.*

Proof. Let us consider the operator $\chi : \mathcal{C}(J, \mathbb{R}) \longrightarrow \mathcal{C}(J, \mathbb{R})$ defined by

$$(\chi u)(t) = \frac{c_2 u_1}{c_1} - u_2 + \phi_{a,b}\sigma_t + \int_a^t K_t^{\alpha}(s)\psi(s)ds$$

where

$$\psi(s) = \Psi(s, u(s), \psi(s)).$$

Step 1: χ is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $\mathcal{C}(J, \mathbb{R})$. Then for each $t \in J$, we have

$$\begin{aligned}\big|(\chi u_n)(t) - (\chi u)(t)\big| &= \Big|\frac{\sigma_t}{\delta}\int_a^b K(s)\big(\psi_n(s) - \psi(s)\big)ds \\ &+ \int_a^t K_t^{\alpha}(s)\big(\psi_n(s) - \psi(s)\big)ds\Big| \\ &\leq \frac{\sigma_b}{|\delta|}\int_a^b |K(s)|\big|\psi_n(s) - \psi(s)\big|ds \\ &+ \int_a^t |K_t^{\alpha}(s)|\big|\psi_n(s) - \psi(s)\big|ds\end{aligned}$$

where

$$\psi_n(s) = \Psi(s, u_n(s), \psi_n(s))$$

. In virtue of $(\mathcal{A}_2)$, we have

$$|\psi_n(s) - \psi(s)| \leq \frac{k}{1-l}|u_n(s) - u(s)|.$$

It follows that

$$\begin{aligned} |\chi u_n(t) - \chi u(t)| &\leq \frac{k}{1-l}\Big(\frac{\sigma_b K^*}{|\delta|} + \int_a^b K_b^\alpha(s)ds\Big)|u_n(s) - u(s)| \\ &\leq \frac{k}{1-l}\Big(\frac{\sigma_b K^*}{|\delta|} + \frac{\rho^{-\alpha}}{\Gamma(\alpha+1)}(b^\rho - a^\rho)^\alpha\Big)\|u_n - u)\|_\infty. \end{aligned}$$

Since $u_n \to u$, we get that $\quad \|\chi u_n - \chi u\|_\infty \to 0 \quad$ as $\quad n \to \infty$. Hence Then χ is continuous.

Step 2: χ maps bounded sets into bounded sets in $\mathcal{C}(J, \mathbb{R})$.

It is enough to show that there exists a positive constant m for $r > 0$ such that for each $u \in \mathcal{D}_r = \{u \in \mathcal{C}(J, \mathbb{R}) : \|u\|_\infty \leq r\}$ we have $\|\chi u\|_\infty \leq m$. Indeed for each $t \in J$, and $u \in \mathcal{D}_r$ we have

$$\begin{aligned} |(\chi u)(t)| &= \Big|\frac{c_2 u_1}{c_1} - u_2 + \phi_{a,b}\sigma_t + \int_a^t K_t^\alpha(s)\psi(s)ds\Big| \\ &\leq \Big|\frac{c_2 u_1}{c_1}\Big| + |u_2| + |\phi_{a,b}|\sigma_b + \int_a^t K_t^\alpha(s)|\psi(s)|ds. \end{aligned}$$

According to $(\mathcal{A}_2)$ we have

$$\begin{aligned} |\psi(s)| &= |\Psi(s, u(s), \psi(s)) - \Psi(s, 0, 0) + \Psi(s, 0, 0)| \\ &\leq \frac{k\|u\|_\infty + \sup_{s\in J}|\Psi(s, 0, 0)|}{1-l} \\ &\leq \frac{kr + \Psi^*}{1-l}, \qquad \text{where} \quad \Psi^* = \sup_{s\in J}|\Psi(s, 0, 0)|. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} |\phi_{a,b}| &= \Big|\frac{1}{\delta}\Big(\frac{c_2 u_1}{c_1} - u_2 + \int_a^b K(s)\psi(s)ds\Big)\Big| \\ &\leq \frac{1}{|\delta|}\Bigg(\Big|\frac{c_2 u_1}{c_1}\Big| + |u_2| + \int_a^b |K(s)||\psi(s)|ds\Bigg) \\ &\leq \frac{1}{|\delta|}\Bigg(\Big|\frac{c_2 u_1}{c_1}\Big| + |u_2| + \frac{kr + \Psi^*}{1-l}\int_a^b |K(s)|ds\Bigg) \\ &\leq \frac{1}{|\delta|}\Bigg(\Big|\frac{c_2 u_1}{c_1}\Big| + |u_2| + \frac{(kr + \Psi^*)K^*}{1-l}\Bigg) := \phi^*_{a,b}. \end{aligned}$$

Then,

$$\begin{aligned} |(\chi u)(t)| &\leq \left|\frac{c_2 u_1}{c_1}\right| + |u_2| + \phi^*_{a,b}\sigma_b + \frac{kr+\Psi^*}{1-l}\int_a^t K_t^\alpha(s)ds \\ &\leq \left|\frac{c_2 u_1}{c_1}\right| + |u_2| + \phi^*_{a,b}\sigma_b + \frac{kr+\Psi^*}{1-l}\int_a^b K_b^\alpha(s)ds := m. \end{aligned}$$

It follows that

$$\|\chi u\|_\infty \leq m$$

which implies that χ maps bounded sets into bounded sets of $\mathcal{C}(J,\mathbb{R})$.

Step 3: χ maps bounded sets into a equicontinuous set of $\mathcal{C}(J,\mathbb{R})$.

Let $u \in \mathcal{D}_r$ (as defined in **Step 2**), and t_1, $t_2 \in J$ with $t_1 < t_2$, then

$$\begin{aligned} &|\chi u(t_2) - \chi u(t_1)| \\ &\leq |\phi_{a,b}||\sigma_{t_2} - \sigma_{t_1}| + \left|\int_a^{t_2} K_{t_2}^\alpha(s)\psi(s)ds - \int_a^{t_1} K_{t_1}^\alpha(s)\psi(s)ds\right| \\ &\leq \phi^*_{a,b}|\sigma_{t_2} - \sigma_{t_1}| + \left|\int_a^{t_1} (K_{t_2}^\alpha - K_{t_1}^\alpha)(s)\psi(s)ds + \int_{t_1}^{t_2} K_{t_2}^\alpha(s)\psi(s)ds\right| \\ &\leq \phi^*_{a,b}|\sigma_{t_2} - \sigma_{t_1}| + \frac{(k\|u\|_\infty + \Psi^*)}{1-l}\left|\int_a^{t_1} (K_{t_2}^\alpha - K_{t_1}^\alpha)(s)ds + \int_{t_1}^{t_2} K_{t_2}^\alpha(s)ds\right| \\ &\leq \phi^*_{a,b}|\sigma_{t_2} - \sigma_{t_1}| + \frac{(kr+\Psi^*)\rho^{-\alpha}}{(1-l)\Gamma(\alpha+1)}\Big[2\big(t_2^\rho - t_1^\rho\big)^\alpha + (t_1^\rho - a^\rho)^\alpha - (t_2^\rho - a^\rho)^\alpha\Big]. \end{aligned}$$

As $t_2 \longrightarrow t_1$ the right-hand side of above inequality tends to zero. As a sequence of Steps 1 to 3 together with Arzelà-Ascoli theorem, we conclude that χ is completely continuous.

Step 4: A priori bounds.

We show there exists an open set $\mathcal{O} \subset \mathcal{C}(J,\mathbb{R})$ with $u \neq \lambda\chi(u)$ where $\lambda \in (0,1)$ and $u \in \partial\mathcal{O}$. Let $u \in \mathcal{C}(J,\mathbb{R})$ and $u = \lambda\chi(u)$, with $\lambda \in (0,1)$, then for each $t \in J$ we have

$$\begin{aligned} |u(t)| &= \lambda\left|\frac{c_2 u_1}{c_1} + u_2 + \phi_{a,b}\sigma_t + \int_a^t K_t^\alpha(s)\psi(s)ds\right| \\ &\leq \left|\frac{c_2 u_1}{c_1}\right| + |u_2| + |\phi_{a,b}|\sigma_b + \int_a^b K_b^\alpha(s)|\psi(s)|ds \\ &\leq \left|\frac{c_2 u_1}{c_1}\right| + |u_2| + \phi^*_{a,b}\sigma_b + \frac{kr+\Psi^*}{1-l}\int_a^b K_b^\alpha(s)ds. \end{aligned}$$

Thus

$$\|u\|_\infty \leq m.$$

Let

$$\mathcal{O} = \{u \in \mathcal{C}(J,\mathbb{R}) : \|u\|_\infty < m+1\}.$$

By our choosing of $\mathcal{O}$, there is no $u \in \partial\mathcal{O}$, such that $u = \lambda\chi(u)$, for $\lambda \in (0,1)$. As a consequence of Theorem 3.1 and the nonlinear alternative of Leray-Schauder's fixed point theorem, χ has a fixed point $u \in O$ which is a solution of our problem (1.1).

The second result is based on Theorem 2.2.

Theorem 3.2. *Assume that* $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, *and*

$$\theta = \frac{k\sigma_b K^*}{|\delta|(1-l)} < 1. \tag{3.2}$$

Then the problem (1.1) *has at least one solution.*

Proof. Let

$$\mathcal{M} = \{u \in \mathcal{C}(J,\mathbb{R}) : \|u\|_\infty \le r_1 + r_2 \le r\},$$

where

$$r_1 = \left|\frac{c_2 u_1}{c_1}\right| + |u_2| + \phi^*_{a,b}\sigma_b, \quad r_2 = \frac{(kr + \Psi^*)(b^\rho - a^\rho)^\alpha \rho^{-\alpha}}{(1-l)\Gamma(\alpha+1)}.$$

We define two operators S_1, and S_2 by

$$\mathrm{S}_1 u(t) = \frac{c_2 u_1}{c_1} - u_2 + \phi_{a,b}\sigma_t$$

$$\mathrm{S}_2 u(t) = \int_a^t K_t^\alpha(s)\psi(s)ds$$

where

$$\psi(s) = \Psi(s, u(s), \psi(s)).$$

Step 1: We will show that $\mathrm{S}_1 u + \mathrm{S}_2 v \in \mathcal{M}$.

Let $u, v \in \mathcal{M}$, and $t \in J$ so we have

$$\begin{aligned} |\mathrm{S}_1 u(t)| &\le \left|\frac{c_2 u_1}{c_1}\right| + |u_2| + |\phi_{a,b}|\sigma_t \\ &\le \left|\frac{c_2 u_1}{c_1}\right| + |u_2| + |\phi_{a,b}|\sigma_b \\ &\le \left|\frac{c_2 u_1}{c_1}\right| + |u_2| + \phi^*_{a,b}\sigma_b \\ &\le r_1, \end{aligned}$$

and

$$\begin{aligned} |\mathrm{S}_2 v(t)| &\leq \int_a^t K_t^\alpha(s)|\psi(s)|ds \\ &\leq \frac{(kr+\Psi^*)}{1-l}\int_a^b K_b^\alpha(s)ds \\ &\leq \frac{(kr+\Psi^*)(b^\rho-a^\rho)^\alpha\rho^{-\alpha}}{(1-l)\Gamma(\alpha+1)} \\ &\leq r_2. \end{aligned}$$

Therefore

$$\begin{aligned} \|\mathrm{S}_1 u+\mathrm{S}_2 v\|_\infty &\leq \|\mathrm{S}_1 u\|_\infty+\|\mathrm{S}_2 v\|_\infty \\ &\leq r_1+r_2 \\ &\leq r. \end{aligned}$$

We deduce that $\mathrm{S}_1 u+\mathrm{S}_2 v\in\mathcal{M}$.

Step 2: S_1 is a contraction on $\mathcal{M}$.

For each $t\in J$, $u,v\in\mathcal{M}$, $\psi(s)=\Psi(s,u(s),\psi(s))$, and $\phi(s)=\Psi(s,v(s),\phi(s))$, we have

$$\begin{aligned} \big|\mathrm{S}_1 u(t)-\mathrm{S}_1 v(t)\big| &= \Big|\frac{\sigma_t}{\delta}\int_a^b K(s)\big(\psi(s)-\phi(s)\big)ds\Big| \\ &\leq \frac{\sigma_b}{|\delta|}\int_a^b |K(s)|\big|\psi(s)-\phi(s)\big|ds \\ &\leq \frac{k\sigma_b}{|\delta|(1-l)}\int_a^b |K(s)||u(s)-v(s)|ds \\ &\leq \frac{k\sigma_b K^*}{|\delta|(1-l)}|u(s)-v(s)|. \end{aligned}$$

Therefore

$$\|\mathrm{S}_1 u-\mathrm{S}_1 v\|_\infty\leq\frac{k\sigma_b K^*}{|\delta|(1-l)}\|u-v\|_\infty.$$

By (3.2) we deduce that S_1 is a contraction.

Step 3: S_2 is compact.

It is clear that S_2 is continuous and uniformly bounded on $\mathcal{M}$ ($\|\mathrm{S}_2 u\|_\infty\leq r_2$).

It remains to show that S_2 maps bounded sets into a equicontinuous set of $\mathcal{C}(J,\mathbb{R})$.

Let $u \in \mathcal{M}$, and $t_1, t_2 \in J$ with $t_1 < t_2$, then

$$\begin{aligned}
|\mathrm{S}_2u(t_2) - \mathrm{S}_2u(t_1)| &= \Big|\int_a^{t_2} K_{t_2}^{\alpha}(s)\psi(s)ds - \int_a^{t_1} K_{t_1}^{\alpha}(s)\psi(s)ds\Big| \\
&= \Big|\int_a^{t_1} (K_{t_2}^{\alpha} - K_{t_1}^{\alpha})(s)\psi(s)ds + \int_{t_1}^{t_2} K_{t_2}^{\alpha}(s)\psi(s)ds\Big| \\
&\leq \frac{(k\|u\|_\infty + \Psi^*)}{1-l}\Big|\int_a^{t_1} (K_{t_2}^{\alpha} - K_{t_1}^{\alpha})(s)ds + \int_{t_1}^{t_2} K_{t_2}^{\alpha}(s)ds\Big| \\
&\leq \frac{(kr + \Psi^*)\rho^{-\alpha}}{(1-l)\Gamma(\alpha+1)}\Big[2\big(t_2^{\rho} - t_1^{\rho}\big)^{\alpha} + (t_1^{\rho} - a^{\rho})^{\alpha} - (t_2^{\rho} - a^{\rho})^{\alpha}\Big].
\end{aligned}$$

It is obvious that since $t_2 \longrightarrow t_1$ we get $|\mathrm{S}_2u(t_2) - \mathrm{S}_2u(t_1)| \longrightarrow 0$. It means that S_2 is compact. By Theorem 3.2 we conclude that our problem (1.1) has a solution in $\mathcal{C}(J, \mathbb{R})$.

4. Examples

Example 4.1. *Let us consider the following boundary problem*

$$\begin{cases}
{}^{\frac{1}{3}}D^{\frac{3}{2}}u(t) = \dfrac{|u(t)|}{5+|u(t)|} + \dfrac{1}{2}\tan\big|{}^{\frac{1}{3}}D^{\frac{3}{2}}u(t)\big|, \quad t \in [0, \dfrac{\pi}{3}], \\
u(0) - u'(0) = \frac{3}{2}, \\
u(\frac{\pi}{3}) + u'(\frac{\pi}{3}) = \frac{\pi}{6}.
\end{cases} \tag{4.1}$$

Let the function Ψ defined by

$$\Psi(t, u, v) = \frac{u}{5+u} + \frac{1}{2}\tan v, \qquad u, v \in \mathbb{R}^+, \quad t \in [0, \frac{\pi}{3}].$$

Obviously the function Ψ is continuous. Now we check assumption $(\mathcal{A}_2)$. Indeed for each $t \in [0, \frac{\pi}{3}]$ and $u, v \in \mathbb{R}^+$, we have

$$\begin{aligned}
|\Psi(t, u_2, v_2) - \Psi(t, u_1, v_1)| &= \Big|\frac{u_2}{5+u_2} - \frac{u_1}{5+u_1} + \frac{1}{2}\big(\tan v_2 - \tan v_1\big)\Big| \\
&\leq \Big|\frac{5(u_2-u_1)}{(5+u_2)(5+u_1)}\Big| + \frac{1}{2}\big|\tan v_2 - \tan v_1\big| \\
&\leq \frac{1}{5}\big|u_2 - u_1\big| + \frac{2}{3}\big|v_2 - v_1\big|.
\end{aligned}$$

Therefore $(\mathcal{A}_2)$ holds for $k = \frac{1}{5}$, and $l = \frac{2}{3}$. Then according to Theorem 3.1 the problem (4.1) has at least one solution.

Example 4.2. *Let us consider the following boundary problem*

$$(4.2)\qquad \begin{cases} {}_{\frac{2}{3}}D^{\frac{5}{2}}u(t) = \dfrac{|u(t)|}{3+\left|{}^{\frac{2}{3}}D^{\frac{5}{2}}u(t)\right|} + \dfrac{\left|{}^{\frac{2}{3}}D^{\frac{5}{2}}u(t)\right|}{3+|u(t)|}, & t\in[0,1],\\ u(0)-u'(0)=1, \\ u(1)+u'(1)=\frac{1}{2}. \end{cases}$$

Set the function Ψ as

$$\Psi(t,u,v) = \frac{u}{3+v} + \frac{v}{3+u}, \qquad u,v\in\mathbb{R}^+, \quad t\in[0,1].$$

It is easy to see that the function Ψ is continuous. On the other hand for each $t\in[0,1]$ and $u,v\in\mathbb{R}^+$, we have

$$\begin{aligned} |\Psi(t,u_2,v_2)-\Psi(t,u_1,u_1)| &= \left|\frac{u_2}{3+v_2}+\frac{v_2}{3+u_2}-\frac{u_1}{3+v_1}-\frac{v_1}{3+u_1}\right| \\ &\le \left|\frac{3u_2+u_2v_1-3u_1-u_1v_2}{(3+u_2)(3+v_2)}\right| + \left|\frac{3v_2+v_2u_1-3v_1-u_2v_1}{(3+v_1)(3+u_1)}\right| \\ &\le \frac{1}{9}\Big(|3u_2-3u_1|+|3v_2-3v_1|\Big) \\ &\le \frac{1}{3}\Big(|u_2-u_1|+|v_2-v_1|\Big). \end{aligned}$$

Therefore the assumption $(\mathcal{A}_2)$ holds for $k=l=\frac{1}{3}$. On the other hand we have

$$\theta = \frac{\frac{1}{3}\times\frac{3}{2}\sqrt{\frac{3}{2}}\times\frac{45}{8\sqrt{\frac{2}{3}}\Gamma(\frac{5}{2})}}{3\times\sqrt{\frac{3}{2}}\times\frac{2}{3}} = \frac{45\times\sqrt{2}}{32\sqrt{3}\sqrt{\pi}} < 1$$

By Theorem 3.2 we conclude that the problem (4.2) has at least one solution.

References

[1] D.R. Anderson, D.J. Ulness, Properties of Katugampola fractional derivative with potential application in quantum mechanics, J. Math. Phys **56** (2015), 063502.

[2] A. Bashir, S. Sivasundaram Some existence results for fractional integro-differential equations with nonlaocl conditions, Commun. Appl. Anal. 12 (2) (2008), 107-112.

[3] M. Benchohra, K. Maazouz, Existence and uniqueness results for implicit fractional differential equations with integral boundary conditions, Commun. Appl. Anal. **20**(2016),355-366.

[4] A. Cabada, K. Maazouz, Results for Fractional Differential Equations with Integral Boundary Conditions Involving the Hadamard Derivative. In: Area I. et al. (eds) Nonlinear Analysis and Boundary Value Problems. NABVP 2018. Springer Proceedings in Mathematics & Statistics, vol 292. Springer, Cham, 2019.

[5] Granas, J. Dugundji, Fixed Point Theory, Springer-Verlag, New York, 2003.

[6] M. Janaki, K. Kanagarajan, E.M. Elsayed, Existence criteria for Katugampola fractional type impulsive differential equations with inclusions, Math. Sci. Model. **2**(1) (2019), 51-63.

[7] M. Janaki, K. Kanagarajan, D. Vivek, Analytic study on fractional implicit differential equations with impulses via Katugampola fractional Derivative, Int. J. Math. And Appl.,, **6**(2-A) (2018), 53-62.

[8] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[9] U.N. Katugampola, New approach to generalized fractional integral, Appl. Math. Comput. **218**(3) (2011), 860-865.

[10] U.N Katugampola, New approach to generalized fractional derivative, Bull. Math. Anal. Appl. **6** (4) (2014), 1-15.

[11] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differenatial Equations. North-Holland Mathematics Studies, 204. Elsevier Science B.V. Amsterdam, 2006.

[12] V. Lakshmikantham, S. Leela and J. Vasundhara, Theory of Fractional Dynamic Systems, Cambridge Academic Publishers, Cambridge, 2009.

[13] F. Mainardi, Fractional Calculus and Waves in Linear Viscoelasticity. An Introduction to Mathematical Models. Imperial College Press, London, 2010.

[14] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Differential Equations, John Wiley, New York, 1993.

[15] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[16] S.G. Samko, A.A. Kilbas and O.I. Marichev, Fractional Integrals and Derivatives. Theory and Applications, Gordon and Breach, Yverdon, 1993.

[17] D. R. Smart, Fixed Point Theorems, Cambridge University Press, Cambridge 1974. derivative, Bull. Math. Anal. Appl. **6** (4) (2014), 1-15.

[18] D. Vivek, E.M. Elsayed, K. Kanagarajan, Dynamics and stability results for impulsive type integro-differential equations with generalized fractional derivative, Math. Nat. Sci. **4** (2019), 1-12.

[19] D. Vivek, K. Kanagarajan, S. Harikrishnan, Theory and analysis of impulsive type pantograph equations with Katugampola fractioanl derivative, J. Vabrat. Test. Syst. Dyn. **2** (1) (2018), 9-20.

[20] D. Vivek, E.M. Elsayed, K. Kanagarajan, Theory of fractional implicit differential equations with complex order, J. Univ. Math. **2** (2) (2019), 154-165.