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ON NORM-ATTAINABLE FUNCTIONALS IN BANACH SPACES

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ABSTRACT. In this paper, we give conditions for norm-attainability of linear functionals in Banach spaces. We show that if $\Omega \in AM$ is a bounded σ-complete abstract space and $B(\Omega)$ is an order-closed Banach space, then $\varphi \in \Omega^*$ is norm-attainable if and only if there exists a subspace E of Ω such that $\varphi^+ = \varphi|_E$ and $\varphi^- = -\varphi|_{E^\perp}$. Moreover, we prove that $\varphi \in extB(\Omega^*)$ if and only if $\varphi(x)\varphi(y) = 0$, for all $x, y \in \Omega$ satisfying $x \wedge y = 0$.

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1. INTRODUCTION

Norm-attainability is also a property which has been give keen attention in Banach spaces [7] and [9]. This property still remains very important as it has a lot of open questions which are unanswered particulary when a super class of Hilbert space operators called supraposinormal operators [6] are considered. In [1] the authors characterized the norm property for elementary operators and gave conditions under which a general elementary operator is norm-attainable. In-depth characterization of norm-attainable operators has also been done in details with considerations given to other properties like orthogonality (see [4] - [8] and the references therein). Regarding derivations, authors in [8] showed that if V_Γ and W_Γ are Γ-Banach algebras and δ an α-inner derivation, then δ is norm-attainable if and only if the adjoint, δ^*, of δ is norm-attainable. Moreover, as a consequence they proved that if δ_N^1 and δ_N^2 are norm-attainable then δ_N is norm attainable if either δ_N^1 and δ_N^2 or both are zero derivations and δ_N^1, and δ_N^2

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are α-inner derivation and α'-inner derivation respectively. In this paper, we characterize the notion of norm-attainability for linear functionals. We give necessary and sufficient conditions for norm-attainability of linear functionals in Banach spaces.

2. Preliminaries

In this section, we outline preliminary concepts which are useful in the sequel.

Definition 2.1. ([5], *Definition 1.1*) *An operator $A \in B(H)$ is said to be norm-attainable if there exists a unit vector $x_0 \in H$ such that $\|Ax_0\| = \|A\|$. The set of all norm-attainable operators on a Hilbert space H is denoted by $NA(H)$.*

Definition 2.2. ([2]) *Let Ω be a Banach lattice then Ω is an abstract M space i.e. $\Omega \in AM$ if $x \wedge y = 0$ implies $\|x + y\| = \max\{\|x\|,\ \|y\|\}$. Also Ω is abstract L space i.e. $\Omega \in AL$ if $x \vee y = 0$ implies $\|x + y\| = \|x\| + \|y\|$.*

Definition 2.3. ([3])*Let Ω be a Banach lattice. Then $\Omega \in AM$ implies $\Omega^* \in AL$ and $\Omega \in AL$ implies $\Omega^* \in AM$. $\Omega \in AM$ if and only if for any $x,\ y \in \Omega, x, y \geq 0$ implies $\|x \vee y\| = \max\{\|x\|,\ \|y\|\}$. $\Omega \in AL$ if and only if for any $x,\ y \in \Omega, x, y \geq 0$ implies $\|x + y\| = \|x\| + \|y\|$.*

Definition 2.4. ([2]) *Let Ω be a Banach lattice. Then Ω is said to be σ complete, if for every order bounded sequence $\{x_n\} \in \Omega, \vee_{n\geq 1} x_n$ exists in Ω. Also Ω is said to be bounded σ complete, provided that the any norm bounded and order monotone sequence in Ω is order convergent.*

Definition 2.5. ([2])*Let Ω be a Banach space. An element $x \in D(\Omega)$ is called an extreme point of $B(\Omega)$ if $x = \lambda y + (1 - \lambda)z,\ y,\ z \in B(\Omega)$ and $\lambda \in (0,\ 1)$, imply $y = z$. In this case, we write $x \in extB(\Omega)$.*

3. Norm-attainability for functionals

In this section, we characterize norm-attainability of functionals in Banach spaces. We regard H^* the dual space of a Hilbert space H to be non-zero throughout this section unless otherwise stated. Let $\varphi \in H^*$. Then φ is said to be norm-attainable at $\frac{\varphi}{\|\varphi\|}$ if there exists $T \in B(H)$ such that $\langle \varphi, T \rangle = \|\varphi\|.\|T\| > 0$. $\frac{\varphi^*}{\|\varphi^*\|}$ is called a support for φ. The following proposition shows that any functional is norm-attainable in non-zero dual spaces.

Proposition 3.1. *Let $B(W)$ be the set of all bounded linear maps on an Orlicz space W then every $\varphi \in H^*_+$ is norm-attainable on $B(W)$.*

Proof. Let W be a bounded linear Orlicz space and $b_n \in W$ such that b_n is monotone decreasing to 0. Given $J_n \in B(W)$ we have $\varphi(J_n) > \|\varphi\| - b_n$ since $\pi(J_n) \leq 1 < \infty$ and $\varphi(W_0) = \{0\}$. Suppose that $\pi(J_n) \leq 2^{-n}$ and let $J(t) = \sup|J_n(t)|$. Then $\pi(J) \leq \sum_{n=1}^{\infty} \pi(J_n) \leq 1$, that is, $J \in B(W)$ and $\varphi(J) \geq \sup_n(|J_n|) = \|\varphi\|$, because $\varphi \in H_+^*$. □

Theorem 3.1. *Let $B(W)$ be the set of all bounded linear maps on an Orlicz space W then $\varphi \in H^*$ is norm-attainable on $B(W)$ if and only there exists L in a subspace $\mathcal{C}$ of $B(W)$ such that $\varphi^+ = \varphi|_L$ and $\varphi^- = \varphi|_{G\setminus L}$.*

Proof. It is known that we have $P, Q \in B(W)$ such that $\varphi|_L(P)\ \varphi^*\ \|\varphi^+\|$ and $\varphi|_{G\setminus L}(Q) = \|\varphi^-\|$ by Proposition 3.1. Assume that $\pi(P|_L) \leq \frac{1}{2}$ and $\pi(Q|_{G\setminus L}) \leq \frac{1}{2}$. Then $\pi(P|_L - Q|_{G\setminus L}) \leq \pi(P) + \pi(Q) \leq 1$. Indeed, $P|_L - Q|_{G\setminus L} \in L(W)$ and $\varphi(P|_L - y|_{G\setminus L}) = \|\varphi^+\| + \|\varphi^-\| = \|\varphi\|$. Suppose that $P \in B(W)$ satisfies $\varphi(P) = \|\varphi\|$. Let $L = \{\ \in G : P(l) \geq 0\}$ then we show that $\varphi|_L,\ -\varphi|_{G\setminus L} \in H_+^*$. Now, if $\varphi|_L \in H_+^*$, then there exists $Q \in W^+$. If $\varphi|_{L\notin H'}$ then there exists $Q \in W^+$ such that $\varphi|_L(Q) < 0$. Now φ being singular, assume $\pi(Q) \leq \frac{1}{2}$ and $\pi(P) \leq \frac{1}{2}$. Then $J = P|_{G\setminus L} - Q|_L \in B(W)$ and so, $\|\varphi^-\| \geq \varphi^-(-J) = -\varphi^+(J) + \varphi(J) \geq \varphi(J) = \varphi(P|_{G\setminus L} - \varphi|_L(Q)) > \varphi(P|_{G\setminus L})$. This is contrary to our earlier assumption. Lastly, $\|\varphi\| = \|\varphi^+\| + \|\varphi^-\| > \varphi|(P|_L) + \varphi(P|_{G\setminus L}) = \varphi|(P) = \|\varphi\|$. $(\varphi|_{G\setminus L}) \in H_1^*$ follows analogously. □

Corollary 3.1. *Let $\varphi \in H^*$ is norm-attainable at $P \in B(W)$, then $\varphi(P|_A)\ \|\varphi|_A\|$ for all $A \in \mathcal{C}$.*

Proof. $\|\varphi\| = \|\varphi|_A\| + \|\varphi|_{G\setminus A}\| \geq \varphi|_A(P) + \varphi|_{G\setminus A}(P) = \varphi(P) = \|\varphi\|$ is enough. □

Theorem 3.2. *Let $\varphi \in H^*$ be singular then the set of all such φ is dense in H^*.*

Proof. Let $\varphi \in H^*$ be singular and $\epsilon > 0$ be given. Then we have $L \in \mathcal{C}$ such that $\|\varphi^+|_{G\setminus L}\| < \epsilon$ and $\|\varphi^-|_L\| \leq \epsilon$. Suppose that $\psi = \varphi^+|_L - \varphi^-|_{G\setminus L}$. Then by Theorem 3.1 ψ is norm-attainable. Also, $\|\varphi - \psi\| \leq \|\varphi^+ - \psi|_l\| + \|\varphi - \psi|_{G\setminus l}\| = \|\varphi^+|_{G\setminus l}\| + \|\varphi^-|_l\| < 2\epsilon$. □

Theorem 3.3. *Let $B(W)$ be the set of all bounded linear maps on an Orlicz space W. Then $\phi = \chi + \varphi$ $(0 \neq \chi \in H_0^*,\ \varphi \in H^*)$ is norm-attainable at $P \in B(W)$ if and only if $\pi(P) = 1$, $\varphi(P) = \|\varphi\|$ and $\int_G k\chi(t)P(t)dt = \pi(x) + \omega(k\chi)$, where $k \in K_N(\chi) = \{k : k^{-1}[1 + \omega(k\chi)] = \|\chi\|_N^0\}$.*

Proof. From the statement of the theorem we have $\|\phi\|^0 = f(P) = k^{-1}\langle k\chi,\ \chi\rangle + \varphi(P) \leq k^{-1}[\pi(P) + \omega(k\chi)] + \varphi(P) \leq k^{-1}[1 + \omega(k\chi)] + \|\varphi\| = \|\chi\|_N^0 + \|\phi\| = \|\phi\|^0$. The converse follows from the fact that ϕ is singular from Theorem 3.2 and an assertion from Proposition 3.1. □

At this point, we consider norm-attainable functionals in Banach lattices. We denote an abstract L space and abstract M space by AL and AM respectively. For details on AL and AM see [2]. We state the following lemma.

Lemma 3.1. *Let $\Omega \in AL$ and $\varphi \in B(\Omega^*)$. Then the following are equivalent.*

(i). *φ is norm-attainable.*

(ii). *Both φ^+ and φ^- are norm-attainable.*

(iii). *φ^+ or φ^- is norm one.*

Proof. $(i) \Rightarrow (ii)$. Choose $x \in B(\Omega)$ such that $\varphi(x) = \|\varphi\| = 1$. From

$$\begin{aligned} 1 = \|\varphi\| &= \varphi(x) = \varphi^+(x^+) + \varphi^-(x^-) - \varphi^+(x^-) - \varphi^-(x^+) \\ &= \|\varphi^+\|\|x^+\| + \|\varphi^-\|\|x^-\| - \varphi^+(x^-) - \varphi^-(x^+) \\ &= \|\varphi\|(\|x^+\| + \|x^-\|) = \|\varphi\|\|x\| = \|\varphi\| = 1 \end{aligned}$$

we obtain $\varphi^+(x^+) = \|\varphi^+\|\|x^+\|; \quad \varphi^-(x^-) = \|\varphi^-\|\|x^-\|$ and $\varphi^+(x^-) = \varphi^-(x^+) = 0$ since $\varphi^\pm(x^\pm) \leq \|\varphi^\pm\|\|x^\pm\|$ and $\varphi^\pm, \ (x^\pm)$ are non-negative.

$(ii) \Rightarrow (iii)$. This follows obviously.

$(iii) \Rightarrow (i)$. We suppose that φ^+ is norm one and norm-attainable. choose $x \in B(\Omega^+)$ such that $\varphi^+(x) = \|\varphi\| = 1$. We have $\varphi^-(x) = 0$. Indeed, $1 = \|\varphi\| \geq |\varphi|(|x|) \geq \varphi^+(x) + \varphi^-(|x|) = 1 + \varphi^-(|x|) \geq 1$, which implies that $\varphi^-(x^+) = \varphi^-(x^-) = 0$. Hence, $\varphi(x) = \varphi^+(x) = \|\varphi\| = 1$. □

Theorem 3.4. *Let $\Omega \in AL$ and $\theta \leq \varphi \in B(\Omega^*)$. Then the following are equivalent.*

(i). *φ is norm-attainable.*

(ii). *There exists $\theta \leq x \neq \theta$ such that $\varphi(y) = \|y\|$ for all $\overline{E}_\Omega$, the norm closure of E_Ω where $E_\Omega = \{y \in \Omega : \ \theta \leq y \leq nx, \text{ for some } n > 0\}$.*

(iii). *There exists $\theta \neq x \in \Omega^+$ such that among $B(\Omega^*) = \{\psi \in \Omega^*; \ \|\psi\| = 1\}$, φ is maximal on $\overline{E}_\Omega$.*

Proof. $(i) \Rightarrow (ii)$. Choose $x \in B(\Omega)$ satisfying $\varphi(x) = \|x\| = 1$. We have $x \in \theta$ for $1 = \varphi(x) = \varphi(x^+) = \varphi(x^+) - \varphi(^-) \leq \varphi(^+) \leq \|x^+\| - \|x^-\| \leq 1$ by Lemma 3.1 which implies $\|x^-\| = 0$. Now if we consider $y \in \overline{E}_\Omega$, we need to prove that $\varphi(y) = \|y\|$. Since φ is continuous, let $y \in \overline{E}_\Omega$, i.e $\theta \leq y \leq nx$ for some $n \geq 0$. But since $n = \varphi(nx) = \varphi(nx - y) + \varphi(y) \leq \|nx - y\| + \|y\| = \|nx\| = n$, we have $\varphi(y) = \|y\|$.

$(ii) \Rightarrow (iii)$. Follows trivially from $(ii) \Rightarrow (iii)$ in Lemma 3.1 and abstractness of AL.

$(iii) \Rightarrow (i)$. Let φ be maximal in $B(\Omega^*)$ on $\overline{E}_\Omega$ for some $\theta \neq x \in \Omega^+$. Choose$\psi \in B(\Omega_*)$ such that $\psi(x) = \|x\|$, then by $\varphi(x) \geq \psi(x) = \|x\|$, it is clear that φ is norm-attainable at $x/\|x\|$. □

Proposition 3.2. *Let $\Omega \in AM$ be σ-complete and $\varphi \in \Omega^*$. Then for any $\epsilon > 0$, there exists a subspace E of $\Omega = E + E^\perp$ and $\|\varphi^+|_{E^\perp}\| < \epsilon, \ \|\varphi^-|_E\| < \epsilon$.*

Proof. Choose $x \in D(\Omega)$ with the property that $\varphi^+(x) > \|\varphi\| - \epsilon$, and $E = (x^\perp)^\perp$. Then $x^+ \in E, \ x^- \in E^\perp$ and $\Omega = E + E^\perp$. Furthermore, for $x \in D(\Omega)$ we have that $\|\varphi^+|_E\| + \|\varphi^+|_{E^\perp}\| + \|\varphi^-|_E\| + \|\varphi^-|_{E^\perp}\| = \|\varphi^+|\| + \|\varphi^+\| = \|\varphi\| < \varphi(x) + \epsilon = \varphi^+|_E(x) + \varphi^+|_{E^\perp}(x) - \varphi^-|_E(x) - \varphi^+|_{E^\perp}(x) + \epsilon$ because $\varphi^+|_{E^\perp}(x) \leq 0$ and $\varphi^-|_E(x) \geq 0$. So, we conclude that $\|\varphi^+|_{E^\perp}\| + \|\varphi^-|_E\| = \|\varphi^+|\| - \|\varphi^+|_E\| + \|\varphi^-|\| - \|\varphi^-|_E\| \leq \|\varphi^+\| - \varphi^+|_E(x) + \|\varphi^-\| - \varphi^-|_{E^\perp}(x) < \varphi^+|_{E^\perp}(x) - \varphi^-|_E(x) + \epsilon \leq \epsilon$. This completes the proof as required. □

Theorem 3.5. *Let a Banach lattice Ω be bounded σ-complete and $B(\Omega)$ order-closed, then every positive bounded linear $\varphi \in \Omega^*$ is norm-attainable.*

Proof. Consider $x_n(\geq 0) \in D(\Omega)$ such that $\varphi(x_n) \to \|\varphi\|$. Since Ω is bounded σ-complete and $B(\Omega)$ is closed under order, $y = \vee_n(x_n)$ exists in Ω and $\|y\| = 1$. Hence, $y \geq x_n \geq 0$ and $\varphi \geq 0$ implies $\|\varphi\| \geq \varphi(y) \geq \varphi(x_n) \to \|\varphi\|$. So, $x \in D(\Omega)$ exists which satisfies the norm-attainability condition, $\varphi(x) = \|\varphi\|$, for functionals and this completes the proof. □

Theorem 3.6. *Let $\Omega \in AM$ be bounded σ-complete and $B(\Omega)$ order-closed, then $\varphi \in \Omega^*$ is norm-attainable if and only if there exists a subspace E of Ω such that $\varphi^+ = \varphi|_E, \ \varphi^- = -\varphi|_{E^\perp}$.*

Proof. Necessity. Let $x \in B(\Omega)$ be having the property that $\varphi|(x) = \|\varphi\|$, and define $E = (x^-)^\perp$. Then $\Omega = E + E^\perp$ and $x^+ \in E, \ x^- \in E^\perp$. Now, $\|\varphi\| = \|\varphi|_E\| + \|\varphi|_{E^\perp}\|$; we need to show that $\varphi^+ = \varphi|_E$ and $\varphi^- = -\varphi|_{E^\perp}$, it is enough that $\varphi|_E \geq 0$ and $-\varphi|_{E^\perp} \geq 0$. Indeed, if $\varphi_E(y) < 0$ for some $y(\geq 0) \in D(\Omega)$ then we let $y \in E$. So, $z = -x^- - y$ satisfies $\|z\| = \max\{\|x\|, \ \|y\|\} = 1$ and hence, $\|\varphi^-\| = \varphi^-(-z) = \varphi(z) - \varphi^+(z) \geq \varphi(z) = \varphi|_{E^\perp}(-x^-) - \varphi|_E(y) > \varphi|_{E^\perp}(-x^-) = -\varphi|_{E^\perp}(-x)$. Now since $\|\varphi^+\| \geq \varphi(x|_E) = \varphi|_E(x)$, this is contrary to $\|\varphi\| = \|\varphi^+\| + \|\varphi^-\| > \varphi|_E(x) - \varphi|_{E^\perp}(x) = \varphi(x) = \|\varphi\|$. Also $-\varphi|_{E^\perp} \geq 0$ follows analogously.

Sufficiency. We know that there exists $x, y(\geq 0) \in D(\Omega)$ such that $\varphi^+(x) = \|\varphi^+\|$ and $\varphi^-(x) = \|\varphi^-\|$ from Theorem 3.5. Since $\varphi^+ = \varphi|_E$ and $\varphi^- = -\varphi|_{E^\perp}$, we let $x \in E$ and $y \in E^\perp$ and $u \in x - y$. Then $u = \|x - y\| = \max\{\|x\|, \|y\|\} = 1$ and so we have $\|\varphi\| = \|\varphi^+\| + \|\varphi^-\| = \varphi^+(x) + \varphi^-(y) = \varphi|_E(x) + \varphi|_{E^\perp}(-y) = \varphi(u)$. □

Corollary 3.2. *Let $\Omega \in AM$ be a σ-complete and $\varphi \in D(\Omega^*)$. Then $\varphi \in extB(\Omega^*)$ if and only if $\varphi(x)\varphi(y) = 0$ for all $x, y \in \Omega$ satisfying $x \wedge y = 0$.*

Proof. Necessity. If there exists $x, y \in \Omega$ satisfying $x \wedge y = 0$ but $\varphi(x) > 0$ and $\varphi(y) > 0$, then we set $E = y^\perp$, and $\Omega = E + E^\perp$. Let $\psi = \varphi|_E$ and $\tau = \varphi|_{E^\perp}$. Then $\|\psi\| > 0, \ \|\psi\| > 0$ since $x \in E, \ y \in E^\perp$. Therefore, $\varphi = \|\psi\|\frac{\psi}{\|\psi\|} + \|\tau\|\frac{\tau}{\|\tau\|}$ and $\|\psi\| + \|\tau\| = \|\varphi\| = 1$. Hence, $\varphi \in extB(\Omega)$.

Sufficiency. First we show $\|\varphi^+\|\|\varphi^-\| = 0$. In fact, for any $\epsilon > 0$, by Theorem 3.5, there exists two orthogonal subspaces $E, \ F \in \Omega$ such that $\Omega = E + F$ and $\|\varphi^-|_E\| < \epsilon, \ \|\varphi^+|_F\| < \epsilon$. Choose $x \in B(\Omega)$ such that $\varphi(x) > \|\varphi\| - \epsilon$, and let $x = u + v$, where $u \in E$ and $v \in F$. Then $\varphi(u)\varphi(v) = 0$ since $u \wedge v = 0$. If $\varphi(v) = 0$ then $\|\varphi\| - \epsilon < \varphi(x) = \varphi^+|_E(u) - \varphi^-|_E(u) \leq \|\varphi^+|_E\| + \|\varphi^-|_E\| < \|\varphi^+\| + \epsilon$. Let $\epsilon \to 0$, we find $\|\varphi^-\| = \|\varphi\| - \|\varphi^+\| = 0$. Similarly, if $\varphi(u) = 0$. Then $\|\varphi^+\| = 0$. Hence, without loss of generality, we assume $\varphi = \varphi^+$. Let $\psi, \ \tau \in D(\Omega^*)$ satisfy $2\varphi = \psi + \tau$. Then $2\varphi = (\psi^+ + \tau^+) - (\psi^- + \tau^-)$ and hence $\|2\varphi\| = \|\psi^+\| + \|\tau^+\| + \|\psi^-\| + \|\tau^-\| = \|\psi\| + \|\tau\| = 2 = \|2\varphi\|$ Thus $\psi^+ + \tau^+ = 2\varphi$ and $\psi^- = \tau^- = 0$. Now we show $\psi = \tau = \varphi$, that is, $\varphi \in extB(\Omega^*)$. This follows if we prove that $\psi(y) = \tau(y) = 0$ whenever $\varphi(y) = 0$ this means $\varphi = a\psi = b\tau$, but $\varphi, \ \psi, \tau \in D(\Omega^*)$ and $2\varphi = \psi + \tau$, so $a = b = 1$, this means assume $y \geq 0$; then from $\psi(y) \geq 0, \tau(y) \geq 0$ and $\psi(y + \tau(y) = 2\varphi(y) = 0$. We have $\psi(y) = \tau(y) = 0$. For the general case, since $\varphi(y) = 0$ and by the condition given in theorem $\varphi(y^+)\varphi(y^-) = 0$, we have $\varphi(y^+) = \varphi(y^-) = 0$ hence the condition $\psi(y) = \tau(y) = 0$ follows from the first case.

□

4. Conclusion

In this work, we have established norm- attainability of functionals in Banach spaces. We have shown that a functional $\varphi \in \Omega^*$ is norm-attainable if and only if there exists a subspace E of Ω such that $\varphi^+ = \varphi|_E$ and $\varphi^- = -\varphi|_{E^\perp}$.

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Author's contribution

The author contributed wholly in writing this article and declares no conflict of interest.

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