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# CERTAIN NEW SUBCLASSES OF ANALYTIC AND BI-UNIVALENT FUNCTIONS USING SALAGEAN OPERATOR

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Abstract. Bi-univalent functions in $\Delta = \{z \in \mathfrak{C} : |z| < 1\}$ consisting of two new subclasses signified by $\mathfrak{M}_{\mathfrak{E}}^{\gamma,h}(\Im)$ and $\mathfrak{M}_{\mathfrak{E}}^{\gamma,h}(\mu)$ which are introduced by applying Salagean differential operator. The coefficient estimates $|V_2|$, $|V_3|$ and $|V_4|$ on the new subclasses is investigated and importance results are indicated.

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## 1. Introduction

We indicate by $\mathfrak{J}$ the subclass of class of function $\mathfrak{T}$ which is of the form

$$\varphi(z) = z + \sum_{\varsigma=2}^{\infty} V_{\varsigma} z^{\varsigma} \tag{1.1}$$

consisting of function which are holomorphic and univalent in unit disk $\Delta = \{z \in \mathfrak{C} : |z| < 1\}$. Let $\mathfrak{S}^*(\rho)$ and $\mathfrak{K}(\rho)$ indicate the familiar classes of starlike and convex function of order $\rho$ $(0 \leq \rho < 1)$ respectively (see [5]).

Let $\varphi^{-1}(z)$ be the inverse of the function $\varphi(z)$ then we have

$$\varphi^{-1}(\varphi(z)) = z,$$

$$\varphi(\varphi^{-1}(l)) = l, \quad |l| < r_0(\varphi); r_0(\varphi) \geq \frac{1}{4}$$

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therefore,

$$\varphi^{-1}(l) = \eta(l) = l - V_2 l^2 + (2V_2^2 - V_3)l^3 - (5V_2^3 - 5V_2V_3 + V_4)l^4 + \cdots \tag{1.2}$$

A function $\varphi(z) \in \mathfrak{T}$ denoted by $\mathfrak{E}$ is said to be bi-univalent in $\Delta$, considering that $\varphi(z)$ and $\varphi^{-1}(z)$ are univalent in $\Delta$, (for more details see; [10], [22], [4], [1], [12], [3], [16], [21], [23], [15]). Also various researches ( [6], [19], [11], [13], [14], [9], [8]) obtain the coefficient $|V_2|$ and $|V_3|$ of bi-univalent function for different subclass of the function class $\mathfrak{E}$.
For $\varphi(z) \in \mathfrak{T}$, Salagean [18] introduced the differential operator $\mathfrak{D}^h$ which is defined by

$$\mathfrak{D}^0\varphi(z) = \varphi(z);$$

$$\mathfrak{D}^1\varphi(z) = \mathfrak{D}\varphi(z) = z\varphi'(z);$$

$$\mathfrak{D}^h\varphi(z) = \mathfrak{D}(\mathfrak{D}^{h-1}\varphi(z)).$$

then,

$$\mathfrak{D}^h\varphi(z) = z + \sum_{\varsigma=2}^{\infty} \varsigma^h V_\varsigma z^\varsigma$$

where $h \in \mathcal{N}_0 = \mathcal{N} \cup \{0\} = 0, 1, 2, 3, \cdots$.

**Definition 1.1.** [2] *Let $\varphi(z) \in \mathcal{L}$, suppose $0 \leq \rho < 1$ and $\gamma \geq 1$ is real. Then $\psi(z) \in L_\gamma(\rho)$ of $\gamma$-pseodu-starlike function of order $\rho$ in $\Delta$ if and only if*

$$\Re\left(\frac{z[\varphi'(z)]^\gamma}{\varphi(z)}\right) > \rho. \tag{1.3}$$

*Babalola [2] verified that, all pseodu-starlike function are Bazilevic of type $\left(1 - \frac{1}{\gamma}\right)$, order $\rho^{\frac{1}{\gamma}}$ and univalent in $\Delta$.*

This study is inspired by the earlier work of Girgaonkar et al. [7], thus we introduce two new subclasses $\mathfrak{M}_{\mathfrak{E}}^{\gamma,h}(\Im)$ and $\mathfrak{M}_{\mathfrak{E}}^{\gamma,h}(\mu)$ of the function class $\mathfrak{E}$ associated with Salagean differential operator and determine the improved estimates on the initial coefficient $|V_2|$, $|V_3|$ and $|V_4|$ for the functions in these subclasses.

**Lemma 1.1.** [17] *If $r(z) \in \mathcal{P}$ and $z \in \nabla$, then $|w_n| \leq 2$ for each $n$. where $\mathcal{P}$ is the family of all function $r$ analytic in $\Delta$ for which $\Re(r(z)) > 0$,*

$$r(z) = 1 + w_1 z + w_2 z^2 + \cdots$$

*where $z \in \Delta$.*

## 2. Coefficient bounds for the Function Class $\mathfrak{M}_{\mathfrak{E}}^{\gamma,h}(\Im)$

**Definition 2.1.** *A function $\varphi(z) \in \mathfrak{T}$ is in the class $\mathfrak{M}_{\mathfrak{E}}^{\gamma,h}(\Im)$ if the following condition are fulfilled:*

$$|\arg[((D^h\varphi(z))')^{\gamma}]| < \frac{\Im\pi}{2} \quad z \in \nabla, \tag{2.1}$$

*and*

$$|\arg[((D^h\eta(l))')^{\gamma}]| < \frac{\Im\pi}{2} \quad l \in \nabla, \tag{2.2}$$

*where $\varphi(z) \in \mathfrak{E}$, $\gamma > 0$, $0 < \Im \leq 1$ and*

$$\eta(l) = l - V_2 l^2 + (2V_2^2 - V_3)l^3 - (5V_2^3 - 5V_2V_3 + V_4)l^4 + \cdots$$

**Remark 2.1.** .

(1) $\mathfrak{M}_{\mathfrak{E}}^{1,0}(\Im)$=$\mathfrak{M}_{\mathfrak{E}}(\Im)$ *which Srivastava et al.* [20] *introduced and studied.*

(2) $\mathfrak{M}_{\mathfrak{E}}^{\gamma,0}(\Im)$=$\mathfrak{M}_{\mathfrak{E}}^{\gamma}(\Im)$ *which Girgaonkar et al.* [7] *introduced and studied.*

Now we have the following theorem and the proof.

**Theorem 2.1.** *A function $\varphi(z) \in \mathfrak{T}$ is in the class $\mathfrak{M}_{\mathfrak{E}}^{\gamma,h}(\Im)$. Then*

$$|V_2| \leq \frac{\sqrt{2}\Im}{\sqrt{2^h\gamma^2(2^{h+1} - 2^h\Im + \Im) - \gamma\Im(2^{2h+1} - 3^{h+1})}}, \tag{2.3}$$

$$|V_3| \leq \frac{3^{h+1}\Im^2 + 2^{2h+1}\Im\gamma}{3^{h+1}2^{2h}\gamma^2} \tag{2.4}$$

*and*

$$\begin{aligned} |V_4| \leq & \frac{\Im}{2^{2h+1}\gamma}\left[1 + 2(\Im - 1)\left(1 + \frac{(\Im - 2)}{3}\right)\right] \\ & + \frac{\sqrt{2}\Im^2}{\sqrt{2^h\gamma^2(2^{h+1} - 2^h\Im + \Im) - \gamma\Im(2^{2h+1} - 3^{h+1})}}\left[\frac{5}{3^{h+1}\gamma} + \right. \\ & \left. \frac{3^{h+1}(1-\gamma)\Im}{2^{2h+1}\gamma^2(2^{h+1} - 2^h\Im + \Im) - 2^{h+1}\gamma\Im(2^{2h+1} - 3^{h+1})}\right] \end{aligned} \tag{2.5}$$

*Proof.* From (2.1) and (2.2) we have,

$$((D^h\varphi(z))')^{\gamma} = [y(z)]^{\Im} \tag{2.6}$$

and

$$((D^h\eta(l))')^{\gamma} = [x(l)]^{\Im} \tag{2.7}$$

where $y(z)$ and $x(l)$ are in the class $\mathcal{P}$ which is of the form

$$y(z) = 1 + y_1 z + y_2 z^2 + y_3 z^3 + \cdots \tag{2.8}$$

$$x(l) = 1 + x_1 l + x_2 l^2 + x_3 l^3 + \cdots \tag{2.9}$$

Hence,

$$[y(z)]^{\Im} = 1 + \Im y_1 z + \left(\Im y_2 + \frac{\Im(\Im-1)y_1^2}{2!}\right) z^2 + \left(\Im y_3 + \Im(\Im-1)y_1 y_2 + \frac{\Im(\Im-1)(\Im-2)}{3!} y_1^3\right) z^3 + \cdots$$

$$[x(l)]^{\Im} = 1 + \Im x_1 l + \left(\Im x_2 + \frac{\Im(\Im-1)x_1^2}{2!}\right) l^2 + \left(\Im x_3 + \Im(\Im-1)x_1 x_2 + \frac{\Im(\Im-1)(\Im-2)}{3!} x_1^3\right) l^3 + \cdots$$

Now, equating the coefficient in (2.6) and (2.7) we get

$$2^{h+1}\gamma V_2 = \Im y_1 \tag{2.10}$$

$$3^{h+1}\gamma V_3 + 2^{2h+1}V_2^2\gamma(\gamma-1) = \Im y_2 + \frac{\Im(\Im-1)y_1^2}{2!} \tag{2.11}$$

$$4^{h+1}\gamma V_4 + 2^{h+1}3^{h+2}\gamma(\gamma-1)V_2V_3 = \Im y_3 + \Im(\Im-1)y_1y_2 + \frac{\Im(\Im-1)(\Im-2)}{3!}y_1^3 \tag{2.12}$$

$$-2^{h+1}\gamma V_2 = \Im x_1 \tag{2.13}$$

$$2\gamma[3^{h+1} - 2^{2h} + 2^{2h}\gamma]V_2^2 - 3^{h+1}\gamma V_3 = \Im x_2 + \frac{\Im(\Im-1)x_1^2}{2!} \tag{2.14}$$

$$(2.15)\quad -4^{g+1}\gamma V_4-(5\cdot 4^{h+1}+2^{h+1}3^{h+1}\gamma-2^{h+2}3^{h+1})V_2^3\gamma+V_2V_3\gamma\Bigg(5\cdot 4^{h+1}+2^{h+1}3^{h+1}\gamma -2^{h+1}3^{h+1}\Bigg)=\Im x_3+\Im(\Im-1)x_1x_2+\frac{\Im(\Im-1)(\Im-2)}{3!}x_1^3$$

From (2.10) and (2.13) we obtain

$$(2.16)\qquad y_1=-x_1$$

and

$$(2.17)\qquad 2^{2h+3}\gamma^2V_2^2=\Im^2(y_1^2+x_1^2)$$

Now, adding (2.11) and (2.14), we have

$$(2.18)\qquad \left[2\gamma^2(2^{2h}+2^h)-2\gamma(2^{2h+1}-3^{h+1})\right]V_2^2=\Im(y_2+x_2)+\frac{\Im(\Im-1)}{2!}(y_1^2+x_1^2)$$

from (2.17), we have

$$(2.19)\qquad V_2^2=\frac{\Im^2(y_2+x_2)}{2^{h+1}\gamma^2(2^{h+1}-2^h\Im+\Im)-2\gamma\Im(2^{2h+1}-3^{h+1})}$$

Applying lemma (1.1), we get

$$|V_2|\leq\frac{\sqrt{2}\Im}{\sqrt{2^h\gamma^2(2^{h+1}-2^h\Im+\Im)-\gamma\Im(2^{2h+1}-3^{h+1})}},$$

Now, to get the bound $|V_3|$, we subtract (2.14) from (2.11) to have

$$(2.20)\quad 2.3^{h+1}\gamma V_3+2^{2h+1}V_2^2\gamma(\gamma-1)-2\gamma[3^{h+1}-2^{2h}+2^{2h}\gamma]V_2^2 =\Im(y_2-x_2)+\frac{\Im(\Im-1)}{2!}(y_1^2-x_1^2)$$

then from (2.16), we have

$$(2.21)\qquad 2.3^{h+1}\gamma V_3+2^{2h+1}V_2^2\gamma(\gamma-1)-2\gamma[3^{h+1}-2^{2h}+2^{2h}\gamma]V_2^2=\Im(y_2-x_2)$$

$$(2.22)\qquad V_3=V_2^2+\frac{\Im(y_2-x_2)}{2.3^{h+1}\gamma}$$

$$(2.23)\qquad V_3=\frac{\Im^2y_1^2}{2^{2h+2}\gamma^2}+\frac{\Im(y_2-x_2)}{2.3^{h+1}\gamma}$$

Applying lemma (1.1), we get

$$|V_3| \leq \frac{\Im^2}{2^{2h}\gamma^2} + \frac{2\Im}{3^{h+1}\gamma} \tag{2.24}$$

$$|V_3| \leq \frac{3^{h+1}\Im^2 + 2^{2h+1}\Im\gamma}{3^{h+1}2^{2h}\gamma^2} \tag{2.25}$$

Subtracting (2.15) from (2.12) we obtain the bound on $|V_4|$

$$\begin{aligned} &4^{h+1}\gamma V_4 + 2^{h+1}3^{h+2}\gamma(\gamma-1)V_2V_3 + 4^{g+1}\gamma V_4 + (5\cdot 4^{h+1} + 2^{h+1}3^{h+1}\gamma - 2^{h+2}3^{h+1})V_2^3\gamma \\ &\quad - V_2V_3\gamma(5\cdot 4^{h+1} + 2^{h+1}3^{h+1}\gamma - 2^{h+1}3^{h+1}) = \Im(x_3 - y_3) + \Im(\Im-1)(x_1x_2 - y_1y_2) \\ &\quad + \frac{\Im(\Im-1)(\Im-2)}{3!}\left(x_1^3 - y_1^3\right) \end{aligned}$$

$$\begin{aligned} &2.4^{h+1}\gamma V_4 + V_2\left[5\cdot 4^{h+1}V_2^2\gamma + 2^{h+2}3^{h+1}\gamma(\gamma-1)V_2^2 - 5\cdot 4^{h+1}V_3\gamma\right] = \Im(x_3 - y_3) \\ &\quad + \Im(\Im-1)(x_1x_2 - y_1y_2) + \frac{\Im(\Im-1)(\Im-2)}{3!}\left(x_1^3 - y_1^3\right) \end{aligned}$$

from (2.22), we get

$$\begin{aligned} &2.4^{h+1}\gamma V_4 - V_2\left[\frac{5\cdot 4^{h+1}\gamma\Im(y_2 - x_2)}{2.3^{h+1}\gamma} + 2^{h+2}3^{h+1}\gamma(1-\gamma)V_2^2\right] = \Im(x_3 - y_3) \\ &\quad + \Im(\Im-1)(x_1x_2 - y_1y_2) + \frac{\Im(\Im-1)(\Im-2)}{3!}\left(x_1^3 - y_1^3\right) \end{aligned}$$

also from (2.19). we have

$$\begin{aligned} 2.4^{h+1}\gamma V_4 &= \frac{\sqrt{2}\Im}{\sqrt{2^h\gamma^2(2^{h+1} - 2^h\Im + \Im) - \gamma\Im(2^{2h+1} - 3^{h+1})}}\left[\frac{5\cdot 4^{h+1}\gamma\Im(y_2 - x_2)}{2.3^{h+1}\gamma}\right. \\ &\quad \left. + 2^{h+2}3^{h+1}\gamma(1-\gamma)\left(\frac{\Im^2(y_2 + x_2)}{2^{h+1}\gamma^2(2^{h+1} - 2^h\Im + \Im) - 2\gamma\Im(2^{2h+1} - 3^{h+1})}\right)\right] + \Im(x_3 - y_3) \\ &\quad + \Im(\Im-1)(x_1x_2 - y_1y_2) + \frac{\Im(\Im-1)(\Im-2)}{3!}\left(x_1^3 - y_1^3\right) \end{aligned}$$

Applying lemma (1.1), we get

$$|V_4| \leq \frac{\Im}{2^{2h+1}\gamma}\left[1+2(\Im-1)\left(1+\frac{(\Im-2)}{3}\right)\right] + \frac{\sqrt{2}\Im^2}{\sqrt{2^h\gamma^2(2^{h+1}-2^h\Im+\Im)-\gamma\Im(2^{2h+1}-3^{h+1})}} \left[\frac{5}{3^{h+1}\gamma}+\frac{3^{h+1}(1-\gamma)\Im}{2^{2h+1}\gamma^2(2^{h+1}-2^h\Im+\Im)-2^{h+1}\gamma\Im(2^{2h+1}-3^{h+1})]}\right]$$

□

## 3. Coefficient bounds for the Function Class $\mathfrak{M}_{\mathfrak{E}}^{\gamma,h}(\mu)$

**Definition 3.1.** *A function $\varphi(z) \in \mathfrak{T}$ is in the class $\mathfrak{K}_{\mathfrak{E}}^{\phi,g}(\mu)$ if the following condition are fulfilled:*

(3.1)
$$\Re\left[\left((D^h\varphi(z))'\right)^{\gamma}\right] > \mu \quad z \in \nabla,$$

*and*

(3.2)
$$\Re\left[\left((D^h\eta(l))'\right)^{\gamma}\right] > \mu \quad l \in \nabla,$$

*where $\varphi(z) \in \mathfrak{E}$, $\gamma > 0$, $0 \leq \mu < 1$ and*

$$\eta(l) = l - V_2l^2 + (2V_2^2 - V_3)l^3 - (5V_2^3 - 5V_2V_3 + V_4)l^4 + \cdots$$

**Remark 3.1.** .

(1) $\mathfrak{M}_{\mathfrak{E}}^{1,0}(\mu)$=$\mathfrak{K}_{\mathfrak{E}}(\mu)$ *which Srivastava et al. [20] introduced and studied.*
(2) $\mathfrak{M}_{\mathfrak{E}}^{\gamma,0}(\mu)$=$\mathfrak{M}_{\mathfrak{E}}^{\gamma}(\mu)$ *which Girgaonkar et al. [7] introduced and studied.*

Now we have the following theorem and the proof.

**Theorem 3.1.** *A function $\varphi(z) \in \mathfrak{T}$ is in the class $\mathfrak{M}_{\mathfrak{E}}^{\gamma,h}(\mu)$. Then*

(3.3)
$$|V_2| \leq \sqrt{\frac{2(1-\mu)}{\gamma^2(2^{2h}+2^h)-\gamma(2^{2h+1}-3^{3h+1})}},$$

(3.4)
$$|V_3| \leq \frac{(1-\mu)[2^{2h+1}\gamma - 3^{h+1}\mu + 3^{2h+1}]}{2^{2h}3^{h+1}\gamma^2}$$

*and*

$$(3.5)\quad |V_4| \le \frac{(1-\mu)}{2^{2h+1}\gamma} + \sqrt{\frac{2(1-\mu)}{\gamma^2(2^{2h}+2^h)-\gamma(2^{2h+1}-3^{3h+1})}}\left[\frac{5(1-\mu)}{3^{h+1}\gamma} + \frac{3^{h+1}(1-\mu)(1-\gamma)}{2^{h+1}\left(\gamma^2(2^{2h}+2^h)-\gamma(2^{2h+1}-3^{3h+1})\right)}\right]$$

*Proof.* From (3.1) and (3.2) we have,

$$\left((D^h\varphi(z))'\right)^\gamma = \mu + (1-\mu)y(z) \tag{3.6}$$

and

$$\left((D^h\eta(l))'\right)^\gamma = \mu + (1-\mu)x(l) \tag{3.7}$$

where $y(z)$ and $x(l)$ in $\mathcal{P}$ given by (2.8) and (2.9), that is

$$\mu + (1-\mu)y(z) = 1 + (1-\mu)y_1z + (1-\mu)y_2z^2 + (1-\mu)y_3z^3\cdots$$

and

$$\mu + (1-\mu)x(l) = 1 + (1-\mu)x_1l + (1-\mu)x_2l^2 + (1-\mu)x_3l^3 + \cdots$$

Equating the coefficients of (3.6) and (3.7) we get

$$2^{h+1}\gamma V_2 = (1-\mu)y_1, \tag{3.8}$$

$$3^{h+1}\gamma V_3 + 2^{2h+1}V_2^2\gamma(\gamma-1) = (1-\mu)y_2, \tag{3.9}$$

$$4^{h+1}\gamma V_4 + 2^{h+1}3^{h+2}\gamma(\gamma-1)V_2V_3 = (1-\mu)y_3 \tag{3.10}$$

$$-2^{h+1}\gamma V_2 = (1-\mu)x_1, \tag{3.11}$$

$$2\gamma[3^{h+1} - 2^{2h} + 2^{2h}\gamma]V_2^2 - 3^{h+1}\gamma V_3 = (1-\mu)x_2 \tag{3.12}$$

$$(3.13)\quad -4^{g+1}\gamma V_4 - (5\cdot 4^{h+1} + 2^{h+1}3^{h+1}\gamma - 2^{h+2}3^{h+1})V_2^3\gamma + V_2V_3\gamma(5\cdot 4^{h+1} + 2^{h+1}3^{h+1}\gamma - 2^{h+1}3^{h+1}) = (1-\mu)x_3$$

From (3.8) and (3.11) we get

$$y_1 = -x_1 \tag{3.14}$$

and

$$2^{2h+3}\gamma^2 V_2^2 = (1-\mu)^2(y_1^2 + x_1^2) \tag{3.15}$$

Now, adding (3.9) and (3.12), we have

$$\left[2\gamma^2(2^{2h} + 2^h) - 2\gamma(2^{2h+1} - 3^{h+1})\right] V_2^2 = (1-\mu)(y_2 + x_2) \tag{3.16}$$

$$|V_2^2| \le \frac{(1-\mu)(|y_2| + |x_2|)}{[2\gamma^2(2^{2h} + 2^h) - 2\gamma(2^{2h+1} - 3^{h+1})]} \tag{3.17}$$

Applying Lemma 1.1, we get

$$|V_2| \le \sqrt{\frac{2(1-\mu)}{\gamma^2(2^{2h} + 2^h) - \gamma(2^{2h+1} - 3^{3h+1})}},$$

Now, to get the bound $|V_3|$, we subtract (3.12) from (3.9) to have

$$2.3^{h+1}\gamma V_3 + 2^{2h+1}V_2^2\gamma(\gamma - 1) - 2\gamma[3^{h+1} - 2^{2h} + 2^{2h}\gamma]V_2^2 = (1-\mu)(y_2 - x_2)$$

$$V_3 = V_2^2 + \frac{(1-\mu)(y_2 - x_2)}{2.3^{h+1}\gamma} \tag{3.18}$$

$$V_3 = \frac{2(1-\mu)^2 y_1^2}{2^{2h+3}\gamma^2} + \frac{(1-\mu)(y_2 - x_2)}{2 \cdot 3^{h+1}\gamma}$$

Applying Lemma 1.1, we get

$$|V_3| \le \frac{(1-\mu)[2^{2h+1}\gamma - 3^{h+1}\mu + 3^{2h+1}]}{2^{2h}3^{h+1}\gamma^2}$$

Subtracting (3.13) from (3.10) we obtain the bound on $|V_4|$

$$4^{h+1}\gamma V_4 + 2^{h+1}3^{h+2}\gamma(\gamma - 1)V_2 V_3 + 4^{g+1}\gamma V_4 + (5 \cdot 4^{h+1} + 2^{h+1}3^{h+1}\gamma - 2^{h+2}3^{h+1})$$
$$V_2^3\gamma - V_2 V_3\gamma(5 \cdot 4^{h+1} + 2^{h+1}3^{h+1}\gamma - 2^{h+1}3^{h+1}) = (1-\mu)(y_3 - x_3)$$

$$2.4^{h+1}\gamma V_4 + V_2\left[5 \cdot 4^{h+1}V_2^2\gamma + 2^{h+2}3^{h+1}\gamma(\gamma - 1)V_2^2 - 5 \cdot 4^{h+1}V_3\gamma\right] = (1-\mu)(y_3 - x_3)$$

from (3.18), we have

$$2.4^{h+1}\gamma|V_4| \leq (1-\mu)(y_3 - x_3) + \sqrt{\frac{(1-\mu)(|y_2|+|x_2|)}{[2\gamma^2(2^{2h}+2^h) - 2\gamma(2^{2h+1}-3^{h+1})]}} \left[\frac{5\cdot 4^{h+1}\gamma(1-\mu)(y_2-x_2)}{2.3^{h+1}\gamma} + 2^{h+2}3^{h+1}\gamma(1-\gamma) \left[\frac{(1-\mu)(|y_2|+|x_2|)}{[2\gamma^2(2^{2h}+2^h) - 2\gamma(2^{2h+1}-3^{h+1})]}\right]\right]$$

Applying Lemma 1.1, we get

$$|V_4| \leq \frac{(1-\mu)}{2^{2h+1}\gamma} + \sqrt{\frac{2(1-\mu)}{\gamma^2(2^{2h}+2^h) - \gamma(2^{2h+1}-3^{3h+1})}} \left[\frac{5(1-\mu)}{3^{h+1}\gamma} + \frac{3^{h+1}(1-\mu)(1-\gamma)}{2^{h+1}\left(\gamma^2(2^{2h}+2^h) - \gamma(2^{2h+1}-3^{3h+1})\right)}\right]$$

□

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