WEAK NONTRIVIAL SOLUTIONS OF DIRICHLET DISCRETE NONLINEAR PROBLEMS

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ABSTRACT. In this paper we prove the existence of solutions to limit value problems. In the literature, these problems are taken as models in different fields of mathematical physics.

The main existence results are obtained by using a critical point theory and the Ekelands variational principle by applying the mountain pass geometry lemma.

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1. INTRODUCTION

Difference equations describe different phenomena in several fields such as economics, physics, mechanics, computer science and finance see for example [1,5,12] and the references cited therein. Because of its importance in applications, there is a large literature about it. For example, based on the method of minimization, in [11], Koné and Ouaro proved the existence of a single weak solution to the following Dirichlet problem

$$-\Delta\left(a\left(k-1, \Delta u\left(k-1\right)\right)\right) = f\left(k\right), \ k \in \mathbb{N}\left[1, T\right],$$

where $T \geq 2$ is a positive integer.

Also, in [14], based on critical point theory, Mihailescu and his collaborators proved the

existence of a continuous spectrum of eigenvalues for the Dirichlet problem

$$-\Delta\left(|\Delta u\left(k-1\right)|^{p(k-1)-2}\Delta u\left(k-1\right)\right) = \lambda|u\left(k\right)|^{q(k)-2}u\left(k\right),\ k\in\mathbb{N}\left[1,T\right],$$

where T is a positive integer, $p:\mathbb{N}\left[0,T\right]\longrightarrow[2,+\infty)$ and $q:\mathbb{N}\left[1,T\right]\longrightarrow[2,+\infty)$ are bounded while λ is positive constant.

M. Galewski and R. Wieteska studied in [7], by using variational methods and a critical point theory, the existence and the multiplicity of positive solutions for a perturbed anisotropic difference equation with dependence on a numerical parameter:

$$\Delta\left(\alpha(k)|\Delta u\left(k-1\right)|^{p(k-1)-2}\Delta u\left(k-1\right)\right) + \lambda f\left(k,u(k)\right) = 0,\ k\in\mathbb{N}\left[1,T\right],$$

with some function α, a nonlinear term f and a numerical non negative parameter λ. The authors derived the intervals of a numerical parameter λ for which the considered boundary value problem has at least one, exactly one, or at least two positive solutions. Some useful discrete inequalities are also derived.

In this paper, motivated by the above facts, we propose the following anisotropic nonlinear discrete Dirichlet problem

(1.1)
$$\begin{cases} -\Delta\left[\alpha(k-1)a(k-1,\Delta u(k-1))\right] = \lambda f(k,u(k)),\ k\in\mathbb{N}[1,T], \\ \\ u(0) = u(T+1) = 0, \end{cases}$$

where $\lambda > 0$ is a numerical parameter. $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator, $\mathbb{N}[1,T] = \{1,\ldots,T\}$, with $T \geq 2$, a, α and f are functions to be defined later.

Our paper is organized in the following way. In section 2 we define the general results that we will use throughout our work. In section 3, under suitable assumptions on the data, we show that problem (1.1) admits at least a weak nontrivial solution. This is subdivided into four subsections. In the first we use the minimization method; in the second we apply mountain pass geometry lemma, then in the third we apply the Ekelands variational principle and finally we managed to show that the problem (1.1) has at least two nontrivial weak solutions. In section 4 we proceed to an extension of the previous results.

2. Preliminaries

By a solution to problem (1.1) we mean such a function $u:\mathbb{N}[0,T+1]\longrightarrow\mathbb{R}$ which satisfies the given equation on $\mathbb{N}[1,T]$ and the boundary conditions. In the T-dimensional Hilbert

space

$$X = \{u : \mathbb{N}[0, T + 1] \longrightarrow \mathbb{R} : u(0) = u(T + 1) = 0\},$$

with the inner product

$$\langle u, v \rangle = \sum_{k=1}^{T+1} \Delta u(k - 1) \Delta v(k - 1), \qquad \forall\, u, v \in X,$$

we consider the norm

(2.1)
$$\|u\| = \left(\sum_{k=1}^{T+1} |\Delta u(k - 1)|^2 \right)^{\frac{1}{2}}.$$

Let the function

(2.2)
$$p : \mathbb{N}[0, T] \longrightarrow [2, +\infty)$$

and denoted by

$$p^- = \min_{k \in \mathbb{N}[0,T]} p(k), \qquad p^+ = \max_{k \in \mathbb{N}[0,T]} p(k).$$

For the data α, a and f, we assume the following.

$(H_1):$
$$\begin{cases} a(k, .) : \mathbb{R} \to \mathbb{R}, \ \ k \in \mathbb{N}[0, T] \text{ and there exists } A(., .) : \mathbb{N}[0, T] \times \mathbb{R} \to \mathbb{R} \\ \text{which satisfies } a(k, \xi) = \frac{\partial}{\partial \xi} A(k, \xi) \text{ and } A(k, 0) = 0, \text{ for all } k \in \mathbb{N}[0, T]. \end{cases}$$

$(H_2):$ For all $k \in \mathbb{N}[0, T]$ and $\xi \neq \eta$

(2.3)
$$(a(k, \xi) - a(k, \eta)) \,.\, (\xi - \eta) > 0.$$

$(H_3):$ For any $k \in \mathbb{N}[0, T]$, $\xi \in \mathbb{R}$, we have

(2.4)
$$p(k) A(k, \xi) \geq a(k, \xi) \xi \geq |\xi|^{p(k)}.$$

$(H_4):$ For any $k \in \mathbb{N}[0, T]$, $\xi \in \mathbb{R}$ it exist $C_1 > 0$ such that

(2.5)
$$|a(k, \xi)| \leq C_1 (1 + |\xi|^{p(k)-1}).$$

$(H_5):$ The function $\alpha : \mathbb{N}[0, T] \longrightarrow (0, +\infty)$ is such that for all $k \in \mathbb{N}[0, T]$,

(2.6)
$$0 < \underline{\alpha} = \inf_{k \in \mathbb{N}[0,T]} (\alpha(k)) \leq \alpha(k) \leq \bar{\alpha} = \sup_{k \in \mathbb{N}[0,T]} (\alpha(k)) < +\infty.$$

$(H_6):$ For each $k \in \mathbb{N}[1, T]$, the function $f(k, .) : \mathbb{R} \longrightarrow \mathbb{R}$ is jointly continuous and there exist the functions $A_1, A_2 : \mathbb{N}[1, T] \longrightarrow (-\infty, 0)$; $B_1, B_2 : \mathbb{N}[1, T] \longrightarrow (0, +\infty)$ and a function $r : \mathbb{N}[1, T] \longrightarrow [2, +\infty)$ such that

(2.7)
$$A_1(k) + B_1(k) |\xi|^{r(k)-1} \leq f(k, \xi) \leq A_2(k) + B_2(k) |\xi|^{r(k)-1},$$

where

$$-\infty < \underline{A_1} = \inf_{k\in\mathbb{N}[1,T]} A_1(k), \quad \overline{A_1} = \sup_{k\in\mathbb{N}[1,T]} A_1(k) < 0;$$

$$-\infty < \underline{A_2} = \inf_{k\in\mathbb{N}[1,T]} A_2(k), \quad \overline{A_2} = \sup_{k\in\mathbb{N}[1,T]} A_2(k) < 0;$$

$$0 < \underline{B_1} = \inf_{k\in\mathbb{N}[1,T]} B_1(k), \quad \overline{B_1} = \sup_{k\in\mathbb{N}[1,T]} B_1(k) < +\infty;$$

$$0 < \underline{B_2} = \inf_{k\in\mathbb{N}[1,T]} B_2(k), \quad \overline{B_2} = \sup_{k\in\mathbb{N}[1,T]} B_2(k) < +\infty;$$

$$r^- = \min_{k\in\mathbb{N}[1,T]} r(k), \quad r^+ = \max_{k\in\mathbb{N}[1,T]} r(k)$$

and there exists

$$\beta_1, \beta_2 > 1 \ \text{ such that } \ \underline{B_1} > \max(\beta_1, \beta_2)|\underline{A_1}|r^+.$$

We denote

(2.8)
$$F(k,\xi) = \int_0^\xi f(k,s)ds \ \text{ for } \ (k,\xi) \in \mathbb{N}[0,T] \times \mathbb{R}.$$

There are many functions satisfying both $(H_1) - (H_6)$. Let us mention the following.

Example 2.1.

- $A(k,\xi) = \dfrac{1}{p(k)}\left(\left(1+|\xi|^2\right)^{p(k)/2} - 1 \right)$

  with $\quad a(k,\xi) = \left(1+|\xi|^2\right)^{(p(k)-2)/2}\xi, \ \ \forall\, k \in \mathbb{N}[0,T], \forall\, \xi \in \mathbb{R},$
- $f(k,\xi) = -1 + \left|\xi\right|^{r(k)-1}, \ \ \forall\, k \in \mathbb{N}[1,T], \ \ \forall\, \xi \in \mathbb{R},$
- $\alpha(k) = 1, \ \ \forall\, k \in \mathbb{N}[0,T].$

Moreover, we may consider X with the following norm

(2.9)
$$|u|_m = \left(\sum_{k=1}^T |u(k)|^m \right)^{\frac{1}{m}}, \quad \forall\, u \in X \ \text{ and } \ m \geq 2.$$

We have the following inequalities (see [3])

(2.10)
$$T^{(2-m)/(2m)}|u|_2 \leq |u|_m \leq T^{1/m}|u|_2, \quad \forall\, u \in X \ \text{ and } \ m \geq 2.$$

We need the following auxiliary results throughout our paper (see [7,8,14]).

Lemma 2.1.

(1) *There exist two positive constants C_2, C_3 such that*

(2.11)
$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq C_2 \|u\|^{p^-} - C_3$$

for all $u \in X$ with $\|u\| > 1$.

(2) *For every $u \in X$ with $\|u\| \leq 1$ we have*

(2.12)
$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq T^{-\frac{p^+-2}{2}} \|u\|^{p^+}.$$

(3) *For any $m \geq 2$ there exists a positive constant c_m such that*

(2.13)
$$\sum_{k=1}^{T} |u(k)|^m \leq c_m \sum_{k=1}^{T+1} |\Delta u(k-1)|^m, \quad \forall\, u \in X.$$

(4) *For every $u \in X$ we have*

(2.14)
$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \leq (T+1)\|u\|^{p^+} + (T+1).$$

(5) *For every $u \in X$ and for every $m \geq 2$ we have*

(2.15)
$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^m \leq 2^m \sum_{k=1}^{T} |u(k)|^m.$$

Theorem 2.1. [13] *Let H be a reflexive Banach space. If a functional $J \in C^1(H, \mathbb{R})$ is weakly lower semi-continuous and coercive, i.e.* $\lim\limits_{\|x\| \to \infty} J(x) = +\infty$, *then there exists x_0 such that*

$$\inf_{x \in H} J(x) = J(x_0)$$

and x_0 is also a critical point of J, i.e. $J'(x_0) = 0$. Moreover, if J is strictly convex, then a critical point is unique.

Theorem 2.2. [6]*(Ekeland's principle) Let E be a complete metric space and $\Phi : E \longrightarrow \mathbb{R}$ a lower semicontinuous function that is bounded below. Let $\varepsilon > 0$ and $\overline{u} \in E$ be given such that*

$$\Phi(\overline{u}) \leq \inf_E \Phi + \frac{\varepsilon}{2}.$$

Then given $\lambda > 0$ there exists $u_\lambda \in E$ such that

(i) $\Phi(u_\lambda) \leq \Phi(\overline{u})$,

(ii) $d(u_\lambda, \overline{u}) < \lambda$,

(iii) $\Phi(u_\lambda) < \Phi(u) + \frac{\varepsilon}{\lambda} d(u, u_\lambda)$ *for all $u \neq u_\lambda$.*

Definition 2.1. *Let H be a real Banach space. We say that a functional $J : H \longrightarrow \mathbb{R}$ satisfies the Palais-Smale condition if every sequence (u_n) such that $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$ has a convergent subsequence.*

Lemma 2.2. [4] *Let H be a Banach space and $J \in C^1(H, \mathbb{R})$ satisfies the Palais-Smale condition. Assume that there exist $x_0, x_1 \in H$ and a bounded open neighborhood Ω of x_0 such that $x_1 \notin \overline{\Omega}$ and*

$$\max \{J(x_0), J(x_1)\} < \inf_{x \in \partial \Omega} J(x).$$

Let

$$\Gamma = \{h \in C([0, 1], H) : h(0) = x_0, \quad h(1) = x_1\}$$

and

$$c = \inf_{h \in \Gamma} \max_{s \in [0,1]} J(h(s)).$$

Then c is a critical value of J; that is, there exists $x^* \in H$ such that $J'(x^*) = 0$ and $J(x^*) = c$, where $c > \max \{J(x_0), J(x_1)\}$.

Theorem 2.3. [2] *Let E be a finite-dimensional Euclidean space, $\eta, \mu_1, \mu_2 : E \longrightarrow \mathbb{R}$ be differentiable functions, and $S = \{x \in E : \mu_1 \leq 0, \mu_2 \leq 0\}$. Moreover, let $\overline{x} \in S$ be such that $\eta(\overline{x}) = \inf_S \eta(x)$. Then there exist numbers $\sigma_0, \sigma_1, \sigma_2 \geq 0$ such that $(\sigma_0)^2 + (\sigma_1)^2 + (\sigma_2)^2 > 0$ and*

$$\sigma_0 \eta'(\overline{x}) + \sigma_1 \mu_1'(\overline{x}) + \sigma_2 \mu_2'(\overline{x}) = 0 \quad \text{and} \quad \sigma_1 \mu_1(\overline{x}) = 0, \quad \sigma_2 \mu_2(\overline{x}) = 0.$$

3. Existence of solutions

In this section, we will prove the existence of at least one weak nontrivial solution to the problem (1.1).

This section is divided into four subsections.

We define the energy functional corresponding to (1.1); $J_\lambda : X \longrightarrow \mathbb{R}$ by

$$(3.1) \qquad J_\lambda(u) = \sum_{k=1}^{T+1} \alpha(k-1)A\big(k-1, \Delta u(k-1)\big) - \lambda \sum_{k=1}^{T} F\big(k, u(k)\big).$$

The functional J_λ is differentiable in sense of Gâteaux and its Gâteaux derivative reads

$$(3.2) \qquad \langle J_\lambda'(u), v \rangle = \sum_{k=1}^{T+1} \alpha(k-1)a(k-1, \Delta u(k-1))\Delta v(k-1) - \lambda \sum_{k=1}^{T} f\big(k, u(k)\big)v(k),$$

for all $u, v \in X$.

A critical point to J_λ, i.e. a point $u \in X$ such that

$$(3.3) \qquad \langle J'_\lambda(u), v \rangle = 0 \qquad \text{for all} \quad v \in X$$

is a weak solution to (1.1). The previous results have been proven in $[9, 10, 15]$.

Lemma 3.1. *Assume that* (2.5), (2.7) *holds and* $r^- > p^+$. *Then for any* $\lambda > 0$ *the functional* J_λ
satisfies the Palais-Smale condition.

Proof Assume that $\{u_n\}$ is such that $\{J_\lambda(u_n)\}$ is bounded and $J'_\lambda(u_n) \to 0$. We have X finitely
dimensional, it is enough to show that $\{u_n\}$ is bounded. Assume that $\{u_n\}$ is unbounded, we
have, for n large enough, $\|u_n\|$ sufficiently large. We consider n large enough and by (2.5),
(2.7), (2.10), (2.14), (2.15) we have

$$
\begin{aligned}
J_\lambda(u_n) \;\leq\; & \bar{\alpha} \sum_{k=1}^{T+1} \int_0^{\Delta u_n(k-1)} |a(k-1, s)| ds - \lambda \left(-|\underline{A_1}| \sum_{k=1}^{T} |u_n(k)| + \frac{B_1}{r^+} \sum_{k=1}^{T} |u_n(k)|^{r(k)} \right) \\
\leq\; & C_1 \bar{\alpha} \sum_{k=1}^{T+1} \left(|\Delta u_n(k-1)| + \frac{|\Delta u_n(k-1)|^{p(k-1)}}{p^-} \right) \\
& -\lambda \left(-|\underline{A_1}| \sqrt{T} \sum_{k=1}^{T} |u_n(k)|^2 + \frac{B_1}{r^+} \sum_{k=1}^{T} |u_n(k)|^{r^-} \right) \\
\leq\; & C_1 \bar{\alpha} \left(\sqrt{T+1} : \|u_n\| + \frac{(T+1)\|u_n\|^{p^+}}{p^-} + \frac{T+1}{p^-} \right) \\
& -\lambda \left(-|\underline{A_1}| \sqrt{T} c_2 \|u_n\|^2 + \frac{2^{-r^-}(T+1)^{\frac{2-r^-}{2}} B_1}{r^+} \|u_n\|^{r^-} \right).
\end{aligned}
$$

Since $r^- > p^+$ and $\|u_n\| \to +\infty$; we have $J_\lambda(u_n) \to -\infty$. This is contradictory. Hence the
sequence $\{u_n\}$ is bounded.

3.1. Case $p^- > r^+$.

In this subsection we will show that the problem (1.1) admits at least one weak nontrivial
solution using the minimisation method.

Proposition 3.1. *Assume that* (2.4), (2.7) *holds and* $p^- > r^+$, *then* J_λ *is coercive for all* $\lambda > 0$.

Proof according to (2.4) and (2.7), we have

$$
\begin{aligned}
J_\lambda(u) \quad &= \sum_{k=1}^{T+1} \alpha(k-1) A\big(k-1, \Delta u(k-1)\big) - \lambda \sum_{k=1}^{T} F\big(k, u(k)\big) \\
&\geq \frac{\alpha}{p^+} \sum_{k=1}^{T+1} \big|\Delta u(k-1)\big|^{p(k-1)} - \lambda \left(\frac{\overline{B_2}}{r^-} \sum_{k=1}^{T} |u(k)|^{r(k)} + |\underline{A_2}| \sum_{k=1}^{T} |u(k)| \right).
\end{aligned}
$$

To prove the coerciveness of the functional J_λ, we consider $\|u\| > 1$, and from the above inequalities (2.10), (2.11) and (2.13) we deduce that

$$
\begin{aligned}
J_\lambda(u) &\geq \frac{C_2\,\alpha}{p^+} \|u\|^{p^-} - \frac{\alpha}{p^+} C_3 - \lambda \left(\frac{\overline{B_2}}{r^-} \sum_{k=1}^{T} |u(k)|^{r^+} + \frac{\overline{B_2}}{r^-} \sum_{k=1}^{T} |u(k)|^{r^-} + |\underline{A_2}| \sqrt{T} \sum_{k=1}^{T} |u(k)|^2 \right) \\
&\geq \frac{C_2\,\alpha}{p^+} \|u\|^{p^-} - \frac{\alpha}{p^+} C_3 \\
&\quad - \lambda \left(\frac{\overline{B_2}}{r^-} c_{r^+} \sum_{k=1}^{T+1} |\Delta u(k-1)|^{r^+} + \frac{\overline{B_2}}{r^-} c_{r^-} \sum_{k=1}^{T+1} |\Delta u(k-1)|^{r^-} + |\underline{A_2}| \sqrt{T} c_2 \sum_{k=1}^{T+1} |\Delta u(k-1)|^2 \right) \\
&\geq \frac{C_2\,\alpha}{p^+} \|u\|^{p^-} - \frac{\alpha}{p^+} C_3 - \lambda \left(\frac{\overline{B_2}}{r^-} c_{r^+} (T+1) |u\|^{r^+} + \frac{\overline{B_2}}{r^-} c_{r^-} (T+1) \|u\|^{r^-} + |\underline{A_2}| \sqrt{T} c_2 \|u\|^2 \right).
\end{aligned}
$$

Hence $p^- > r^+$, J_λ is coercive.

Theorem 3.1. *Assume that (2.5) and (2.7) holds and $p^- > r^+$. Then, there exists $\lambda_0 \in (0, +\infty)$ such that for all $\lambda > \lambda_0$ the problem (1.1) has at least one weak nontrivial solution.*

Proof. As in [11], $J_\lambda \in C^1(X, \mathbb{R})$ and weakly lower semicontinuous. Moreover by Proposition 3.1 we prove the Theorem 2.1. Let $u_\varepsilon \in X$ a global minimizer of J_λ, a weak solution of problem (1.1). We show u_ε is not trivial for all $p^- > r^+$ and $\lambda > \lambda_0$.

For $t_0 > 1$ be a fixed real and $k_0 \in \mathbb{N}[0, T+1]$, we define $u_0 : \mathbb{N}[0, T+1] \longrightarrow \mathbb{R}$ such that $u_0(k_0) = t_0$ and $u_0(k) = 0$ for any $k \in \mathbb{N}[0, T+1] \backslash \{k_0\}$, we have $u_0 \in X$. By (2.5) and (2.7)

$$
\begin{aligned}
J_\lambda(u_0) \quad &\leq C_1 \overline{\alpha} \left(2t_0 + \frac{t_0^{p(k_0-1)} + t_0^{p(k_0)}}{p^-} \right) - \lambda \left(-|\underline{A_1}| t_0 + \frac{B_1}{r^+} t_0^{r^-} \right) \\
&\leq 4 C_1 \overline{\alpha} t_0^{p^+} - \lambda t_0 \left(-|\underline{A_1}| + \frac{B_1}{r^+} \right) \\
&\leq 4 C_1 \overline{\alpha} t_0^{p^+} - \lambda t_0 \left(-\max(\beta_1, \beta_2) |\underline{A_1}| + \frac{B_1}{r^+} \right).
\end{aligned}
$$

where

$$
\lambda_0 = \frac{4 C_1 \overline{\alpha} t_0^{p^+ - 1}}{-\max(\beta_1, \beta_2) |\underline{A_1}| + \frac{B_1}{r^+}}.
$$

We have $J_\lambda(u_0) < 0$ for any $\lambda \in (\lambda_0, +\infty)$.

It follows that $J_\lambda(u_\varepsilon) < 0$ for any $\lambda > \lambda_0$, u_ε is a weak nontrivial solution of problem (1.1) for λ large enough. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.2. **Case** $r^- > p^+$.

We have previously shown that if $r^- > p^+$ J_λ satisfies the Palais-Smale condition. We can thus use the mountain pass geometry Lemma.

Theorem 3.2. *Let $r^- > p^+$ and the condition (2.7) holds. Then, there exists $\lambda_1 \in (0, +\infty)$ such that for $\lambda \in (0, \lambda_1)$ the problem (1.1) has at least one weak nontrivial solution.*

Proof. Let

$$\Omega = \{u \in X : \|u\| \leq \beta\}$$

with $\beta \in (0, 1)$.

For $u \in \Omega$, by (2.10), (2.12) and (2.13)

$$
\begin{aligned}
J_\lambda(u) &\geq \frac{\alpha}{p^+} \sum_{k=1}^{T+1} \left|\Delta u(k-1)\right|^{p(k-1)} - \lambda \left(\frac{\overline{B_2}}{r^-} \sum_{k=1}^{T} |u(k)|^{r(k)} + |\underline{A_2}| \sum_{k=1}^{T} |u(k)|\right) \\
&\geq \frac{\alpha}{p^+} T^{\frac{2-p^+}{2}} \|u\|^{p^+} - \lambda \left(\frac{\overline{B_2}}{r^-} \sum_{k=1}^{T} |u(k)|^{r^+} + \frac{\overline{B_2}}{r^-} \sum_{k=1}^{T} |u(k)|^{r^-} + |\underline{A_2}|\sqrt{T}\left(\sum_{k=1}^{T} |u(k)|^2\right)^{\frac{1}{2}}\right) \\
&\geq \frac{\alpha}{p^+} T^{\frac{2-p^+}{2}} \|u\|^{p^+} - \lambda \left(\frac{\overline{B_2}}{r^-} c_{r^+}(T+1)\|u\|^{r^+} + \frac{\overline{B_2}}{r^-} c_{r^-}(T+1)\|u\|^{r^-} + |\underline{A_2}|\sqrt{Tc_2}\|u\|\right).
\end{aligned}
$$

For $u \in \partial\Omega$, we obtain

$$J_\lambda(u) \geq \frac{\alpha}{p^+} T^{-\frac{p^+-2}{2}} \beta^{p^+} - \lambda\beta \left(\frac{2\overline{B_2}}{r^-}(T+1)\max(c_{r^-}, c_{r^+}) + |\underline{A_2}|\sqrt{Tc_2}\right).$$

So for every $\lambda \in (0, \lambda_1)$

$$J_\lambda(u) > 0 \quad \text{for all } u \in \partial\Omega$$

with

$$\lambda_1 = \frac{\frac{\alpha}{p^+} \beta^{(p^+-1)} T^{-\frac{p^+-2}{2}}}{\frac{2\overline{B_2}}{r^-}(T+1)\max(c_{r^-}, c_{r^+}) + |\underline{A_2}|\sqrt{Tc_2}}.$$

Take $u \in X$ such that $u(k) > 1$, for $k \in \mathbb{N}[1,T]$

$$J_\lambda(u) \le C_1\overline{\alpha} \sum_{k=1}^{T+1} \left(|\Delta u(k-1)| + \frac{|\Delta u(k-1)|^{p(k-1)}}{p^-}\right) - \lambda\left(-|\underline{A_1}| \sum_{k=1}^{T} u(k) + \frac{B_1}{r^+} \sum_{k=1}^{T} |u(k)|^{r^-}\right).$$

Let $u_t \in X$ defined in the following way : $u_t(k) = t$ for $k \in \mathbb{N}[1,T]$ and
$u_t(0) = u_t(T+1) = 0$.

For $t > 1$, we have

$$
\begin{aligned}
J_\lambda(u_t) &\le C_1\overline{\alpha}\left(2t + \frac{t^{p(0)} + t^{p(T)}}{p^-}\right) - \lambda T\left(-|\underline{A_1}|t + \frac{B_1}{r^+}t^{r^-}\right) \\
&\le 4C_1\overline{\alpha}t^{p^+} - \lambda T\left(-|\underline{A_1}|t + \frac{B_1}{r^+}t^{r^-}\right)
\end{aligned}
$$

Since $r^- > p^+$, $\lim\limits_{t\to+\infty} J_\lambda(u_t) = -\infty$; then there exists t_0 such that for $u_{t_0} \in X\backslash\Omega$

$$J_\lambda(u_{t_0}) < \min_{u\in\partial\Omega} J_\lambda(u).$$

$J_\lambda \in C^1(X,\mathbb{R})$, and according to Lemma 2.2 the problem (1.1) has at least one weak nontrivial solution. □

3.3. **Case $p^- > r^-$.**

In this part we apply Ekeland's variational principle.

Proof. Take $\lambda \in (0, \lambda_1)$. In the proof of Theorem 3.2, for every $u \in \partial\Omega$, we have $J_\lambda(u) > 0$.
Using Weierstrass theorem we obtain

$$\inf_{u\in\partial\Omega} J_\lambda(u) > 0.$$

Taking $u(k) \in (0, \beta)$, we have

$$J_\lambda(u) \le C_1\overline{\alpha} \sum_{k=1}^{T+1}\left(|\Delta u(k-1)| + \frac{|\Delta u(k-1)|^{p(k-1)}}{p^-}\right) - \lambda\left(-|\underline{A_1}| \sum_{k=1}^{T} |u(k)| + \frac{B_1}{r^+} \sum_{k=1}^{T} |u(k)|^{r(k)}\right).$$

For $t \in (0, \beta)$, assume that

$$t < \sqrt[(p^- - r^-)]{\frac{\lambda\left(-\max(\beta_1, \beta_2)|\underline{A_1}| + \frac{B_1}{r^+}\right)}{2C_1\overline{\alpha}\left(\alpha_1 + \frac{1}{p^-}\right)}}.$$

We choose $k_0 \in \mathbb{N}[1,T]$ such that $r(k_0) = r^-$.

Let $u_0 \in X$ be a function such that $u_0(k_0) = t$ and $u_0(k) = 0$ for any $k \in \mathbb{N}[1,T]\setminus\{k_0\}$.

We obtain

$$
\begin{aligned}
J_\lambda(u_0) &\leq C_1 \overline{\alpha} \left(2t + \frac{t^{p(k_0-1)} + t^{p(k_0)}}{p^-} \right) - \lambda \left(-|\underline{A_1}|t + \frac{B_1}{r^+} t^{r^-} \right) \\
&\leq 2C_1 \overline{\alpha} \left(t + \frac{t^{p^-}}{p^-} \right) - \lambda \left(-|\underline{A_1}|t + \frac{B_1}{r^+} t^{r^-} \right).
\end{aligned}
$$

There exists $\alpha_1, \beta_1 \geq 1$ such that $\alpha_1 t^{p^-} \geq t$ and $\beta_1 t^{r^-} \geq t$.

We have

$$
\begin{aligned}
J_\lambda(u_0) &\leq 2C_1 \overline{\alpha} t^{p^-} \left(\alpha_1 + \frac{1}{p^-} \right) - \lambda \left(-\beta_1 |\underline{A_1}| + \frac{B_1}{r^+} \right) t^{r^-} \\
&\leq 2C_1 \overline{\alpha} t^{p^-} \left(\alpha_1 + \frac{1}{p^-} \right) - \lambda \left(-\max(\beta_1, \beta_2)|\underline{A_1}| + \frac{B_1}{r^+} \right) t^{r^-} < 0.
\end{aligned}
$$

Thus, $J_\lambda(u_0) < 0$ for $u_0 \in Int(\Omega)$.

Therefore,

$$
-\infty < \inf_{u \in Int(\Omega)} J_\lambda(u) < 0.
$$

So, we have

$$
\inf_{u \in Int(\Omega)} J_\lambda(u) < \inf_{u \in \partial\Omega} J_\lambda(u).
$$

Using the proof of [14] we have

$$
0 < \varepsilon < \inf_{u \in \partial\Omega} J_\lambda(u) - \inf_{u \in Int(\Omega)} J_\lambda(u).
$$

Applying Ekeland's variationnal principle to the functional $J_\lambda : \Omega \longrightarrow \mathbb{R}$ we find $u_\varepsilon \in \Omega$ such that

$$
\begin{aligned}
J_\lambda(u_\varepsilon) &< \inf_{u \in \Omega} J_\lambda(u) + \varepsilon \\
&< J_\lambda(u) + \varepsilon \|u - u_\varepsilon\| \quad for \quad u \neq u_\varepsilon.
\end{aligned}
$$

Since

$$
J_\lambda(u_\varepsilon) < \inf_{u \in \Omega} J_\lambda(u) + \varepsilon \leq \inf_{u \in Int(\Omega)} J_\lambda(u) + \varepsilon < \inf_{u \in \partial\Omega} J_\lambda(u),
$$

we deduce $u_\varepsilon \in Int(\Omega)$.

Now, we define $L_\lambda : \Omega \longrightarrow \mathbb{R}$ by

$$
L_\lambda(u) = J_\lambda(u) + \varepsilon \|u - u_\varepsilon\| \quad for \quad u \neq u_\varepsilon.
$$

We have u_ε as a minimum of L_λ and therefore

$$
\frac{L_\lambda(u_\varepsilon + tv) - L_\lambda(u_\varepsilon)}{t} + \varepsilon \|v\| \geq 0
$$

for any $v \in \Omega$ and a small enough positive t.

We deduce that

$$\frac{J_\lambda(u_\varepsilon + tv) - J_\lambda(u_\varepsilon)}{t} + \varepsilon\|v\| \geq 0.$$

Letting $t \to 0$, it follows that

$$\langle J_\lambda'(u_\varepsilon), v \rangle + \varepsilon\|v\| > 0$$

and we obtain

$$\|J_\lambda'(u_\varepsilon)\| \leq \varepsilon.$$

There exists a sequence $\{z_n\} \subset Int(\Omega)$ such that

$$J_\lambda(z_n) \to \inf_{u \in \Omega} J_\lambda(u) \quad and \quad J_\lambda'(z_n) \to 0$$

Since $\{z_n\}$ is bounded in X there exists $z_0 \in X$ such that, up to a subsequence, $\{z_n\}$ converges to $z_0 \in X$. Thus

$$J_\lambda(z_0) = \inf_{u \in \Omega} J_\lambda(u) \quad and \quad J_\lambda'(z_0) = 0.$$

z_0 is one weak nontrivial solution for problem (1.1). □

3.4. **Multiple solutions.**

In this section we prove the existence of at least two weak nontrivial solutions of the problem (1.1).

Theorem 3.3. *Assume that* (2.4), (2.7) *holds. Let* $r^- > p^+$ *and* $\delta > 1$.
For any $\lambda \in \left(0, \frac{\alpha\left(C_2\delta^{p^-} - C_3\right) + \sigma_1\delta^2}{\delta^{r^+}\left(|A_2|\sqrt{Tc_2} + 2\max(c_{r^+}, c_{r^-})\overline{B_2}(T+1)\right)}\right)$ *the problem* (1.1) *has at least two weak nontrivial solutions where one solution satisfies* $\|u\| > 1$.

Proof. Let

$$\Omega_\delta := \{u \in X : \|u\| \leq \delta\}; \qquad \Omega_\nu := \{u \in X : \|u\| \geq \nu\},$$

where $\nu \in (1, \delta)$.

Assume that $u_0 \in X$ is a local minimizer of J_λ in $\Omega = \Omega_\delta \cap \Omega_\nu$.
If $u_0 \in Int(\Omega)$ by using Lemma 2.2 we get $J_\lambda(u_0) < \min_{u \in \partial\Omega_\delta} J_\lambda(u)$.
Now suppose that $u_0 \in \partial\Omega_\delta$, by Theorem 2.3 there exist σ_0, σ_1, $\sigma_2 \geq 0$ with $\sigma_0^2 + \sigma_1^2 + \sigma_2^2 > 0$ such that for all $v \in X$

(3.4) $$\sigma_1\left(\|u_0\|^2 - \delta^2\right) = 0 \quad and \quad \sigma_2\left(\nu^2 - \|u_0\|^2\right) = 0$$

and

$$\sigma_0 \langle J'_\lambda(u_0), v \rangle + \sigma_1 \langle u_0, v \rangle - \sigma_2 \langle u_0, v \rangle = 0.$$

Since $u_0 \in \partial\Omega_\delta$, we have $\|u_0\| = \delta$ and $\sigma_2 = 0$.

Taking $v = u_0$ and $\sigma_0 = 1$ we see that

$$\sum_{k=1}^{T+1} \alpha(k-1)a(k-1, \Delta u_0(k-1))\Delta u_0(k-1) + \sigma_1\|u_0\|^2 = \lambda \sum_{k=1}^{T} f(k, u_0(k))u_0(k).$$

By (2.4) and (2.11)

$$(3.5) \qquad \sum_{k=1}^{T+1} \alpha(k-1)a(k-1, \Delta u_0(k-1))\Delta u_0(k-1) + \sigma_1\|u_0\|^2 \geq \underline{\alpha}\left(C_2\delta^{p^-} - C_3\right) + \sigma_1\delta^2.$$

Using (2.7), (2.10) and (2.13)

$$\lambda \sum_{k=1}^{T} f(k, u_0(k))u_0(k)$$

$$\leq \lambda\left(|\underline{A_2}|\sum_{k=1}^{T}|u_0(k)| + \overline{B_2}\sum_{k=1}^{T}|u_0(k)|^{r^+} + \overline{B_2}\sum_{k=1}^{T}|u_0(k)|^{r^-}\right)$$

$$\leq \lambda\left(|\underline{A_2}|\sqrt{T}\sqrt{\sum_{k=1}^{T}|u_0(k)|^2} + c_{r+}\overline{B_2}\sum_{k=1}^{T+1}|\Delta u_0(k)|^{r^+} + c_{r-}\overline{B_2}\sum_{k=1}^{T+1}|\Delta u_0(k)|^{r^-}\right)$$

$$\leq \lambda\left(|\underline{A_2}|\sqrt{Tc_2}\|u_0\| + c_{r+}\overline{B_2}(T+1)\|u_0\|^{r^+} + c_{r-}\overline{B_2}(T+1)\|u_0\|^{r^-}\right)$$

$$\leq \lambda\delta^{r^+}\left(|\underline{A_2}|\sqrt{Tc_2} + 2\max(c_{r+}, c_{r-})\overline{B_2}(T+1)\right).$$

So

$$\underline{\alpha}\left(C_2\delta^{p^-} - C_3\right) + \sigma_1\delta^2 \leq \lambda\delta^{r^+}\left(|\underline{A_2}|\sqrt{Tc_2} + 2\max(c_{r+}, c_{r-})\overline{B_2}(T+1)\right).$$

This is contradictory.

Hence $u_0 \in Int(\Omega)$ is a nontrivial minimizer of J_λ.

In the proof of Theorem (3.2) we show that there exists $u_1 \in X\backslash\Omega$ such that $J_\lambda(u_1) < \min_{u\in\partial\Omega_\delta} J_\lambda(u)$.

According to Lemma 2.2, there exists $u^* \in X$ such that u^* is a critical point of J_λ.

Finally, u_0 and u^* are two different weak nontrivial solutions to the problem (1.1) and since $u_0 \in Int(\Omega)$, it is easy to see that $\|u_0\| > 1$. $\qquad\square$

4. AN EXTENSION

In this section, we show that the existence result obtained in (1.1) can be extended to more general discrete boundary value problems of the form

(4.1)
$$
\begin{cases}
-\Delta\big[\alpha(k-1)a(k-1,\Delta u(k-1))\big] + |u|^{q(k)-2}u(k) = \lambda f(k,u(k)), \ k \in \mathbb{N}[1,T], \\
\\
u(0) = u(T+1) = 0,
\end{cases}
$$

with $q : \mathbb{N}[1,T] \longrightarrow (2,+\infty)$.

A function $u \in X$ is a solution of problem (4.1) if for any $v \in X$,

(4.2)
$$
\sum_{k=1}^{T+1} \alpha(k-1)a(k-1,\Delta u(k-1))\Delta v(k-1) + \sum_{k=1}^{T} |u(k)|^{q(k)-2}u(k)v(k) - \lambda \sum_{k=1}^{T} f\big(k,u(k)\big)v(k) = 0
$$

Theorem 4.1. *Assume that* (2.5) *and* (2.7) *hold. For all* $p^- > r^+$ *there exists* $\lambda_2 \in (0,+\infty)$ *such that for all* $\lambda > \lambda_2$ *the problem* (4.1) *has at least one weak nontrivial solution.*

Proof. In this part we use the proof of Theorem 3.1. For $u \in X$, we define the energy functional J_λ by

$$
J_\lambda(u) = \sum_{k=1}^{T+1} \alpha(k-1)A\big(k-1,\Delta u(k-1)\big) + \sum_{k=1}^{T} \frac{1}{q(k)}|u|^{q(k)} - \lambda \sum_{k=1}^{T} F\big(k,u(k)\big).
$$

The functional J_λ is well defined, weakly lower semi continuous and is of class $C^1(X,\mathbb{R})$ with a derivative given by

$$
\langle J'_\lambda(u),v\rangle = \sum_{k=1}^{T+1} \alpha(k-1)a(k-1,\Delta u(k-1))\Delta v(k-1) + \sum_{k=1}^{T} |u|^{q(k)-2}u(k)v(k) - \lambda \sum_{k=1}^{T} f\big(k,u(k)\big)v(k),
$$

for all $u,v \in X$.

Since

$$
\sum_{k=1}^{T} \frac{1}{q(k)}|u|^{q(k)} \geq 0
$$

we have

(4.3)
$$
J_\lambda(u) \geq \sum_{k=1}^{T+1} \alpha(k-1)A\big(k-1,\Delta u(k-1)\big) - \lambda \sum_{k=1}^{T} F\big(k,u(k)\big).
$$

According to Proposition 3.1 we deduce that the functional J_λ is coercive.

Let u_λ be a global minimizer of J_λ, taking u_0 such that $u_0(k_0) = t_0$ and $u_0(k) = 0$, for

$k \in \mathbb{N}[0, T+1] \backslash \{k_0\}$ where $t_0 > 1$ is a fixed real, we have

$$J_\lambda(u_0) \leq 4C_1\overline{\alpha}t_0^{p^+} + \frac{t_0^{q^+}}{q^-} - \lambda t_0 \left(-\max(\beta_1, \beta_2)|\underline{A_1}| + \frac{B_1}{r^+} \right),$$

where

$$\lambda_2 = \frac{4C_1\overline{\alpha}t_0^{p^+-1} + \dfrac{t_0^{q^+-1}}{q^-}}{-\max(\beta_1, \beta_2)|\underline{A_1}| + \dfrac{B_1}{r^+}}.$$

It follows that $J_\lambda(u_\lambda) < 0$ for any $\lambda > \lambda_2$. u_λ is one weak nontrivial solution of problem (4.1).

Lemma 4.1. *Assume that* (2.5), (2.7) *hold and* $r^- > \max(p^+, q^+)$. *Then for any* $\lambda > 0$ *the functional* J_λ *satisfies the Palais-Smale condition.*

Proof. By mimicking the proof of Lemma 3.1, we obtain

$$\begin{aligned}
J_\lambda(u_n) &\leq C_1\overline{\alpha}\left(\sqrt{T+1}\|u_n\| + \frac{(T+1)\|u_n\|^{p^+}}{p^-} + \frac{T+1}{p^-} \right) + \sum_{k=1}^{T} \frac{1}{q(k)}|u_n(k)|^{q(k)} \\
&\quad - \lambda \left(-|\underline{A_1}|\sqrt{T}c_2\|u_n\|^2 + \frac{2^{-r^-}(T+1)^{\frac{2-r^-}{2}}B_1}{r^+}\|u_n\|^{r^-} \right) \\
&\leq C_1\overline{\alpha}\left(\sqrt{T+1}\|u_n\| + \frac{(T+1)\|u_n\|^{p^+}}{p^-} + \frac{T+1}{p^-} \right) + \frac{c_{q^+}(T+1)\|u_n\|^{q^+}}{q^-} \\
&\quad - \lambda \left(-|\underline{A_1}|\sqrt{T}c_2\|u_n\|^2 + \frac{2^{-r^-}(T+1)^{\frac{2-r^-}{2}}B_1}{r^+}\|u_n\|^{r^-} \right).
\end{aligned}$$

Since $r^- > \max(p^+, q^+)$ then the sequence $\{u_n\}$ is bounded. $\qquad \square$

In what follow, we apply the mountain pass geometry Lemma.

Theorem 4.2. *Let* $r^- > \max(p^+, q^+)$ *and the condition* (2.7) *holds. Then for* $\lambda \in (0, \lambda_1)$, *the problem* (4.1) *has at least one weak nontrivial solution.*

Proof. We will refer to the proof of theorem 3.2.
Let

$$\Omega = \{u \in X : \|u\| \leq \beta\}$$

with $\beta \in (0, 1)$.

For $u \in \Omega$

$$J_\lambda(u) \geq \frac{\alpha}{p^+} \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} - \lambda \left(\frac{\overline{B_2}}{r^-} \sum_{k=1}^{T} |u(k)|^{r(k)} + |\underline{A_2}| \sum_{k=1}^{T} |u(k)| \right).$$

For all $\lambda \in (0, \lambda_1)$ we obtain $\quad \inf\limits_{u \in \partial\Omega} J_\lambda(u) > 0.$

Let $u_t \in X$ be defined as follows : $\quad u_t(k) = t$ for $\quad k \in \mathbb{N}[1, T]$ and $\quad u_t(0) = u_t(T+1) = 0.$

For $t > 1$ we have

$$J_\lambda(u) \leq 4 C_1 \overline{\alpha} t^{p^+} + T t^{q^+} - \lambda T \left(-|\underline{A_1}|t + \frac{B_1}{r^+} t^{r^-} \right).$$

Since $r^- > \max(p^+, q^+)$, $\lim\limits_{t \to +\infty} J_\lambda(u_t) = -\infty$; then it exist t_1 such that

$$u_{t_1} \in X \backslash \Omega \quad J_\lambda(u_{t_1}) < \min_{u \in \partial\Omega} J_\lambda(u).$$

The functional $J_\lambda \in C^1(X, \mathbb{R})$, and the assumptions of Lemma 2.2 are verified, then the problem (4.1) has at least one weak nontrivial solution. $\qquad \square$

We apply Ekeland's variational principle with $\min(p^-, q^-) > r^-$; we will use the result of case $p^- > r^-$.

For $\lambda \in (0, \lambda_1)$

$$\inf_{u \in \partial\Omega} J_\lambda(u) > 0.$$

For $t \in (0, \beta)$, assume that

$$t < {}^{\left(\min(p^-, q^-) - r^- \right)} \sqrt{ \frac{\lambda \left(-\max(\beta_1, \beta_2)|\underline{A_1}| + \frac{B_1}{r^+} \right)}{2 C_1 \overline{\alpha} \left(\alpha_2 + \frac{1}{p^-} + \frac{1}{2 C_1 \overline{\alpha} q^-} \right)} }$$

we choose $k_0 \in \mathbb{N}[1, T]$ such that $r(k_0) = r^-$.

Let $u_0 \in X$ be a function such that $u_0(k_0) = t$ and $u_0(k) = 0$, for any $k \in \mathbb{N}[1, T] \backslash \{k_0\}$. We obtain

$$J_\lambda(u_0) \leq 2 C_1 \overline{\alpha} \left(t + \frac{t^{p^-}}{p^-} \right) + \frac{t^{q^-}}{q^-} - \lambda \left(-|\underline{A_1}|t + \frac{B_1}{r^+} t^{r^-} \right).$$

There exists $\alpha_2, \beta_2 > 1$ such that $\alpha_2 t^{\min(p^-, q^-)} \geq t$ and $\beta_2 t^{r^-} > t$.

We have

$$J_\lambda(u_0) \leq 2 C_1 \overline{\alpha} t^{\min(p^-, q^-)} \left(\alpha_2 + \frac{1}{p^-} + \frac{1}{2 C_1 \overline{\alpha} q^-} \right) - \lambda \left(-\max(\beta_1, \beta_2)|\underline{A_1}| + \frac{B_1}{r^+} \right) t^{r^-} < 0.$$

Thus, $J_\lambda(u_0) < 0$ for $u_0 \in Int(\Omega)$.

By the same reasoning we prove that the problem (4.1) has at least one weak nontrivial

solution.

Now we will prove that problem (4.1) has at least two weak nontrivial solution.

In the case of multiple solutions we will use the Theorem 3.3 and the same sets defined previously.

Indeed, let $u_0 \in X$ a local minimizer of J_λ on $\Omega = \Omega_\delta \cap \Omega_\nu$. If $u_0 \in Int(\Omega)$ we have $J_\lambda(u_0) < \min\limits_{u \in \partial\Omega_\delta} J_\lambda(u)$.

Assume that $u_0 \in \partial\Omega_\delta$, we have $\sigma_2 = 0$, taking $v = u_0$ and $\sigma_0 = 1$ we get

$$\sum_{k=1}^{T+1} \alpha(k-1)a(k-1,\Delta u_0(k-1))\Delta u_0(k-1) + \sum_{k=1}^{T} |u_0|^{q(k)} + \sigma_1 \|u_0\|^2 = \lambda \sum_{k=1}^{T} f(k,u_0(k))u_0(k).$$

By the same reasoning we prove that problem (4.1) has at least two weak nontrivial solutions.

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