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NONLINEAR ELLIPTIC ANISOTROPIC PROBLEM WITH NON-LOCAL BOUNDARY CONDITIONS AND L^1-DATA

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Abstract. We study a nonlinear anisotropic elliptic problem with non-local boundary conditions and L^1-data. We prove an existence and uniqueness result of entropy solution.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ such that $\partial\Omega$ is Lipschitz and $\partial\Omega = \Gamma_D \cup \Gamma_{Ne}$ with $\Gamma_D \cap \Gamma_{Ne} = \emptyset$. Our aim is to study the following problem.

$$(1.1)\qquad P(\rho, f, d)\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \frac{\partial}{\partial x_i} u) + |u|^{P_M(x)-2}u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \left.\begin{array}{l} \rho(u) + \displaystyle\sum_{i=1}^{N} \int_{\Gamma_{Ne}} a_i(x, \frac{\partial}{\partial x_i} u)\eta_i = d \\ u \equiv constant \end{array}\right\} & \text{on } \Gamma_{Ne}, \end{cases}$$

where the right-hand side $f \in L^1(\Omega)$ and $\eta_i,\ i \in \{1, ..., N\}$ are the components of the outer normal unit vector. For any $\Omega \subset \mathbb{R}^N$, we set

$$(1.2)\qquad C_+(\bar{\Omega}) = \{h \in C(\bar{\Omega}) : \inf_{x\in\Omega} h(x) > 1\},$$

and we denote

$$(1.3)\qquad h^+ = \sup_{x\in\Omega} h(x),\ h^- = \inf_{x\in\Omega} h(x).$$

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For the exponents, $\vec{p}(.) : \bar{\Omega} \to \mathbb{R}^N$, $\vec{p}(.) = (p_1(.), ..., p_N(.))$ with $p_i \in C_+(\bar{\Omega})$ for every $i \in \{1, ..., N\}$ and for all $x \in \bar{\Omega}$. We put $p_M(x) = \max\{p_1(x), ..., p_N(x)\}$ and $p_m(x) = \min\{p_1(x), ..., p_N(x)\}$.

We assume that for $i = 1, ..., N$, the function $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$ is Carathéodory and satisfies the following conditions:

- (H_1): $a_i(x, \xi)$ is the continuous derivative with respect to ξ of the mapping $A_i = A_i(x, \xi)$, that is, $a_i(x, \xi) = \frac{\partial}{\partial \xi} A_i(x, \xi)$ such that the following equality holds.

$$A_i(x, 0) = 0, \tag{1.4}$$

  for almost every $x \in \Omega$.

- (H_2) : There exists a positive constant C_1 such that

$$|a_i(x, \xi)| \leq C_1(j_i(x) + |\xi|^{p_i(x)-1}), \tag{1.5}$$

  for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$, where j_i is a non-negative function in $L^{p'_i(.)}(\Omega)$, with $\frac{1}{p_i(x)} + \frac{1}{p'_i(x)} = 1$.

- (H_3) : there exists a positive constant C_2 such that

$$(a_i(x, \xi) - a_i(x, \eta)).(\xi - \eta) \geq \begin{cases} C_2|\xi - \eta|^{p_i(x)} & \text{if } |\xi - \eta| \geq 1, \\ C_2|\xi - \eta|^{p_i^-} & \text{if } |\xi - \eta| < 1, \end{cases} \tag{1.6}$$

  for almost every $x \in \Omega$ and for every $\xi, \eta \in \mathbb{R}^N$, with $\xi \neq \eta$.

- (H_4) : For almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$,

$$|\xi|^{p_i(x)} \leq a_i(x, \xi).\xi \leq p_i(x) A_i(x, \xi). \tag{1.7}$$

- (H_5) : We also assume that the variable exponents $p_i(.) : \bar{\Omega} \to [2, N)$ are continuous functions for all $i = 1, ..., N$ such that

$$\frac{\bar{p}(N-1)}{N(\bar{p}-1)} < p_i^- < \frac{\bar{p}(N-1)}{N-\bar{p}}, \ \sum_{i=1}^{N} \frac{1}{p_i^-} > 1 \text{ and } \frac{p_i^+ - p_i^- - 1}{p_i^-} < \frac{\bar{p}-N}{\bar{p}(N-1)}, \tag{1.8}$$

  where $\frac{1}{\bar{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i^-}$.

We put for all $x \in \partial\Omega$,

$$p^{\partial}(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

We introduce the numbers

$$q = \frac{N(\bar{p}-1)}{N-1}, \ q^* = \frac{Nq}{N-q} = \frac{N(\bar{p}-1)}{N-\bar{p}}. \tag{1.9}$$

Non-local boundary value problems of various kinds for partial differential equations are of great interest by now in several fields of application. In a typical non-local problem, the partial differential equation (resp. boundary conditions) for an unknown function u at any point in a domain Ω involves not only the local behavior of u in a neighborhood of that point but also the non-local behavior of u elsewhere in Ω. For example, at any point in Ω the partial differential equation and/or the boundary conditions may contains integrals of the unknown u over parts of Ω, values of u elsewhere in D or, generally speaking, some non-local operator on u. Beside the mathematical interest of nonlocal conditions, it seems that this type of boundary condition appears in petroleum engineering model for well modeling in a $3D$ stratified petroleum reservoir with arbitrary geometry (see [5] and [6]). All papers on problems like (1.1) considered cases of generally boundary value condition. Indeed, in [8], Bonzi et al. studied the following problems:

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i \left(x, \frac{\partial}{\partial x_i} u \right) + |u|^{P_M(x)-2} u = f & \text{in } \Omega \\ \sum_{i=1}^{N} a_i \left(x, \frac{\partial}{\partial x_i} u) \eta_i \right) = -|u|^{r(x)-2} u & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

which correspond to the Robin type boundary condition. The authors used minimization techniques used in [4] to prove the existence and uniqueness of entropy solution. By the same techniques, Koné and al. proved the existence and uniqueness of entropy solution for the following problem:

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i (x, \frac{\partial}{\partial x_i} u) + |u|^{P_M(x)-2} u = f & \text{in } \Omega \\ \sum_{i=1}^{N} a_i \left(x, \frac{\partial}{\partial x_i} u) \eta_i \right) + \lambda u = g & \text{on } \partial\Omega, \end{cases} \tag{1.11}$$

which correspond to the Fourier type boundary condition.

2. Preliminary

This part is related to anisotropic Lebesgue and Sobolev spaces with variable exponent and some of their properties.

Given a measurable function $p(.) : \Omega \to [1, \infty)$. We define the Lebesgue space with variable exponent $L^{p(.)}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)} dx$$

is finite.

If the exponent is bounded, i.e., if $p_+ < \infty$, then the expression

$$|u|_{p(.)} := \inf \left\{ \lambda > 0 : \rho_{p(.)}(\frac{u}{\lambda}) \leq 1 \right\}$$

defines a norm in $L^{p(.)}(\Omega)$, called the Luxembourg norm. The space $(L^{p(.)}(\Omega), |.|_{p(.)})$ is a separable Banach space. Then, $L^{p(.)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p'(.)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, for all $x \in \Omega$.

Finally, we have the Hölder type inequality.

Proposition 2.1. *(see [9])*

(i) *For any $u \in L^{p(.)}(\Omega)$ and $v \in L^{p'(.)}(\Omega)$, we have*

$$\left| \int_\Omega uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(.)} |v|_{p'(.)}.$$

(ii) *If $p_1, p_2 \in C_+(\bar{\Omega})$, $p_1(x) \leq p_2(x)$ for any $x \in \bar{\Omega}$, then $L^{p_2(.)}(\Omega) \hookrightarrow L^{p_1(.)}(\Omega)$ and the imbedding is continuous.*

We have the following properties (see [9]) on the modular $\rho_{p(.)}$.
If $u \in L^{p(.)}(\Omega)$ and $p < \infty$, then

$$|u|_{p(.)} < 1 \Rightarrow |u|_{p(.)}^{p^+} \leq \rho_{p(.)}(u) \leq |u|_{p(.)}^{p^-}, \tag{2.1}$$

$$|u|_{p(.)} > 1 \Rightarrow |u|_{p(.)}^{p^-} \leq \rho_{p(.)}(u) \leq |u|_{p(.)}^{p^+}, \tag{2.2}$$

$$|u|_{p(.)} < 1 (= 1; > 1) \Rightarrow \rho_{p(.)}(u) < 1 (= 1; > 1), \tag{2.3}$$

and

$$|u|_{p(.)} \to 0 (\, |u|_{p(.)} \to \infty) \Leftrightarrow \rho_{p(.)}(u) \to 0 (\, \rho_{p(.)}(u) \to \infty). \tag{2.4}$$

If in addition, $(u_n)_{n\in\mathbb{N}} \subset L^{p(.)}(\Omega)$, then $\lim_{n\to\infty} |u_n - u|_{p(.)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{p(.)}(u_n - u) = 0 \Leftrightarrow (u_n)_{n\in\mathbb{N}}$ converges to u in measure and $\lim_{n\to\infty} \rho_{p(.)}(u_n) = \rho_{p(.)}(u)$.
We introduce the definition of the isotropic Sobolev space with variable exponent,

$$W^{1,p(.)}(\Omega) := \left\{ u \in L^{p(.)}(\Omega) : |\nabla u| \in L^{p(.)}(\Omega) \right\},$$

which is a Banach space equipped with the norm

$$\|u\|_{1,p(.)} := |u|_{p(.)} + |\nabla u|_{p(.)}.$$

Now, we present the anisotropic Sobolev space with variable exponent which is used for the study of $P(\rho, d, f)$.
The anisotropic variable exponent Sobolev space $W^{1,\vec{p}(.)}(\Omega)$ is defined as follow.

$$W^{1,\vec{p}(.)}(\Omega) := \left\{ u \in L^{p_M(.)}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i(.)}(\Omega), \text{ for all } i \in \{1, ..., N\} \right\}.$$

Endowed with the norm

$$\|u\|_{\vec{p}(.)} := |u|_{p_M(.)} + \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|_{p_i(.)},$$

the space $\left(W^{1,\vec{p}(.)}(\Omega), \|.\|_{\vec{p}(.)}\right)$ is a reflexive Banach space (see [10], Theorem 2.1 and Theorem 2.2).
As consequence, we have the following.

Theorem 2.1. *(see [10]) Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$ $(N \geq 3)$ be a bounded open set and for all $i \in \{1, ..., N\}$, $p_i \in L^\infty(\Omega)$, $p_i(x) \geq 1$ a.e. in Ω. Then, for any $r \in L^\infty(\Omega)$ with $r(x) \geq 1$ a.e. in Ω such that*

$$ess \inf_{x\in\Omega} (p_M(x) - r(x)) > 0,$$

we have the compact embedding

$$W^{1,\vec{p}(.)}(\Omega) \hookrightarrow L^{r(.)}(\Omega).$$

We also need the following trace theorem due to [4].

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$ $(N \geq 2)$ be a bounded open set with smooth boundary and let $\vec{p}(.) \in C(\bar{\Omega})$ satisfy the condition*

$$1 \leq r(x) < \min_{x\in\partial\Omega}\{p_1^{\partial}(x), ..., p_N^{\partial}(x)\}, \ \forall x \in \partial\Omega. \tag{2.5}$$

Then, there is a compact boundary trace embedding

$$W^{1,\vec{p}(.)}(\Omega) \hookrightarrow L^{r(.)}(\partial\Omega).$$

Let us introduce the following notation:

$$\vec{p}_- = (p_1^-, ..., p_N^-).$$

Finally, in this paper, we will use the Marcinkiewicz spaces $\mathcal{M}^q(\Omega)$ $(1 < q < \infty)$ with constant exponent. Note that the Marcinkiewicz spaces $\mathcal{M}^{q(.)}(\Omega)$ in the variable exponent setting was introduced for the first time by Sanchon and Urbano (see [11]).

Marcinkiewicz spaces $\mathcal{M}^q(\Omega)$ $(1 < q < \infty)$ contain all measurable function $h : \Omega \to \mathbb{R}$ for which the distribution function

$$\lambda_h(\gamma) := \text{meas}(\{x \in \Omega : |h(x)| > \gamma\}), \ \gamma \geq 0,$$

satisfies an estimate of the form $\lambda_h(\gamma) \leq C\gamma^{-q}$, for some finite constant $C > 0$.

The space $\mathcal{M}^q(\Omega)$ is a Banach space under the norm

$$\|h\|^*_{\mathcal{M}^q(\Omega)} = \sup_{t>0} t^{\frac{1}{q}} \left(\frac{1}{t}\int_0^t h^*(s)ds\right),$$

where h^* denotes the nonincreasing rearrangement of h:

$$h^*(t) := \inf\left\{C : \lambda_h(\gamma) \leq C\gamma^{-q}, \ \forall\gamma > 0\right\},$$

which is equivalent to the norm $\|h\|^*_{\mathcal{M}^q(\Omega)}$ (see [1]).

We need the following Lemma (see [2], Lemma A-2).

Lemma 2.1. *Let $1 \leq q < p < \infty$. Then, for every measurable function u on Ω,*

(i) $\dfrac{(p-1)^p}{p^{p+1}}\|u\|^p_{\mathcal{M}^p(\Omega)} \leq \sup_{\lambda>0}\{\lambda^p meas[x \in \Omega : |u| > \lambda]\} \leq \|u\|^p_{\mathcal{M}^p(\Omega)}$.

Moreover,

(ii) $\displaystyle\int_K |u|^q dx \leq \frac{p}{p-q}(\frac{p}{q})^{\frac{q}{p}}\|u\|^q_{\mathcal{M}^p(\Omega)}(meas(K))^{\frac{p-q}{p}}$, *for every measurable subset $K \subset \Omega$.*

In particular, $\mathcal{M}^p(\Omega) \subset L^q_{loc}(\Omega)$, with continuous embedding and $u \in \mathcal{M}^p(\Omega)$ implies $|u|^q \in \mathcal{M}^{\frac{p}{q}}(\Omega)$.

The following result is due to Troisi (see [12]).

Theorem 2.3. *Let $p_1, ..., p_N \in [1, \infty)$, $\vec{p} = (p_1, ..., p_N)$; $g \in W^{1,\vec{p}}(\Omega)$, and let*

$$\begin{cases} q = \bar{p}^* & \text{if } \bar{p}^* < N, \\ q \in [1, \infty) & \text{if } \bar{p}^* \geq N; \end{cases} \tag{2.6}$$

where $p^* = \dfrac{N}{\sum_{i=1}^{N}\frac{1}{p_i} - 1}$, $\sum_{i=1}^{N}\frac{1}{p_i} > 1$ *and* $\bar{p}^* = \dfrac{N\bar{p}}{N-\bar{p}}$.
Then, there exists a constant $C > 0$ *depending on* $N, p_1, ..., p_N$ *if* $\bar{p} < N$ *and also on* q *and* $meas(\Omega)$ *if* $\bar{p} \geq N$ *such that*

$$\|g\|_{L^q(\Omega)} \leq c\prod_{i=1}^{N}\left[\|g\|_{L^{p_M}(\Omega)} + \|\frac{\partial g}{\partial x_i}\|_{L^{p_i}(\Omega)}\right]^{\frac{1}{N}}, \tag{2.7}$$

where $p_M = \max\{p_1, ..., p_N\}$ *and* $\frac{1}{\bar{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}$. *In particular, if* $u \in W_0^{1,\vec{p}}(\Omega)$, *we have*

$$\|g\|_{L^q(\Omega)} \leq c\prod_{i=1}^{N}\left[\left\|\frac{\partial g}{\partial x_i}\right\|_{L^{p_i}(\Omega)}\right]^{\frac{1}{N}}. \tag{2.8}$$

In the sequel, we consider the following spaces.

$$W_D^{1,\vec{p}(.)}(\Omega) = \{\xi \in W^{1,\vec{p}(.)}(\Omega)\ :\ \xi = 0 \text{ on } \Gamma_D\}$$

and

$$W_{Ne}^{1,\vec{p}(.)}(\Omega) = \{\xi \in W_D^{1,\vec{p}(.)}(\Omega)\ :\ \xi \equiv \text{constant on } \Gamma_{Ne}\}.$$

$$\mathcal{T}_D^{1,\vec{p}(.)}(\Omega) = \{\xi \text{ measurable on } \Omega \text{ such that } \forall k > 0,\ T_k(\xi) \in\ W_D^{1,\vec{p}(.)}(\Omega)\}$$

and

$$\mathcal{T}_{Ne}^{1,\vec{p}(.)}(\Omega) = \{\xi \text{ measurable on } \Omega \text{ such that } \forall k > 0,\ T_k(\xi) \in\ W_{Ne}^{1,\vec{p}(.)}(\Omega)\},$$

where

$$T_k(s) = \begin{cases} k & \text{if } s > k, \\ s & \text{if } |s| \leq k, \\ -k & \text{if } s < -k. \end{cases}$$

For any $v \in W_{Ne}^{1,\vec{p}(.)}(\Omega)$, we set $v_N = v_{Ne} := v|_{\Gamma_{Ne}}$.

Definition 2.1. *A measurable function* $u : \Omega \to \mathbb{R}$ *is an entropy solution of* $P(\rho, f, d)$ *if* $u \in \mathcal{T}_{Ne}^{1,\vec{p}(.)}(\Omega)$ *and*

$$\begin{cases} \displaystyle\int_\Omega \left(\sum_{i=1}^{N} a_i(x, \frac{\partial}{\partial x_i}u)\frac{\partial}{\partial x_i}T_k(u-\xi)\right)dx + \int_\Omega |u|^{P_M(x)-2}uT_k(u-\xi)dx \leq \\ \displaystyle\int_\Omega fT_k(u-\xi)dx + (d - \rho(u_{Ne}))T_k(u_{Ne}-\xi), \end{cases} \tag{2.9}$$

for all $\xi \in W_{Ne}^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$.

Our main result in this paper is the following theorem.

Theorem 2.4. *For any* $(f, d) \in L^1(\Omega) \times \mathbb{R}$, *the problem* $P(\rho, f, d)$ *admits a unique entropy solution* u.

Before proving Theorem 2.4, we study an auxiliary problem from which, we deduce useful a priori estimates. The paper is organized as follows. In Section 3, we study an approximated problem and in Section 4, we prove by using the results of the Section 3, the existence and uniqueness of entropy solution of problem $P(\rho, f, d)$.

3. The approximated problem corresponding to $P(\rho, f, d)$

We define a new bounded domain $\tilde{\Omega}$ in $\mathbb{R}^N$ as follow.
We fix $\theta > 0$ and we set $\tilde{\Omega} = \Omega \cup \{x \in \mathbb{R}^N / dist(x, \Gamma_{Ne}) < \theta\}$. Then, $\partial\tilde{\Omega} = \Gamma_D \cup \tilde{\Gamma}_{Ne}$ is Lipschitz with $\Gamma_D \cap \tilde{\Gamma}_{Ne} = \emptyset$.

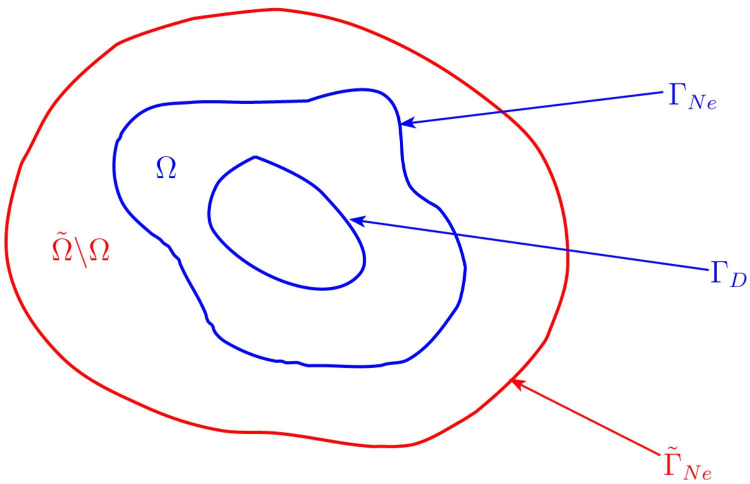

Figure 1. Figure 1: Domains representation

Let us consider $\tilde{a}_i(x, \xi)$ (to be defined later) Carathéodory and satisfying (1.4), (1.5), (1.6) and (1.7), for all $x \in \tilde{\Omega}$.
We also consider a function $\tilde{d}$ in $L^1(\tilde{\Gamma}_{Ne})$ such that

$$\int_{\tilde{\Gamma}_{Ne}} \tilde{d} d\sigma = d. \tag{3.1}$$

For any $\epsilon > 0$, we set $f_\epsilon = T_{\frac{1}{\epsilon}}(f)$ and $\tilde{f}_\epsilon = f_\epsilon \chi_\Omega(x)$, $\tilde{d}_\epsilon = T_{\frac{1}{\epsilon}}(\tilde{d})$ and we consider the problem

$$P_\epsilon(\tilde{\rho}, \tilde{f}_\epsilon, \tilde{d}_\epsilon) \begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \tilde{a}_i(x, \frac{\partial}{\partial x_i} u_\epsilon) + |u_\epsilon|^{P_M(x)-2} u_\epsilon \chi_\Omega(x) = \tilde{f}_\epsilon & \text{in } \tilde{\Omega} \\ u_\epsilon = 0 & \text{on } \Gamma_D \\ \tilde{\rho}(u_\epsilon) + \sum_{i=1}^{N} \tilde{a}_i(x, \frac{\partial}{\partial x_i} u_\epsilon)\eta_i = \tilde{d}_\epsilon & \text{on } \tilde{\Gamma}_{Ne}, \end{cases} \tag{3.2}$$

where the function $\tilde{\rho}$ is defined as follow.

- $\tilde{\rho}(s) = \dfrac{1}{|\tilde{\Gamma}_{Ne}|}\rho(s)$, where $|\tilde{\Gamma}_{Ne}|$ denotes the Hausdorff measure of $\tilde{\Gamma}_{Ne}$.

We obviously have $\forall \epsilon > 0$, $\tilde{f}_\epsilon \in L^\infty(\tilde{\Omega})$, $\tilde{d}_\epsilon \in L^\infty(\tilde{\Gamma}_{Ne})$.
The following definition gives the notion of solution for the problem $P_\epsilon(\tilde{\rho}, \tilde{f}_\epsilon, \tilde{d}_\epsilon)$.

Definition 3.1. *A measurable function $u_\epsilon : \tilde{\Omega} \to \mathbb{R}$ is a solution to problem $P_\epsilon(\tilde{\rho}, \tilde{f}_\epsilon, \tilde{d}_\epsilon)$ if $u_\epsilon \in W_D^{1,\vec{p}(\cdot)}(\tilde{\Omega})$ and*

$$\int_{\tilde{\Omega}} \sum_{i=1}^{N} \tilde{a}_i(x, \frac{\partial}{\partial x_i} u_\epsilon) \frac{\partial}{\partial x_i} \tilde{\xi} dx + \int_\Omega |u_\epsilon|^{P_M(x)-2} u_\epsilon \tilde{\xi} dx = \int_\Omega f_\epsilon \tilde{\xi} dx + \int_{\tilde{\Gamma}_{Ne}} (\tilde{d}_\epsilon - \tilde{\rho}(u_\epsilon)) \tilde{\xi} d\sigma, \tag{3.3}$$

for any $\tilde{\xi} \in W_D^{1,\vec{p}(\cdot)}(\tilde{\Omega}) \cap L^\infty(\Omega)$.

Theorem 3.1. *The problem $P_\epsilon(\tilde{\rho}, \tilde{f}_\epsilon, \tilde{d}_\epsilon)$ admits at least one solution in the sense of Definition 3.1.*

Step 1: Approximate problem. we study an existence result to the following problem. For any $k > 0$ we consider

$$
(3.4)\qquad P_{\epsilon,k}(\tilde{\rho}, \tilde{f}_\epsilon, \tilde{d}_\epsilon)\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\tilde{a}_i(x, \frac{\partial}{\partial x_i}u_{\epsilon,k}) + T_k(b(u_{\epsilon,k}))\chi_\Omega(x) = \tilde{f}_\epsilon & \text{in } \tilde{\Omega} \\ u_{\epsilon,k} = 0 & \text{on } \Gamma_D \\ T_k(\tilde{\rho}(u_{\epsilon,k})) + \sum_{i=1}^{N} \tilde{a}_i(x, \frac{\partial}{\partial x_i}u_{\epsilon,k})\eta_i = \tilde{d}_\epsilon & \text{on } \tilde{\Gamma}_{Ne}, \end{cases}
$$

where $b(u) = |u|^{P_M(x)-2}u$.

We have to prove that $P_{\epsilon,k}(\tilde{\rho}, \tilde{f}_\epsilon, \tilde{d}_\epsilon)$ admits at least one solution in the following sense.

$$
(3.5)\qquad \begin{cases} u_{\epsilon,k} \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \text{ and for all } \tilde{\xi} \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}), \\ \int_{\tilde{\Omega}} \sum_{i=1}^{N} \tilde{a}_i(x, \frac{\partial}{\partial x_i}u_{\epsilon,k})\frac{\partial}{\partial x_i}\tilde{\xi}dx + \int_\Omega T_k(b(u_{\epsilon,k}))\tilde{\xi}dx = \int_\Omega f_\epsilon\tilde{\xi}dx + \int_{\tilde{\Gamma}_{Ne}} (\tilde{d}_\epsilon - T_k(\tilde{\rho}(u_{\epsilon,k}))\tilde{\xi}d\sigma. \end{cases}
$$

For any $k > 0$, let us introduce the operator $\Lambda_k : W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \to (W_D^{1,\vec{p}(.)}(\tilde{\Omega}))'$ such that for any $(u, v) \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \times W_D^{1,\vec{p}(.)}(\tilde{\Omega})$,

$$
(3.6)\qquad \langle \Lambda_k(u), v\rangle = \int_{\tilde{\Omega}} \left(\sum_{i=1}^{N} \tilde{a}_i(x, \frac{\partial}{\partial x_i}u)\frac{\partial}{\partial x_i}v \right) dx + \int_\Omega T_k(b(u))vdx + \int_{\tilde{\Gamma}_{Ne}} T_k(\tilde{\rho}(u))vd\sigma.
$$

We need to prove that for any $k > 0$, the operator Λ_k is bounded, coercive, of type M and therefore, surjective.

(i) Boundedness of Λ_k. Let $(u, v) \in F \times W_D^{1,\vec{p}(.)}(\tilde{\Omega})$ with F a bounded subset of $W_D^{1,\vec{p}(.)}(\tilde{\Omega})$. We have

$$
\begin{cases} |\langle \Lambda_k(u), v\rangle| \le \sum_{i=1}^{N} \left(\int_{\tilde{\Omega}} \left|\tilde{a}_i(x, \frac{\partial}{\partial x_i}u)\right| \left|\frac{\partial}{\partial x_i}v\right| dx \right) + \int_{\tilde{\Omega}} |T_k(b(u))||v|dx \\ + \int_{\tilde{\Gamma}_{Ne}} |T_k(\tilde{\rho}(u))||v|d\sigma \\ = I_1 + I_2 + I_3, \end{cases}
$$

where we denote by I_1, I_2 and I_3 the three terms on the right hand side of the first inequality.

By $(H_2)$and the Hölder type inequality, we have

$$
\begin{cases} I_1 \le C_1 \sum_{i=1}^{N} \left(\int_{\tilde{\Omega}} |j_i(x)| \left|\frac{\partial}{\partial x_i}v\right| dx + \int_{\tilde{\Omega}} \left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)-1} \left|\frac{\partial}{\partial x_i}v\right| dx \right) \\ \le C_1 \sum_{i=1}^{N} \left(\frac{1}{p_i'^-} + \frac{1}{p_i^-} \right) |j_i|_{p_i'(.)} \left|\frac{\partial}{\partial x_i}v\right|_{p_i(.)} + \sum_{i=1}^{N} \left(\frac{1}{p_i'^-} + \frac{1}{p_i^-} \right) \left|\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)-1}\right|_{p_i'(.)} \left|\frac{\partial}{\partial x_i}v\right|_{p_i(.)}. \end{cases}
$$

As $u \in F, \forall\, i \in \{1, ..., N\}$, there exists a constant $M > 0$ such that

$$
\sum_{i=1}^{N} \left|\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)-1}\right|_{p_i'(.)} < M;
$$

so

$$
\left|\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)-1}\right|_{p_i'(.)} < M,\ \forall\, i \in \{1, ..., N\}.
$$

Let $C_4 = \max\limits_{i=1,\ldots,N}\left\{\left|\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)-1}\right|_{p_i'(.)}\right\}$.
As $j_i \in L^{p_i'(.)}(\tilde{\Omega})$, we have

$$I_1 \leq C_5(C_1, p_i^-, (p_i')^-, C_3(j_i))\sum_1^N\left|\frac{\partial}{\partial x_i}v\right|_{p_i(.)} + C_6(C_1, p_i^-, (p_i')^-, C_4)\sum_{i=1}^N\left|\frac{\partial}{\partial x_i}v\right|_{p_i(.)}.$$

It is easy to see that

$$I_2 \leq k\int_{\tilde{\Omega}}|v|dx.$$

Using Theorem 2.1, we have

$$\|v\|_{L^1(\tilde{\Omega})} \leq C_7\|v\|_{W_D^{1,\vec{p}(.)}(\tilde{\Omega})}.$$

So,

$$I_2 \leq kC_7\|v\|_{W_D^{1,\vec{p}(.)}(\tilde{\Omega})}.$$

Similarly, by using Theorem 2.2, we have

$$I_3 \leq kC_8\|v\|_{W_D^{1,\vec{p}(.)}(\tilde{\Omega})}\Box$$

Therefore, Λ_k maps bounded subsets of $W_D^{1,\vec{p}(.)}(\tilde{\Omega})$ into bounded subsets of $(W_D^{1,\vec{p}(.)}(\tilde{\Omega}))'$. Thus, Λ_k is bounded on $W_D^{1,\vec{p}(.)}(\tilde{\Omega})$.

(ii) Coerciveness of Λ_k. We have to show that for any $k > 0$, $\dfrac{\langle\Lambda_k(u), u\rangle}{\|u\|_{W_D^{1,\vec{p}(.)}(\tilde{\Omega})}} \to \infty$ as $\|u\|_{W_D^{1,\vec{p}(.)}(\tilde{\Omega})} \to \infty$.

For any $u \in W_D^{1,\vec{p}(.)}(\tilde{\Omega})$, we have

$$\langle\Lambda_k(u), u\rangle = \langle\Lambda(u), u\rangle + \int_\Omega T_k(b(u))udx + \int_{\tilde{\Gamma}_{Ne}} T_k(\tilde{\rho}(u))ud\sigma, \tag{3.7}$$

where $\langle\Lambda(u), u\rangle = \sum\limits_{i=1}^N\left(\int_{\tilde{\Omega}}\tilde{a}_i(x, \frac{\partial}{\partial x_i}u)\frac{\partial}{\partial x_i}udx\right)$.
The last two terms on the right-hand side of (3.7) are non-negative by the monotonicity of T_k, b and $\tilde{\rho}$. We can assert that

$$\begin{cases}\langle\Lambda_k(u), u\rangle \geq \langle\Lambda(u), u\rangle\\ \geq \dfrac{1}{N^{p_m^- - 1}}\|u\|^{p_m^-}_{W_D^{1,\vec{p}(.)}(\tilde{\Omega})} - N.\end{cases}$$

Indeed, since $\int_{\tilde{\Omega}}|T_k(b(u))||u|dx + \int_{\tilde{\Gamma}_{Ne}}|T_k(\tilde{\rho}(u))||u|d\sigma \geq 0$, for all $u \in W_D^{1,\vec{p}(.)}(\tilde{\Omega})$, we have

$$\langle\Lambda_k(u), u\rangle \geq \langle\Lambda(u), u\rangle.$$

So,

$$\begin{aligned}\langle\Lambda_k(u), u\rangle &\geq \sum_{i=1}^N\left(\int_{\tilde{\Omega}}\tilde{a}_i(x, \frac{\partial}{\partial x_i}u)\frac{\partial}{\partial x_i}udx\right)\\ &\geq \sum_{i=1}^N\left(\int_{\tilde{\Omega}}\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)}dx\right).\end{aligned}$$

We make the following notations:
$\mathcal{I} = \left\{ i \in \{1, ..., N\} : \left|\frac{\partial}{\partial x_i} u\right|_{p_i(.)} \leq 1 \right\}$ and $\mathcal{J} = \left\{ i \in \{1, ..., N\} : \left|\frac{\partial}{\partial x_i} u\right|_{p_i(.)} > 1 \right\}$.
We have

$$\begin{aligned}
\langle \Lambda_k(u), u \rangle &\geq \sum_{i\in\mathcal{I}} \left(\int_{\tilde{\Omega}} \left|\frac{\partial}{\partial x_i} u\right|^{p_i(x)} dx \right) + \sum_{i\in\mathcal{J}} \left(\int_{\tilde{\Omega}} \left|\frac{\partial}{\partial x_i} u\right|^{p_i(x)} dx \right) \\
&\geq \sum_{i\in\mathcal{I}} \left(\left|\frac{\partial}{\partial x_i} u\right|_{p_i(.)}^{p_i^+} \right) + \sum_{i\in\mathcal{J}} \left(\left|\frac{\partial}{\partial x_i} u\right|_{p_i(.)}^{p_i^-} \right) \\
&\geq \sum_{i\in\mathcal{J}} \left(\left|\frac{\partial}{\partial x_i} u\right|_{p_i(.)}^{p_i^-} \right) \\
&\geq \sum_{i\in\mathcal{J}} \left(\left|\frac{\partial}{\partial x_i} u\right|_{p_i(.)}^{p_m^-} \right) \\
&\geq \sum_{i=1}^{N} \left(\left|\frac{\partial}{\partial x_i} u\right|_{p_i(.)}^{p_m^-} \right) - \sum_{i\in\mathcal{I}} \left(\left|\frac{\partial}{\partial x_i} u\right|_{p_i(.)}^{p_m^-} \right) \\
&\geq \sum_{i=1}^{N} \left(\left|\frac{\partial}{\partial x_i} u\right|_{p_i(.)}^{p_m^-} \right) - N.
\end{aligned}$$

We now use Jensen's inequality on the convex function $Z: \mathbb{R}^+ \to \mathbb{R}^+$, $Z(t) = t^{p_m^-}$, $p_m^- > 1$ to get

$$\begin{cases} \langle \Lambda_k(u), u \rangle \geq \langle \Lambda(u), u \rangle \\ \geq \dfrac{1}{N^{p_m^- - 1}} \|u\|^{p_m^-}_{W_D^{1,\vec{p}(.)}(\tilde{\Omega})} - N. \end{cases}$$

Hence, Λ_k is coercive (as $p_m^- > 1$).

(iii) The operator Λ_k is of type M.

Lemma 3.1. *(Cf [13]) Let $\mathcal{A}$ and $\mathcal{B}$ be two operators. If $\mathcal{A}$ is of type M and $\mathcal{B}$ is monotone and weakly continuous, then $\mathcal{A} + \mathcal{B}$ is of type M.*

Now , we set $\langle \mathcal{A}u, v \rangle := \langle \Lambda(u), v \rangle$ and $\langle \mathcal{B}_k u, v \rangle := \int_\Omega T_k(b(u))v dx + \int_{\tilde{\Gamma}_{Ne}} T_k(\tilde{\rho}(u))v d\sigma$.
Then, for every $k > 0$, we have $\Lambda_k = \mathcal{A} + \mathcal{B}_k$. We now have to show that for every $k > 0$, $\mathcal{B}_k$ is monotone and weakly continuous, because it is well-known that $\mathcal{A}$ is of type M. For the monotonicity of $\mathcal{B}_k$, we have to show that

$$\langle \mathcal{B}_k u - \mathcal{B}_k v, u - v \rangle \geq 0 \text{ for all } (u, v) \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \times W_D^{1,\vec{p}(.)}(\tilde{\Omega}).$$

We have

$$\begin{aligned}
\langle \mathcal{B}_k u - \mathcal{B}_k v, u - v \rangle &= \int_\Omega (T_k(b(u)) - T_k(b(v)))(u - v) dx \\
&+ \int_{\tilde{\Gamma}_{Ne}} (T_k(\tilde{\rho}(u)) - T_k(\tilde{\rho}(v)))(u - v) d\sigma.
\end{aligned}$$

From the monotonicity of b, $\tilde{\rho}$ and the map T_k, we conclude that

$$\langle \mathcal{B}_k u - \mathcal{B}_k v, u - v \rangle \geq 0. \tag{3.8}$$

We need now to prove that for each $k > 0$ the operator $\mathcal{B}_k$ is weakly continuous, that is, for all sequences $(u_n)_{n\in\mathbb{N}} \subset W_D^{1,\vec{p}(.)}(\tilde{\Omega})$ such that $u_n \rightharpoonup u$ in $W_D^{1,\vec{p}(.)}(\tilde{\Omega})$, we have $\mathcal{B}_k u_n \rightharpoonup \mathcal{B}_k u$ as $n \to \infty$.
For all $\phi \in W_D^{1,\vec{p}(.)}(\tilde{\Omega})$, we have

$$\langle \mathcal{B}_k u_n, \phi\rangle := \int_\Omega T_k(b(u_n))\phi dx + \int_{\tilde{\Gamma}_{Ne}} T_k(\tilde{\rho}(u_n))\phi d\sigma. \tag{3.9}$$

Passing to the limit in (3.9) as n goes to ∞ and using the Lebesgue dominated convergence theorem, since $u_n \rightharpoonup u$ in $W_D^{1,\vec{p}(.)}(\tilde{\Omega})$; up to a subsequence, we have $u_n \to u$ in $L^1(\tilde{\Omega})$ and a.e. on $\tilde{\Omega}$. As $|T_k(b(u_n))\phi| \le k|\phi|$ and $\phi \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \hookrightarrow L^1(\tilde{\Omega})$, for the first term on the right-hand side of (3.9), we obtain

$$\lim_{n\to\infty} \int_\Omega T_k(b(u_n))\phi dx = \int_\Omega T_k(b(u))\phi dx. \tag{3.10}$$

Furthermore, since $u_n \rightharpoonup u$ in $W_D^{1,\vec{p}(.)}(\tilde{\Omega})$; up to a subsequence, we have $u_n \to u$ in $L^1(\partial\tilde{\Omega})$ and a.e. on $\partial\tilde{\Omega}$. As $|T_k(\tilde{\rho}(u_n))\phi| \le k|\phi|$ and $\phi \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \hookrightarrow L^1(\partial\tilde{\Omega})$, we deduce by the Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty} \int_{\tilde{\Gamma}_{Ne}} T_k(\tilde{\rho}(u_n))\phi dx = \int_{\tilde{\Gamma}_{Ne}} T_k(\tilde{\rho}(u))\phi dx. \tag{3.11}$$

From (3.10) and (3.11) we conclude that for every $k > 0$, $\mathcal{B}_k(u_n) \to \mathcal{B}_k(u)$ as $n \to \infty$.
The operator $\mathcal{A}$ is type M and as $\mathcal{B}_k$ is monotone and weakly continuous, thanks to Lemma 3.1, we conclude that the operator Λ_k is of type M. Then for any $L \in (W_D^{1,\vec{p}(.)}(\tilde{\Omega}))'$, there exists $u_{\epsilon,k} \in W_D^{1,\vec{p}(.)}(\tilde{\Omega})$, such that $\Lambda_k(u_{\epsilon,k}) = L$.
We now consider $L \in (W_D^{1,\vec{p}(.)}(\tilde{\Omega}))'$ defined by $L(v) = \int_\Omega f_\epsilon v dx + \int_{\tilde{\Gamma}_{Ne}} \tilde{d}_\epsilon v d\sigma$, for $v \in W_D^{1,\vec{p}(.)}(\tilde{\Omega})$ and we obtain (3.5)□

Step 2: A priori estimates.

Lemma 3.2. *Let $u_{\epsilon,k}$ a solution of $P_{\epsilon,k}(\tilde{\rho}, \tilde{f}_\epsilon, \tilde{d}_\epsilon)$. Then*

$$\begin{cases} |\tilde{\rho}(u_{\epsilon,k})| \le k_1 := \max\{\|\tilde{d}\|_\infty, (\tilde{\rho}\circ b^{-1})(\|f_\epsilon\|_\infty)\} \text{ a.e. on } \tilde{\Gamma}_{Ne}, \\ |b(u_{\epsilon,k})| \le k_2 := \max\{|f_\epsilon\|_\infty; (b\circ\rho_0^{-1})(|\tilde{\Gamma}_{Ne}|\|\tilde{d}\|_\infty)\} \text{ a.e. in } \Omega. \end{cases} \tag{3.12}$$

Proof. For any $\tau > 0$, let us introduce the function $H_\tau : \mathbb{R} \to \mathbb{R}$ by

$$H_\tau(s) = \begin{cases} 0 & \text{if } s < 0, \\ \dfrac{s}{\tau} & \text{if } 0 \le s \le \tau, \\ 1 & \text{if } s > \tau. \end{cases}$$

In (3.5) we set $\tilde{\xi} = H_\tau(u_{\epsilon,k} - M)$, where $M > 0$ is to be fixed later. We get

$$\begin{cases} \displaystyle\int_{\tilde{\Omega}} \sum_{i=1}^N \tilde{a}_i(x, \frac{\partial}{\partial x_i}u_{\epsilon,k})\frac{\partial}{\partial x_i}H_\tau(u_{\epsilon,k} - M)dx + \int_\Omega T_k(b(u_{\epsilon,k}))H_\tau(u_{\epsilon,k} - M)dx = \\ \displaystyle\int_\Omega f_\epsilon H_\tau(u_{\epsilon,k} - M)dx + \int_{\tilde{\Gamma}_{Ne}} (\tilde{d} - T_k(\tilde{\rho}(u_\epsilon, k)))H_\tau(u_{\epsilon,k} - M)d\sigma. \end{cases} \tag{3.13}$$

The first term in (3.13) is non-negative. Indeed,

$$\int_{\tilde{\Omega}} \sum_{i=1}^N \tilde{a}_i(x, \frac{\partial}{\partial x_i}u_{\epsilon,k})\frac{\partial}{\partial x_i}H_\tau(u_{\epsilon,k} - M)dx = \frac{1}{\tau}\int_{\{0\le u_{\epsilon,k}-M\le\tau\}} \sum_{i=1}^N \tilde{a}_i(x, \frac{\partial}{\partial x_i}u_{\epsilon,k})\frac{\partial}{\partial x_i}u_{\epsilon,k}dx \ge 0.$$

From (3.13) we obtain

$$\int_\Omega T_k(b(u_{\epsilon,k}))H_\tau(u_{\epsilon,k}-M)dx \le \int_\Omega f_\epsilon H_\tau(u_{\epsilon,k}-M)dx + \int_{\tilde{\Gamma}_{Ne}} (\tilde{d}-T_k(\tilde{\rho}(u_\epsilon,k)))H_\tau(u_{\epsilon,k}-M)d\sigma.$$

Then, one has

$$\begin{cases} \int_\Omega (T_k b(u_{\epsilon,k}) - T_k(b(M)))H_\tau(u_{\epsilon,k}-M)dx + \\ \int_{\tilde{\Gamma}_{Ne}} (T_k(\tilde{\rho}(u_\epsilon,k)) - T_k(\tilde{\rho}(M)))H_\tau(u_{\epsilon,k}-M)dx \le \\ \int_\Omega (f_\epsilon - T_k(b(M)))H_\tau(u_{\epsilon,k}-M)dx + \int_{\tilde{\Gamma}_{Ne}} (\tilde{d}-T_k(\tilde{\rho}(M)))H_\tau(u_{\epsilon,k}-M)d\sigma. \end{cases}$$

Letting τ goes to 0 in the above inequality, we get

$$\begin{cases} \int_\Omega (T_k(b(u_{\epsilon,k})) - T_k(b(M)))^+dx + \int_{\tilde{\Gamma}_{Ne}} (T_k(\tilde{\rho}(u_{\epsilon,k})) - T_k(\tilde{\rho}(M)))^+d\sigma \le \\ \int_\Omega (f_\epsilon - T_k(b(M)))sign_0^+(u_k-M)dx + \int_{\tilde{\Gamma}_{Ne}} (\tilde{d}-T_k(\tilde{\rho}(M)))sign_0^+(u_{\epsilon,k}-M)d\sigma. \end{cases}$$

As $Im(b) = Im(\rho) = \mathbb{R}$, we can fix $M = M_0 = \max\{b^{-1}(\|f_\epsilon\|_\infty), \rho_0^{-1}(|\tilde{\Gamma}_{Ne}|\|\tilde{d}\|_\infty)\}$. From the above inequality we obtain

$$\begin{cases} \int_\Omega (T_k(b(u_{\epsilon,k})) - T_k(b(M_0)))^+dx + \int_{\tilde{\Gamma}_{Ne}} (T_k(\tilde{\rho}(u_{\epsilon,k}) - T_k(\tilde{\rho}(M_0)))^+d\sigma \le \\ \int_\Omega (f_\epsilon - T_k(\|f_\epsilon\|_\infty))sign_0^+(u_{\epsilon,k}-M_0)dx + \int_{\tilde{\Gamma}_{Ne}} (\tilde{d}-T_k(\|\tilde{d}\|_\infty))sign_0^+(u_{\epsilon,k}-M_0)d\sigma. \end{cases}$$

For $k > k_0 := \max\{\|f_\epsilon\|, \|\tilde{d}\|_\infty\}$,
it follows that

$$\int_\Omega (T_k(b(u_{\epsilon,k})) - T_k(b(M_0)))^+dx + \int_{\tilde{\Gamma}_{Ne}} (T_k(\tilde{\rho}(u_{\epsilon,k})) - T_k(\tilde{\rho}(M_0)))^+d\sigma \le 0. \tag{3.14}$$

From (3.14), we deduce that

$$\begin{cases} T_k(\tilde{\rho}(u_{\epsilon,k})) \le T_k(\tilde{\rho}(M_0)) \text{ a.e. on } \tilde{\Gamma}_{Ne}, \\ T_k(b(u_{\epsilon,k})) \le T_k(b(M_0)) \text{ a.e. in } \Omega. \end{cases} \tag{3.15}$$

From (3.15), we deduce that for every $k > k_1 := \max\{\|\tilde{d}\|_\infty, \|f_\epsilon\|_\infty,\ b(M_0),\ \tilde{\rho}(M_0)\}$,

$$\tilde{\rho}(u_{\epsilon,k}) \le \tilde{\rho}(M_0) \text{ a.e. on } \tilde{\Gamma}_{Ne}$$

and

$$b(u_{\epsilon,k}) \le b(M_0) \text{ a.e. in } \Omega.$$

Note that with the choice of M_0 and the fact that $D(\rho) = D(b) = \mathbb{R}$, for every $k > k_1 := \max\{\|\tilde{d}\|_\infty, \|f_\epsilon\|_\infty,\ b(M_0),\ \tilde{\rho}(M_0)\}$, we have

$$\begin{cases} b(u_{\epsilon,k}) \le \max\{\|f_\epsilon\|_\infty, b\circ\rho_0^{-1}(|\tilde{\Gamma}_{Ne}|\|\tilde{d}\|_\infty)\ \} \text{ a.e. in } \Omega, \\ \tilde{\rho}(u_{\epsilon,k}) \le \max\{\|\tilde{d}\|_\infty, (\tilde{\rho}\circ b^{-1})(\|f_\epsilon\|_\infty)\} \text{ a.e. on } \tilde{\Gamma}_{Ne}. \end{cases} \tag{3.16}$$

We need to show that for k large enough,

$$\begin{cases} b(u_{\epsilon,k}) \geq -\max\{\|f_\epsilon\|_\infty, b\circ\rho_0^{-1}(|\tilde{\Gamma}_{Ne}|\|\tilde{d}\|_\infty)\} \text{ a.e. in } \Omega, \\ \tilde{\rho}(u_{\epsilon,k}) \geq -\max\{\|\tilde{d}\|_\infty, (\tilde{\rho}\circ b^{-1})(\|f_\epsilon\|_\infty)\} \text{ a.e. on } \tilde{\Gamma}_{Ne}. \end{cases} \tag{3.17}$$

It is easy to see that if $u_{\epsilon,k}$ is a solution of $P_{\epsilon,k}(\tilde{\rho},\tilde{f}_\epsilon,\tilde{d}_\epsilon)$, then $(-u_{\epsilon,k})$ is a solution of

$$P_k(\hat{\rho},\hat{f}_\epsilon,\hat{d}_\epsilon)\ \begin{cases} -\displaystyle\sum_{i=1}^N \frac{\partial}{\partial x_i}\hat{a}_i(x,\frac{\partial}{\partial x_i}u_{\epsilon,k}) + T_k(\hat{b}(u_{\epsilon,k}))\chi_\Omega(x) = \hat{f}_\epsilon & \text{in } \tilde{\Omega} \\ u_{\epsilon,k} = 0 & \text{on } \Gamma_D \\ T_k(\hat{\rho}(u_{\epsilon,k})) + \displaystyle\sum_{i=1}^N \hat{a}_i(x,\frac{\partial}{\partial x_i}u_{\epsilon,k})\eta_i = \hat{d}_\epsilon & \text{on } \tilde{\Gamma}_{Ne}, \end{cases}$$

where $\hat{a}_i(x,\xi) = -\tilde{a}_i(x,-\xi)$, $\hat{\rho}(s) = -\tilde{\rho}(-s)$, $\hat{b}(x,s) = -b(x,-s)\chi_\Omega(x)$, $\hat{f}_\epsilon = -\tilde{f}_\epsilon$ and $\hat{d} = -\tilde{d}$. Then, for every $k > k_2 := \max\{\|\tilde{d}\|_\infty, \|f_\epsilon\|_\infty,\ -b(-M_0),\ -\tilde{\rho}(-M_0)\}$, we have

$$\begin{cases} -b(u_{\epsilon,k}) \leq \max\{\|f_\epsilon\|_\infty, b\circ\rho_0^{-1}(|\tilde{\Gamma}_{Ne}|\|\tilde{d}\|_\infty)\} \text{ a.e. in } \Omega, \\ -\tilde{\rho}(u_{\epsilon,k}) \leq \max\{\|\tilde{d}\|_\infty, (\tilde{\rho}\circ b^{-1})(\|f_\epsilon\|_\infty)\} \text{ a.e. on } \tilde{\Gamma}_{Ne}. \end{cases}$$

which implies (3.17).
From (3.16) and (3.17), we deduce (3.12). □

Step 3. Convergence. Since $u_{\epsilon,k}$ is a solution of $P_{\epsilon,k}(\tilde{\rho},\tilde{f}_\epsilon,\tilde{d}_\epsilon)$, thanks to Lemma 3.2 and the fact that Ω is bounded, we have $\tilde{\rho}(u_{\epsilon,k}) \in L^1(\tilde{\Gamma}_{Ne})$ and $b(u_{\epsilon,k}) \in L^1(\Omega)$. For $k = 1 + \max(k_1,k_2)$ fixed, by Lemma 3.2, one sees that problem $P_\epsilon(\tilde{\rho},\tilde{f}_\epsilon,\tilde{d}_\epsilon)$ admits at least one solution u_ϵ □

Remark 1. *Using the relation (3.12) and the fact that the functions b and ρ are non-decreasing, it follows that for k large enough, the solution of the problem $P(\tilde{\rho},\tilde{f}_\epsilon,\tilde{d}_\epsilon)$ belongs to $L^\infty(\Omega)\cap L^\infty(\tilde{\Gamma}_{Ne})$ and $|u_\epsilon| \leq c(b,k_1)$ a.e. in Ω and $|u_\epsilon| \leq c(\rho,k_2)$ a.e. on $\tilde{\Gamma}_{Ne}$.*

Now, we set $\tilde{a}_i(x,\xi) = a_i(x,\xi)\chi_\Omega(x) + \frac{1}{\epsilon^{p_i(x)}}|\xi|^{p_i(x)-2}\xi\chi_{\tilde{\Omega}\setminus\Omega}(x)$ for all $(x,\xi) \in \tilde{\Omega}\times\mathbb{R}^N$ and we consider the following problem denoted by $P_\epsilon(\tilde{\rho},\tilde{f}_\epsilon,\tilde{d}_\epsilon)$.

$$\begin{cases} -\displaystyle\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(a_i(x,\frac{\partial}{\partial x_i}u_\epsilon)\chi_\Omega(x) + \frac{1}{\epsilon^{p_i(x)}}\left|\frac{\partial}{\partial x_i}u_\epsilon\right|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon\chi_{\tilde{\Omega}\setminus\Omega}(x)\right) + & \\ |u_\epsilon|^{P_M(x)-2}u_\epsilon\chi_\Omega = \tilde{f}_\epsilon & \text{in } \tilde{\Omega} \\ u_\epsilon = 0 & \text{on } \Gamma_D \\ \tilde{\rho}(u_\epsilon) + \displaystyle\sum_{i=1}^N \tilde{a}_i(x,\frac{\partial}{\partial x_i}u_\epsilon)\eta_i = \tilde{d}_\epsilon & \text{on } \tilde{\Gamma}_{Ne}. \end{cases} \tag{3.18}$$

Thanks to Theorem 3.1, $P_\epsilon(\tilde{\rho},\tilde{f}_\epsilon,\tilde{d}_\epsilon)$ has at least one solution. So, there exists at least one measurable function $u_\epsilon : \tilde{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} \displaystyle\sum_{i=1}^N \int_\Omega a_i(x,\frac{\partial}{\partial x_i}u_\epsilon)\frac{\partial}{\partial x_i}\tilde{\xi}dx + \sum_{i=1}^N \int_{\tilde{\Omega}\setminus\Omega}\left(\frac{1}{\epsilon^{p_i(x)}}\left|\frac{\partial}{\partial x_i}u_\epsilon\right|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon\frac{\partial}{\partial x_i}\tilde{\xi}\right)dx \\ + \displaystyle\int_\Omega |u_\epsilon|^{P_M(x)-2}u_\epsilon\tilde{\xi}dx = \int_\Omega f_\epsilon\tilde{\xi}dx + \int_{\tilde{\Gamma}_{Ne}}(\tilde{d}_\epsilon - \tilde{\rho}(u_\epsilon))\tilde{\xi}d\sigma, \end{cases} \tag{3.19}$$

where $u_\epsilon \in W_D^{1,\vec{p}(.)}(\tilde{\Omega})$ and $\tilde{\xi} \in W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \cap L^\infty(\Omega)$.

The next result gives a priori estimates on the solution u_ϵ of the problem $P_\epsilon(\tilde{\rho}, \tilde{f}_\epsilon, \tilde{d}_\epsilon)$.

Proposition 3.1. *Let u_ϵ be a solution of the problem $P_\epsilon(\tilde{\rho}, \tilde{f}_\epsilon, \tilde{d}_\epsilon)$. Then, the following statements hold.*

(i) $\forall k > 0$,

$$\sum_{i=1}^N \int_\Omega \left(\frac{\partial}{\partial x_i}|T_k(u_\epsilon)|\right)^{p_i(x)} dx + \sum_{i=1}^N \int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon}|\frac{\partial}{\partial x_i}T_k(u_\epsilon)|\right)^{p_i(x)} dx \le k(\|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})} + \|f\|_{L^1(\Omega)});$$

(ii)

$$\int_\Omega |u_\epsilon|^{P_M(x)-1}dx + \int_{\tilde{\Gamma}_{Ne}} |\tilde{\rho}(u_\epsilon)|dx \le (\|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})} + \|f\|_{L^1(\Omega)});$$

(iii) $\forall k > 0$,

$$\sum_{i=1}^N \int_{\tilde{\Omega}} \left(\frac{\partial}{\partial x_i}|T_k(u_\epsilon)|\right)^{p_i(x)} dx \le k(\|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})} + \|f\|_{L^1(\Omega)}).$$

Proof. For any $k > 0$, we set $\tilde{\xi} = T_k(u_\epsilon)$ in (3.19), to get

$$(3.20) \qquad \begin{cases} \displaystyle\sum_{i=1}^N \int_\Omega \left(a_i(x, \frac{\partial}{\partial x_i}u_\epsilon)\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right) dx + \sum_{i=1}^N \int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial}{\partial x_i}u_\epsilon|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right) dx \\ \displaystyle\int_\Omega |u_\epsilon|^{P_M(x)-2}u_\epsilon T_k(u_\epsilon)dx = \int_\Omega f_\epsilon T_k(u_\epsilon)dx + \int_{\tilde{\Gamma}_{Ne}} (\tilde{d}_\epsilon - \tilde{\rho}(u_\epsilon))T_k(u_\epsilon)d\sigma. \end{cases}$$

(i) Obviously, we have

$$\sum_{i=1}^N \int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial}{\partial x_i}u_\epsilon|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right) dx = \sum_{i=1}^N \int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial}{\partial x_i}T_k(u_\epsilon)|^{p_i(x)}\right) dx \ge 0,$$

$\int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}(u_\epsilon)T_k(u_\epsilon)d\sigma \ge 0$ and $\int_\Omega |u_\epsilon|^{P_M(x)-2}u_\epsilon T_k(u_\epsilon) \ge 0$.

Moreover,

$$(3.21) \qquad \begin{cases} \displaystyle\int_\Omega f_\epsilon T_k(u_\epsilon)dx + \int_{\tilde{\Gamma}_{Ne}} \tilde{d}_\epsilon T_k(u_\epsilon)d\sigma & \displaystyle\le k\int_\Omega |f_\epsilon|dx + k\int_{\tilde{\Gamma}_{Ne}} |\tilde{d}_\epsilon|d\sigma \\ & \displaystyle\le k\left(\int_\Omega |f|dx + \int_{\tilde{\Gamma}_{Ne}} |\tilde{d}|d\sigma\right). \end{cases}$$

Using the inequalities above and (1.7),

it follows that

$$(3.22) \qquad \sum_{i=1}^N \int_\Omega \left|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right|^{p_i(x)} dx \le k\left(\int_\Omega |f|dx + \int_{\tilde{\Gamma}_{Ne}} |\tilde{d}|d\sigma\right).$$

In (3.20), as $\displaystyle\sum_{i=1}^N \int_\Omega \left(a_i(x, \frac{\partial}{\partial x_i}u_\epsilon)\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right) dx \ge 0$, $\displaystyle\int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}(u_\epsilon)T_k(u_\epsilon)d\sigma \ge 0$ and

$\displaystyle\int_\Omega |u_\epsilon|^{P_M(x)-2}u_\epsilon T_k(u_\epsilon) \ge 0$;

therefore we obtain

$$(3.23) \qquad \sum_{i=1}^N \int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial}{\partial x_i}T_k(u_\epsilon)|^{p_i(x)}\right) dx \le k\left(\int_\Omega |f|dx + \int_{\tilde{\Gamma}_{Ne}} |\tilde{d}|d\sigma\right).$$

Adding (3.22) and (3.23), we obtain *(i)*.

(ii) The two first terms in (3.20) are non-negative and using (3.21), then we have from (3.20) the following

$$\int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}(u_\epsilon)T_k(u_\epsilon)d\sigma + \int_\Omega |u_\epsilon|^{P_M(x)-2}u_\epsilon T_k(u_\epsilon)dx \le k\left(\int_\Omega |f|dx + \int_{\tilde{\Gamma}_{Ne}} |\tilde{d}|d\sigma\right).$$

We divide the above inequality by $k>0$ and we let k goes to zero, to get

$$\begin{aligned}\int_{\tilde{\Gamma}_{Ne}}\tilde{\rho}(u_\epsilon)sign(u_\epsilon)d\sigma+\int_{\Omega}|u_\epsilon|^{P_M(x)-2}u_\epsilon sign(u_\epsilon)dx &= \int_{\tilde{\Gamma}_{Ne}}|\tilde{\rho}(u_\epsilon)|d\sigma+\int_{\Omega}|u_\epsilon|^{P_M(x)-1}dx\\ &\leq \left(\int_{\Omega}|f|dx+\int_{\tilde{\Gamma}_{Ne}}|\tilde{d}|d\sigma\right).\end{aligned}$$

(iii) For all $k>0$, we have

$$\begin{cases}\sum_{i=1}^{N}\int_{\tilde{\Omega}}\left(\frac{\partial}{\partial x_i}|T_k(u_\epsilon)|^{p_i(x)}\right)dx\leq\sum_{i=1}^{N}\int_{\Omega}\left(\frac{\partial}{\partial x_i}|T_k(u_\epsilon)|^{p_i(x)}\right)dx\\ +\sum_{i=1}^{N}\int_{\tilde{\Omega}\setminus\Omega}\left(\frac{1}{\epsilon}|\frac{\partial}{\partial x_i}T_k(u_\epsilon)|\right)^{p_i(x)}dx,\end{cases}$$

for any $0<\epsilon<1$.

According to (*i*), we deduce that

$$\sum_{i=1}^{N}\int_{\tilde{\Omega}}\left(\frac{\partial}{\partial x_i}|T_k(u_\epsilon)|^{p_i(x)}\right)dx\leq k(\|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}+\|f\|_{L^1(\Omega)})$$

□

We have the following results.

Lemma 3.3. *Assume (1.4)-(1.8), $f\in L^1(\Omega)$ and $\tilde{d}\in L^1(\tilde{\Gamma}_{Ne})$. Let u_ϵ be a solution of the problem $P_\epsilon(\tilde{\rho},\tilde{f}_\epsilon,\tilde{d}_\epsilon)$. If there is a positive constant M such that*

(3.24)
$$\sum_{i=1}^{N}\int_{\{|u_\epsilon|>k\}}k^{q_i(x)}dx\leq M,\ \textit{for all } k>0,$$

then,

$$\sum_{i=1}^{N}\int_{\left\{\left|\frac{\partial}{\partial xi}u_\epsilon\right|^{\alpha_i(.)}>k\right\}}k^{q_i(x)}dx\leq\|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}+\|f\|_{L^1(\Omega)}+M,\ \textit{for all } k>0,$$

where $\alpha_i(.)=\dfrac{p_i(.)}{q_i(.)+1}$, for all $i=1,...,N$.

Proof. Having in mind that by Proposition 3.1-*(iii)*, for all $k>0$,

$$\sum_{i=1}^{N}\int_{\tilde{\Omega}}\left|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right|^{p_i(x)}dx\leq k\left(\int_{\Omega}|f|dx+\int_{\tilde{\Gamma}_{Ne}}|\tilde{d}|d\sigma\right).$$

Therefore, defining $\psi:=T_k(u_\epsilon)/k$, we have , for all $k>0$,

$$\sum_{i=1}^{N}\int_{\tilde{\Omega}}k^{p_i(x)-1}\left|\frac{\partial\psi}{\partial x_i}\right|^{p_i(x)}dx=\sum_{i=1}^{N}\frac{1}{k}\int_{\tilde{\Omega}}\left|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right|^{p_i(x)}dx\leq\left(\int_{\Omega}|f|dx+\int_{\tilde{\Gamma}_{Ne}}|\tilde{d}|d\sigma\right).$$

From the above inequality, the definition of $\alpha_i(.)$ and (3.24), we have

$$\begin{cases} \sum_{i=1}^{N}\int_{\left\{\left|\frac{\partial}{\partial x_i}u_\epsilon\right|^{\alpha_i(.)}>k\right\}} k^{q_i(x)}dx \leq \sum_{i=1}^{N}\int_{\left\{\left|\frac{\partial}{\partial x_i}u_\epsilon\right|^{\alpha_i(.)}>k\right\}\cap\{|u_\epsilon|\leq k\}} k^{q_i(x)}dx \\ +\sum_{i=1}^{N}\int_{\{|u_\epsilon|>k\}} k^{q_i(x)}dx \\ \leq \sum_{i=1}^{N}\int_{\left\{\left|\frac{\partial}{\partial x_i}u_\epsilon\right|^{\alpha_i(.)}>k;\,|u_\epsilon|\leq k\right\}} k^{q_i(x)}\left(\frac{\left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{\alpha_i(.)}}{k}\right)^{\frac{p_i(x)}{\alpha_i(x)}} dx+M \\ \leq \sum_{i=1}^{N}\int_{\left\{\left|\frac{\partial}{\partial xi}u_\epsilon\right|^{\alpha_i(.)}>k;\,|u_\epsilon|\leq k\right\}} k^{q_i(x)-\frac{p_i(x)}{\alpha_i(x)}}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i(x)} dx+M \\ \leq \sum_{i=1}^{N}\int_{\left\{\left|\frac{\partial}{\partial x_i}u_\epsilon\right|^{\alpha_i(.)}>k;\,|u_\epsilon|\leq k\right\}} k^{q_i(x)-\frac{p_i(x)}{\frac{p_i(x)}{q_i(x)+1}}}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i(x)} dx+M \\ \leq \sum_{i=1}^{N}\int_{\left\{\left|\frac{\partial}{\partial x_i}u_\epsilon\right|^{\alpha_i(.)}>k;\,|u_\epsilon|\leq k\right\}} k^{q_i(x)-\frac{p_i(x)(q_i(x)+1)}{p_i(x)}}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i(x)} dx+M \\ \leq \sum_{i=1}^{N}\frac{1}{k}\int_{\left\{\left|\frac{\partial}{\partial x_i}u_\epsilon\right|^{\alpha_i(.)}>k;\,|u_\epsilon|\leq k\right\}} \left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i(x)} dx+M \\ \leq \left(\int_\Omega |f|dx+\int_{\tilde{\Gamma}_{Ne}}|\tilde{d}|d\sigma\right)+M. \end{cases}$$

□

Lemma 3.4. *Assume (1.4)-(1.8), $f\in L^1(\Omega)$ and $\tilde{d}\in L^1(\tilde{\Gamma}_{Ne})$. Let u_ϵ be a solution of the problem $P_\epsilon(\tilde{\rho},\tilde{f}_\epsilon,\tilde{d}_\epsilon)$. There is a positive constant D such that*

$$meas\{|u_\epsilon|>k\}\leq D^{p_m^-}\frac{(1+k)}{k^{p_m^--1}},\ \forall k>0.$$

Proof. Let $k>0$; by using Proposition 3.1-*(iii)*, we have

$$\begin{aligned} \sum_{i=1}^{N}\int_{\tilde{\Omega}}\left|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right|^{p_m^-}dx &\leq \sum_{i=1}^{N}\int_{\left\{\left|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right|>1\right\}}\left|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right|^{p_m^-}dx+Nmeas(\tilde{\Omega}) \\ &\leq \sum_{i=1}^{N}\int_{\tilde{\Omega}}\left|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right|^{p_i(x)}dx+Nmeas(\tilde{\Omega}) \\ &\leq k\left(\int_\Omega|f|dx+\int_{\tilde{\Gamma}_{Ne}}|\tilde{d}|d\sigma\right)+Nmeas(\tilde{\Omega}) \\ &\leq C'(k+1), \end{aligned}$$

with $C'=max\left(\left(\int_\Omega|f|dx+\int_{\tilde{\Gamma}_{Ne}}|\tilde{d}|d\sigma\right);Nmeas(\tilde{\Omega})\right)$.

We can write the above inequality as

$$\sum_{i=1}^{N}\left\|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right\|_{p_m^-}^{p_m^-}\leq C'(1+k) \text{ or } \|T_k(u_\epsilon)\|_{W_D^{1,p_m^-}(\tilde{\Omega})}\leq [C'(1+k)]^{\frac{1}{p_m^-}}.$$

By the Poincaré inequality in constant exponent, we obtain

$$\|T_k(u_\epsilon)\|_{L^{p_m^-}(\tilde{\Omega})}\leq D(1+k)^{\frac{1}{p_m^-}}.$$

The above inequality implies that

$$\int_{\tilde{\Omega}}|T_k(u_\epsilon)|^{p_m^-}dx\leq D^{p_m^-}(1+k),$$

from which we obtain

$$meas\left\{|u_\epsilon|>k\right\}\leq D^{p_m^-}\frac{(1+k)}{k^{p_m^-}},$$

because, as

$$\int_{\tilde{\Omega}}|T_k(u_\epsilon)|^{p_m^-}dx=\int_{\{|u_\epsilon|>k\}}|T_k(u_\epsilon)|^{p_m^-}dx+\int_{\{|u_\epsilon|\leq k\}}|T_k(u_\epsilon)|^{p_m^-}dx,$$

we have

$$\int_{\{|u_\epsilon|>k\}}|T_k(u_\epsilon)|^{p_m^-}dx\leq\int_{\tilde{\Omega}}|T_k(u_\epsilon)|^{p_m^-}dx$$

and

$$k^{p_m^-}meas\left\{|u_\epsilon|>k\right\}\leq\int_{\tilde{\Omega}}|T_k(u_\epsilon)|^{p_m^-}dx\leq D^{p_m^-}(1+k)$$

□

Lemma 3.5. *Assume (1.4)-(1.8), $f\in L^1(\Omega)$ and $\tilde{d}\in L^1(\tilde{\Gamma}_{Ne})$. Let u_ϵ be a solution of the problem $P_\epsilon(\tilde{\rho},\tilde{f}_\epsilon,\tilde{d}_\epsilon)$. There is a positive constant C such that*

$$\sum_{i=1}^{N}\int_{\tilde{\Omega}}\left(\left|\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right|^{p_i^-}\right)dx\leq C(k+1),\ \forall k>0. \tag{3.25}$$

Proof. Let $k>0$, we set $\Omega_1=\left\{|u|\leq k;\left|\frac{\partial}{\partial x_i}u_\epsilon\right|\leq 1\right\}$ and
$\Omega_2=\left\{|u|\leq k;\left|\frac{\partial}{\partial x_i}u_\epsilon\right|>1\right\}$, two subset of $\tilde{\Omega}$.
Using Proposition 3.1-*(iii)*, we have

$$\begin{aligned}\sum_{i=1}^{N}\int_{\tilde{\Omega}}\left(\left|\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right|^{p_i^-}\right)dx &= \sum_{i=1}^{N}\int_{\Omega_1}\left(\left|\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right|^{p_i^-}\right)dx\\ &\quad+\sum_{i=1}^{N}\int_{\Omega_2}\left(\left|\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right|^{p_i^-}\right)dx\\ &\leq Nmeas(\tilde{\Omega})+\sum_{i=1}^{N}\int_{\tilde{\Omega}}\left(\left|\frac{\partial}{\partial x_i}T_k(u_\epsilon)\right|^{p_i(x)}\right)dx\\ &\leq Nmeas(\tilde{\Omega})+k\left(\|f\|_{L^1(\Omega)}+\|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}\right)\leq C(k+1),\end{aligned}$$

with $C=\max\left\{Nmeas(\tilde{\Omega});\left(\|f\|_{L^1(\Omega)}+\|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}\right)\right\}$. □

Lemma 3.6. *Assume (1.4)-(1.8), $f\in L^1(\Omega)$ and $\tilde{d}\in L^1(\tilde{\Gamma}_{Ne})$. Let u_ϵ be a solution of the problem $P_\epsilon(\tilde{\rho},\tilde{f}_\epsilon,\tilde{d}_\epsilon)$. For all $k>0$, there is two constants C_1 and C_2 such that*

(i) $\|u_\epsilon\|_{\mathcal{M}^{q^*}(\tilde{\Omega})} \leq C_1$;

(ii) $\|\dfrac{\partial}{\partial x_i} u_\epsilon\|_{\mathcal{M}^{p_i^- q/p}(\tilde{\Omega})} \leq C_2$.

Proof. *(i)* By Lemma 3.5, we have

$$\sum_{i=1}^{N} \int_{\tilde{\Omega}} \left| \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right|^{p_i^-} dx \leq C(1+k), \ \forall k > 0 \text{ and } i = 1, ..., N.$$

• If $k > 1$, we have

$$\sum_{i=1}^{N} \int_{\tilde{\Omega}} \left| \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right|^{p_i^-} dx \leq C'k,$$

which means $T_k(u_\epsilon) \in W^{1,(p_1^-,...,p_N^-)}(\tilde{\Omega})$.

Using relation (2.8), we deduce that

$$\|T_k(u_\epsilon)\|_{L^{(\bar{p})^*}(\tilde{\Omega}} \leq C_1 \prod_{i=1}^{N} \left\| \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right\|_{L^{p_i^-}(\tilde{\Omega})}^{\frac{1}{N}}.$$

So,

$$\begin{aligned}
\int_{\tilde{\Omega}} |T_k(u_\epsilon)|^{(\bar{p})^*} dx &\leq C \left[\prod_{i=1}^{N} \left(\int_{\tilde{\Omega}} \left| \frac{\partial}{\partial x_i} T_k(u_\epsilon) \right|^{p_i^-} dx \right)^{\frac{1}{Np_i^-}} \right]^{(\bar{p})^*} \\
&\leq C'' \left[\prod_{i=1}^{N} (k)^{\frac{1}{Np_i^-}} \right]^{(\bar{p})^*} \\
&\leq C'' \left[k^{\sum_{i=1}^{N} \frac{1}{Np_i^-}} \right]^{(\bar{p})^*} \\
&\leq C'' k^{\frac{(\bar{p})^*}{\bar{p}}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_{\{|u_\epsilon|>k\}} |T_k(u_\epsilon)|^{(\bar{p})^*} dx &\leq \int_{\tilde{\Omega}} |T_k(u_\epsilon)|^{(\bar{p})^*} dx \\
&\leq C'k^{\frac{(\bar{p})^*}{\bar{p}}}
\end{aligned}$$

and so,

$$(k)^{(\bar{p})^*} meas\{x \in \tilde{\Omega} : |u_\epsilon| > k\} \leq C'k^{\frac{(\bar{p})^*}{\bar{p}}};$$

which means that

$$\lambda_{u_\epsilon}(k) \leq C'k^{(\bar{p})^*(\frac{1}{\bar{p}}-1)} = C'k^{-q^*}, \ \forall k \geq 1.$$

• If $0 < k < 1$, we have

$$\begin{aligned}\lambda_{u_\epsilon}(k) &= meas\left\{x \in \tilde{\Omega} : |u_\epsilon| > k\right\} \\ &\leq meas(\tilde{\Omega}) \\ &\leq meas(\tilde{\Omega})k^{-q^*}.\end{aligned}$$

So,

$$\lambda_{u_\epsilon}(k) \leq (C' + meas(\tilde{\Omega}))k^{-q^*} = C_1 k^{-q^*}.$$

Therefore,

$$\|u_\epsilon\|_{\mathcal{M}^{q^*}(\tilde{\Omega})} \leq C_1.$$

(ii)• Let $\alpha \geq 1$. For all $k \geq 1$, we have

$$\begin{aligned}\lambda_{\frac{\partial u_\epsilon}{\partial x_i}}(\alpha) &= meas\left(\left\{\left|\frac{\partial u_\epsilon}{\partial x_i}\right| > \alpha\right\}\right) \\ &= meas\left(\left\{\left|\frac{\partial u_\epsilon}{\partial x_i}\right| > \alpha; |u_\epsilon| \leq k\right\}\right) + meas\left(\left\{\left|\frac{\partial u_\epsilon}{\partial x_i}\right| > \alpha;\ |u_\epsilon| > k\right\}\right) \\ &\leq \int_{\left\{\left|\frac{\partial u_\epsilon}{\partial x_i}\right| > \alpha; |u_\epsilon| \leq k\right\}} dx + \lambda_{u_\epsilon}(k) \\ &\leq \int_{\{|u_\epsilon| \leq k\}} \left(\frac{1}{\alpha}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|\right)^{p_i^-} dx + \lambda_{u_\epsilon}(k) \\ &\leq \alpha^{-p_i^-} C'k + Ck^{-q^*} \\ &\leq B\left(\alpha^{-p_i^-} k + k^{-q^*}\right),\end{aligned}$$

with $B = \max(C'; C)$.
Let $g : [1; \infty) \to \mathbb{R}$, $x \mapsto g(x) = \dfrac{x}{\alpha^{p_i^-}} + x^{-q^*}$.
We have $g'(x) = 0$ with $x = \left(q^* \alpha^{p^-}\right)^{\dfrac{1}{q^* + 1}}$.
We set $k = \left(q^* \alpha^{p_i^-}\right)^{\dfrac{1}{q^* + 1}} \geq 1$ in the above inequality to get,

$$\begin{aligned}\lambda_{\frac{\partial u_\epsilon}{\partial x_i}}(\alpha) &\leq B\left[\alpha^{-p_i^-} \times \left(q^* \alpha^{p_i^-}\right)^{\frac{1}{q^* + 1}} + \left(q^* \alpha^{p_i^-}\right)^{\frac{-q^*}{q^* + 1}}\right] \\ &\leq B\left[(q^*)^{\frac{1}{q^* + 1}} \times \alpha^{-p_i^-\left(1 - \frac{1}{q^* + 1}\right)} + (q^*)^{\frac{-q^*}{q^* + 1}} \times \alpha^{\frac{-p_i^- q^*}{q^* + 1}}\right]\end{aligned}$$

$$
\begin{aligned}
&\leq B\left[(q^*)^{\dfrac{1}{q^*+1}} \times \alpha^{-p_i^-\left(\dfrac{q^*}{q^*+1}\right)} + (q^*)^{\dfrac{-q^*}{q^*+1}} \times \alpha^{\dfrac{-p_i^- q^*}{q^*+1}}\right] \\
&\leq M\alpha^{-p_i^- \dfrac{q^*}{q^*+1}} \\
&\leq M\alpha^{-p_i^- \dfrac{q}{\bar{p}}},
\end{aligned}
$$

where $M = B \times \max\left((q^*)^{\dfrac{1}{q^*+1}}; (q^*)^{\dfrac{-q^*}{q^*+1}}\right)$ and as $q^* = \dfrac{N(\bar{p}-1)}{N-\bar{p}}$, $q = \dfrac{N(\bar{p}-1)}{N-1}$.

So,

$$
\begin{aligned}
\frac{q^*}{q^*+1} &= \frac{q^*(N-\bar{p})}{N(\bar{p}-1)+N-\bar{p}} \\
&= \frac{q^*(N-\bar{p})}{N\bar{p}-\bar{p}} \\
&= \frac{N(\bar{p}-1)}{(N-1)\bar{p}} \\
&= \frac{q}{\bar{p}}.
\end{aligned}
$$

• If $0 \leq \alpha < 1$, we have.

$$
\begin{aligned}
\lambda_{\dfrac{\partial u_\epsilon}{\partial x_i}}(\alpha) &= meas\left(\left\{x \in \tilde{\Omega} : \left|\frac{\partial u_\epsilon}{\partial x_i}\right| > \alpha\right\}\right) \\
&\leq meas(\tilde{\Omega})\alpha^{-p_i^- \dfrac{q}{\bar{p}}}.
\end{aligned}
$$

Therefore ,

$$
\lambda_{\dfrac{\partial u_\epsilon}{\partial x_i}}(\alpha) \leq \left(M + meas(\tilde{\Omega})\right)\alpha^{-p_i^- \dfrac{q}{\bar{p}}}, \ \forall\, \alpha \geq 0.
$$

So,

$$
\left\|\frac{\partial u_\epsilon}{\partial x_i}\right\|_H \leq C_2,
$$

where $H = \mathcal{M}(\tilde{\Omega})^{\dfrac{p_i^- q}{\bar{p}}}$ □

Proposition 3.2. *Assume (1.4)-(1.8), $f \in L^1(\Omega)$ and $\tilde{d} \in L^1(\tilde{\Gamma}_{Ne})$. Let u_ϵ be a solution of the problem $P_\epsilon(\tilde{\rho}, \tilde{f}_\epsilon, \tilde{d}_\epsilon)$. Then,*

(i) *$u_\epsilon \to u$ in measure, a.e. in Ω and a.e. on $\tilde{\Gamma}_{Ne}$;*
(ii) *For all $i = 1, ...N$, $\dfrac{\partial T_k(u_\epsilon)}{\partial x_i} \rightharpoonup \dfrac{\partial T_k(u)}{\partial x_i} = 0$ in $L^{p_i^-}(\tilde{\Omega} \setminus \Omega)$.*

Proof. (i) By Proposition 3.1 (i), we deduce that $(T_k(u_\epsilon))_{\epsilon>0}$ is bounded in $W_D^{1,\vec{p}(\cdot)}(\tilde{\Omega}) \hookrightarrow L^{p_m(\cdot)}(\tilde{\Omega}) \hookrightarrow L^{p_m^-}(\tilde{\Omega})$ compact. Therefore, up to a subsequence, we can assume that as $\epsilon \to 0$, $(T_k(u_\epsilon))_{\epsilon>0}$ converges strongly to some function in $L^{p_m^-}(\tilde{\Omega})$, $a.e.$ in $\tilde{\Omega}$ and $a.e.$ on $\tilde{\Gamma}_{Ne}$.

Let us see that the sequence $(u_\epsilon)_{\epsilon>0}$ is Cauchy in measure.

Indeed, let $s > 0$ and define:

$E_1 = [|u_{\epsilon_1}| > k]$, $E_2 = [|u_{\epsilon_2}| > k]$ and $E_3 = [|T_k(u_{\epsilon_1}) - T_k(u_{\epsilon_2})| > s]$,

where $k > 0$ is fixed. We note that

$$[|u_{\epsilon_1} - u_{\epsilon_2}| > s] \subset E_1 \cup E_2 \cup E_3;$$

hence,

$$meas([|u_{\epsilon_1} - u_{\epsilon_2}| > s]) \leq \sum_{i=1}^{3} E_i. \tag{3.26}$$

Let $\theta > 0$, using Lemma 3.4, we choose $k = k(\theta)$ such that

$$meas(E_1) \leq \frac{\theta}{3} \text{ and } meas(E_2) \leq \frac{\theta}{3}. \tag{3.27}$$

Since $(T_k(u_\epsilon))_{\epsilon>0}$ converges strongly in $L^{p_m^-}(\tilde{\Omega})$, then, it is a Cauchy sequence in $L^{p_m^-}(\tilde{\Omega})$.

Thus,

$$meas(E_3) \leq \frac{1}{s^{p_m^-}} \int_\Omega |T_k(u_{\epsilon_1}) - T_k(u_{\epsilon_2})|^{p_m^-} dx \leq \frac{\theta}{3}, \tag{3.28}$$

for all $\epsilon_1, \epsilon_2 \geq n_0(s, \theta)$. Finally, from (3.26), (3.27) and (3.28), we obtain

$$meas([|u_{\epsilon_1} - u_{\epsilon_2}| > s]) \leq \theta \text{ for all } \epsilon_1, \epsilon_2 \geq n_0(s, \theta); \tag{3.29}$$

which means that the sequence $(u_\epsilon)_{\epsilon>0}$ is Cauchy in measure, so $u_\epsilon \to u$ in measure and up to a subsequence, we have $u_\epsilon \to u$ $a.e.$ in $\tilde{\Omega}$. Hence, $\sigma_k = T_k(u)$ $a.e.$ in $\tilde{\Omega}$ and so, $u \in \mathcal{T}_D^{1,\vec{p}(.)}(\Omega)$.

(ii) According to the proof of (i), we have $T_k(u_\epsilon) \rightharpoonup T_k(u)$ in $W_D^{1,\vec{p}(.)}(\tilde{\Omega}) \hookrightarrow W_D^{1,\vec{p}_-}(\tilde{\Omega})$ which implies on one hand that for all $i = 1, ...N$, $\dfrac{\partial T_k(u_\epsilon)}{\partial x_i} \rightharpoonup \dfrac{\partial T_k(u)}{\partial x_i}$ in $L^{p_i^-}(\tilde{\Omega})$ and on the other hand that for all $i = 1, ...N$, $\dfrac{\partial T_k(u_\epsilon)}{\partial x_i} \rightharpoonup \dfrac{\partial T_k(u)}{\partial x_i}$ in $L^{p_i^-}(\tilde{\Omega})$

and then for all $i = 1, ...N$, $\dfrac{\partial T_k(u_\epsilon)}{\partial x_i} \rightharpoonup \dfrac{\partial T_k(u)}{\partial x_i}$ in $L^{p_i^-}(\tilde{\Omega} \setminus \Omega)$.

Let $i = 1, ..., N$, by Proposition 3.1-(i), we can assert $\left(\dfrac{1}{\epsilon}\dfrac{\partial T_k(u_\epsilon)}{\partial x_i}\right)_{\epsilon>0}$ is bounded in $L^{p_i^-}(\tilde{\Omega} \setminus \Omega)$. Indeed, let $k > 0$, we set $\Omega^1 = \left\{x \in \tilde{\Omega} \setminus \Omega; |u(x)| \leq k; \left|\dfrac{\partial}{\partial x_i} u_\epsilon(x)\right| \leq \epsilon\right\}$ and $\Omega^2 = \left\{x \in \tilde{\Omega} \setminus \Omega; |u| \leq k; \left|\dfrac{\partial}{\partial x_i} u_\epsilon(x)\right| > \epsilon\right\}$; using Proposition 3.1-*(i)*, we have

$$\begin{aligned}
\sum_{i=1}^{N} \int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon}\left|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right|^{p_i^-}\right) dx &= \sum_{i=1}^{N} \int_{\Omega^1} \left(\frac{1}{\epsilon}\left|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right|^{p_i^-}\right) dx + \\
&\quad \sum_{i=1}^{N} \int_{\Omega^2} \left(\frac{1}{\epsilon}\left|\frac{\partial}{\partial x_i} T_k(u_\epsilon)\right|^{p_i^-}\right) dx \\
&\leq N meas(\tilde{\Omega} \setminus \Omega) + \sum_{i=1}^{N} \int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon}\left|\frac{\partial}{\partial x_i} T_k(u_\epsilon)\right|^{p_i(x)}\right) dx \\
&\leq N meas(\tilde{\Omega} \setminus \Omega) + k\left(\|f\|_{L^1(\Omega)} + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}\right) \\
&\leq C'(k+1),
\end{aligned}$$

with $C' = \max\left\{Nmeas(\tilde{\Omega}\setminus\Omega); \left(\|f\|_{L^1(\Omega)} + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}\right)\right\}$.

To end,

$$\int_{\tilde{\Omega}\setminus\Omega}\left(\frac{1}{\epsilon}\left|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right|^{p_i^-}\right)dx \leq \sum_{i=1}^{N}\int_{\tilde{\Omega}\setminus\Omega}\left(\frac{1}{\epsilon}\left|\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right|^{p_i^-}\right)dx, \text{ for any i=1,...,N.}$$

Therefore, there exists $\Theta_k \in L^{p_i^-}(\tilde{\Omega}\setminus\Omega)$ such that

$$\frac{1}{\epsilon}\frac{\partial T_k(u_\epsilon)}{\partial x_i} \rightharpoonup \Theta_k \text{ in } L^{p_i^-}(\tilde{\Omega}\setminus\Omega) \text{ as } \epsilon \to 0.$$

For any $\psi \in L^{(p_i')^-}(\tilde{\Omega}\setminus\Omega)$, we have

$$\int_{\tilde{\Omega}\setminus\Omega}\frac{\partial T_k(u_\epsilon)}{\partial x_i}\psi dx = \int_{\tilde{\Omega}\setminus\Omega}\left(\frac{1}{\epsilon}\frac{\partial T_k(u_\epsilon)}{\partial x_i} - \Theta_k\right)(\epsilon\psi)dx + \epsilon\int_{\tilde{\Omega}\setminus\Omega}\Theta_k\psi dx. \tag{3.30}$$

As $(\epsilon\psi)_{\epsilon>0}$ converges strongly to zero in $L^{(p_i')^-}(\tilde{\Omega}\setminus\Omega)$, we pass to the limit as $\epsilon \to 0$ in (3.30), to get

$$\frac{\partial T_k(u_\epsilon)}{\partial x_i} \rightharpoonup 0 \text{ in } L^{p_i^-}(\tilde{\Omega}\setminus\Omega).$$

Hence, one has

$$\frac{\partial T_k(u_\epsilon)}{\partial x_i} \rightharpoonup \frac{\partial T_k(u)}{\partial x_i} = 0 \text{ in } L^{p_i^-}(\tilde{\Omega}\setminus\Omega),$$

for any $i = 1, ..., N$□

□

Lemma 3.7. *$b(u) \in L^1(\Omega)$ and $\tilde{\rho}(u) \in L^1(\tilde{\Gamma}_{Ne})$.*

Proof. Having in mind that by Proposition 3.1-*(ii)*,

$$\int_{\Omega}|b(u_\epsilon)|dx + \int_{\tilde{\Gamma}_{Ne}}|\tilde{\rho}(u_\epsilon)|d\sigma \leq (\|f\|_{L^1(\Omega)} + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}).$$

We deduce that

$$\int_{\Omega}|b(u_\epsilon)|dx \leq (\|f\|_{L^1(\Omega)} + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}) \tag{3.31}$$

and

$$\int_{\tilde{\Gamma}_{Ne}}|\tilde{\rho}(u_\epsilon)|d\sigma \leq (\|f\|_{L^1(\Omega)} + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}). \tag{3.32}$$

By Fatou's Lemma, the continuity of b, $\tilde{\rho}$ and using Proposition 3.2, we have

$$\liminf_{\epsilon\to 0}\int_{\Omega}|b(u_\epsilon)|dx \geq \int_{\Omega}|b(u)|dx, \tag{3.33}$$

$$\liminf_{\epsilon\to 0}\int_{\tilde{\Gamma}_{Ne}}|\tilde{\rho}(u_\epsilon)|d\sigma \geq \int_{\tilde{\Gamma}_{Ne}}|\tilde{\rho}(u)|d\sigma. \tag{3.34}$$

Using (3.31)-(3.34), we deduce that

$$\int_{\Omega}|b(u)|dx \leq (\|f\|_{L^1(\Omega)} + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})})$$

and

$$\int_{\tilde{\Gamma}_{Ne}}|\tilde{\rho}(u)|d\sigma \leq (\|f\|_{L^1(\Omega)} + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}).$$

Therefore, $b(u) \in L^1(\Omega)$ and $\tilde{\rho}(u) \in L^1(\tilde{\Gamma}_{Ne})$. □

Lemma 3.8. *Assume (1.4)-(1.8) hold and u_ϵ be a weak solution of the problem $P_\epsilon(\tilde{\rho}, \tilde{f}_\epsilon, \tilde{d}_\epsilon)$. Then,*

(i) $\dfrac{\partial}{\partial x_i} u_\epsilon$ *converges in measure to* $\dfrac{\partial}{\partial x_i} u$.

(ii) $a_i(x, \dfrac{\partial T_k(u_\epsilon)}{\partial x_i}) \to a_i(x, \dfrac{\partial T_k(u)}{\partial x_i})$ *strongly in $L^1(\Omega)$ and weakly in $L^{p_i'(.)}(\Omega)$, for all* $i = 1, ..., N$.

In order to give the proof of Lemma 3.8, we need the following lemmas.

Lemma 3.9 (Cf [8])**.** *Let $u \in \mathcal{T}^{1,\vec{p}(.)}(\Omega)$. Then, there exists a unique measurable function $\nu_i : \Omega \to \mathbb{R}$ such that*

$$\nu_i \chi_{\{|u|<k\}} = \frac{\partial}{\partial x_i} T_k(u) \text{ for a.e. } x \in \Omega,\ \forall k > 0 \text{ and } i = 1, ..., N;$$

where χ_A denotes the characteristic function of a measurable set A.
The functions ν_i are denoted $\dfrac{\partial}{\partial x_i} u$. Moreover, if u belongs to $W^{1,\vec{p}(.)}(\Omega)$, then $v_i \in L^{p_i(.)}(\Omega)$ and coincides with the standard distributional gradient of u i.e. $\nu_i = \dfrac{\partial}{\partial x_i} u$.

Lemma 3.10 (Cf [11], lemma 5.4)**.** *Let $(v_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions. If v_n converges in measure to v and is uniformly bounded in $L^{p(.)}(\Omega)$ for some $1 << p(.) \in L^\infty(\Omega)$, then $v_n \to v$ strongly in $L^1(\Omega)$.*

The third technical lemma is a standard fact in measure theory (Cf [7]).

Lemma 3.11. *Let $(X, \mathcal{M}, \mu)$ be a measurable space such that $\mu(X) < \infty$.*
Consider a measurable function $\gamma : X \to [0; \infty]$ such that

$$\mu(\{x \in X : \gamma(x) = 0\}) = 0.$$

Then, for every $\epsilon > 0$, there exists δ such that

$$\mu(A)) < \epsilon, \text{ for all } A \in \mathcal{M} \text{ with } \int_A \gamma dx < \delta.$$

Proof of Lemma 3.8. (i) We claim that $\left(\dfrac{\partial}{\partial x_i} u_\epsilon\right)_{\epsilon\in\mathbb{N}}$ is Cauchy in measure. Indeed, let $s > 0$, consider
$A_{n,m} := \left\{\left|\dfrac{\partial}{\partial x_i} u_n\right| > h\right\} \cup \left\{\left|\dfrac{\partial}{\partial x_i} u_m\right| > h\right\}$,
$B_{n,m} := \{|u_n - u_m| > k\}$ and
$C_{n,m} := \left\{\left|\dfrac{\partial}{\partial x_i} u_n\right| \le h, \left|\dfrac{\partial}{\partial x_i} u_m\right| \le h, |u_n - u_m| \le k, \left|\dfrac{\partial}{\partial x_i} u_n - \dfrac{\partial}{\partial x_i} u_m\right| > s\right\}$, where h and k will be chosen later. One has

$$\left\{\left|\frac{\partial}{\partial x_i} u_n - \frac{\partial}{\partial x_i} u_m\right| > s\right\} \subset A_{n,m} \cup B_{n,m} \cup C_{n,m}. \tag{3.35}$$

Let $\vartheta > 0$. By Lemma 3.6, we can choose $h = h(\vartheta)$ large enough such that $meas(A_{n,m}) \le \dfrac{\vartheta}{3}$ for all $n, m \ge 0$. On the other hand, by Proposition 3.2, we have that $meas(B_{n,m}) \le \dfrac{\vartheta}{3}$ for all $n, m \ge n_0(k, \vartheta)$. Moreover, by assumption (H_3), there exists a real valued function $\gamma : \Omega \to [0, \infty]$ such that $meas\{x \in \Omega : \gamma(x) = 0\} = 0$ and

$$(a_i(x, \xi) - a_i(x, \xi')).(\xi - \xi') \ge \gamma(x), \tag{3.36}$$

for all $i = 1, ..., N$, $|\xi|, |\xi'| \le h$, $|\xi - \xi'| \ge s$, for a.e. $x \in \Omega$. Indeed, let's set $K = \{(\xi, \eta) \in \mathbb{R}^N \times \mathbb{R}^N : |\xi| \le h, |\eta| \le h, |\xi - \eta| \ge s\}$. We have $K \subset B(0, h) \times B(0, h)$ and so K is a compact set because it is closed in a compact set.

For all $x \in \Omega$ and for all $i = 1, ..., N$, let us define $\phi : K \to [0; \infty]$ such that

$$\psi(\xi, \eta) = (a_i(x, \xi) - a_i(x, \eta)).(\xi - \eta).$$

As for $a.e.\ x \in \Omega$, $a_i(x, .)$ is continuous on $\mathbb{R}^N$, ψ is continuous on the compact K, by Weierstrass theorem, there exists $(\xi_0, \eta_0) \in K$ such that

$$\forall (\xi, \eta) \in K,\ \ \psi(\xi, \eta) \geq \psi(\xi_0, \eta_0).$$

Now let us define γ on Ω as follows.

$$\gamma(x) = \psi_i(\xi_0, \eta_0) = (a_i(x, \xi_0) - a_i(x, \eta)).(\xi - \eta_0).$$

Since $s > 0$, the function γ is such that $meas\,(\{x \in \Omega : \gamma(x) = 0\}) = 0$. Let $\delta = \delta(\epsilon)$ be given by Lemma 3.11, replacing ϵ and A by $\frac{\epsilon}{3}$ and $C_{n,m}$ respectively. Taking respectively $\tilde{\xi} = T_k(u_n - u_m)$ and $\tilde{\xi} = T_k(u_m - u_n)$ for the weak solution u_n and u_m in (3.19) and after adding the two relations, we have

$$\begin{cases} \displaystyle\sum_{i=1}^{N} \int_{\{|u_n - u_m| < k\}} \left(a_i\left(x, \frac{\partial}{\partial x_i} u_n\right) - a_i\left(x, \frac{\partial}{\partial x_i} u_m\right)\right) \left(\frac{\partial}{\partial x_i}(u_n - u_m)\right) dx \\ \displaystyle+ \int_Q \left(\left(\frac{1}{\epsilon^{p_i(x)}} \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u_n}{\partial x_i}\right) - \left(\frac{1}{\epsilon^{p_i(x)}} \left|\frac{\partial u_m}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u_m}{\partial x_i}\right) \left(\frac{\partial (u_n - u_m)}{\partial x_i}\right)\right) dx \\ \displaystyle+ \int_\Omega (|u_n|^{p_M(x)-2} u_n - u_m|^{p_M(x)-2} u_m)(T_k(u_n - u_m) dx \\ \displaystyle+ \int_{\tilde{\Gamma}_{Ne}} (\tilde{\rho}(u_n) - \tilde{\rho}(u_m)) T_k(u_n - u_m) d\sigma \\ \displaystyle= 2 \left(\int_\Omega f_\epsilon T_k(u_n - u_m) dx + \int_{\tilde{\Gamma}_{Ne}} \tilde{d}_\epsilon T_k(u_n - u_m) d\sigma \right), \end{cases}$$

where $Q = \{\tilde{\Omega} \setminus \Omega \cap \{|u_n - u_m| < k\}\}$. As the three last terms on the left hand side are non-negative and

$$\int_\Omega f_\epsilon T_k(u_n - u_m) dx + \int_{\tilde{\Gamma}_{Ne}} \tilde{d}_\epsilon T_k(u_n - u_m) d\sigma \leq k(\|f\|_{L^1(\Omega)} + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}),$$

we deduce that

$$\begin{cases} \displaystyle\sum_{i=1}^{N} \int_{\{|u_n - u_m| < k\}} \left(a_i\left(x, \frac{\partial u_n}{\partial x_i}\right) - a_i\left(x, \frac{\partial u_m}{\partial x_i}\right)\right) \left(\frac{\partial (u_n - u_m)}{\partial x_i}\right) dx \\ \leq 2k(\|f\|_{L^1(\Omega)} + \|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})}). \end{cases}$$

Therefore, using (H_3) we have

$$\begin{aligned} \int_{C_{n,m}} \gamma dx \ &\leq\ \int_{C_{n,m}} \left(a_i\left(x, \frac{\partial}{\partial x_i} u_n\right) - a_i\left(x, \frac{\partial}{\partial x_i} u_m\right)\right) \frac{\partial}{\partial x_i}(u_n - u_m)\, dx \\ &\leq\ \sum_{i=1}^{N} \int_{C_{n,m}} \left(a_i\left(x, \frac{\partial}{\partial x_i} u_n\right) - a_i\left(x, \frac{\partial}{\partial x_i} u_m\right)\right) \frac{\partial}{\partial x_i}(u_n - u_m) dx \\ &\leq\ 2k(\|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})} + \|f\|_{L^1(\Omega)}) < \delta, \end{aligned}$$

by choosing $k = \delta / 4(\|\tilde{d}\|_{L^1(\tilde{\Gamma}_{Ne})} + \|f\|_{L^1(\Omega)})$. From Lemma 3.11 again, it follows that $meas(C_{n,m}) < \frac{\vartheta}{3}$. Thus, using (3.36) and the estimates obtained for $A_{n,m}$, $B_{n,m}$ and $C_{n,m}$, it follows that

(3.37) $$meas\left(\left\{\left|\frac{\partial}{\partial x_i} u_n - \frac{\partial}{\partial x_i} u_m\right| > s\right\}\right) \leq \vartheta,$$

for all $n, m \geq n_0(s, \vartheta)$, and then the claim is proved.
As consequence, $\left(\frac{\partial}{\partial x_i} u_\epsilon\right)_{\epsilon\in\mathbb{N}}$ converges in measure to some measurable function ν_i.
In order to end the proof of Lemma 3.8, we need the following lemma.

Lemma 3.12. (a) *For a.e.* $k \in \mathbb{R}$, $\frac{\partial}{\partial x_i} T_k(u_\epsilon)$ *converges in measure to* $\nu_i \chi_{\{|u|<k\}}$.

(b) *For a.e.* $k \in \mathbb{R}$, $\frac{\partial}{\partial x_i} T_k(u) = \nu_i \chi_{\{|u|<k\}}$.

(c) $\frac{\partial}{\partial x_i} T_k(u) = \nu_i \chi_{\{|u|<k\}}$ *holds for all* $k \in \mathbb{R}$.

Proof. (a) We know that $\frac{\partial}{\partial x_i} u_\epsilon \to \nu_i$ in measure. Thus $\frac{\partial}{\partial x_i} u_\epsilon \chi_{\{|u|<k\}} \to \nu_i \chi_{\{|u|<k\}}$ in measure.
Now, let us show that $\left(\chi_{\{|u_\epsilon|<k\}} - \chi_{\{|u|<k\}}\right) \frac{\partial}{\partial x_i} u_\epsilon \to 0$ in measure.
For that, it is sufficient to show that $\left(\chi_{\{|u_\epsilon|<k\}} - \chi_{\{|u|<k\}}\right) \to 0$ in measure. Now, for all $\delta > 0$, $\left\{ \left|\chi_{\{|u_\epsilon|<k\}} - \chi_{\{|u|<k\}}\right| \left|\frac{\partial}{\partial x_i} u_\epsilon\right| > \delta \right\} \subset \left\{ \left|\chi_{\{|u_\epsilon|<k\}} - \chi_{\{|u|<k\}}\right| \neq 0 \right\} \subset \{|u| = k\} \cup \{u_\epsilon < k < u\} \cup \{u < k < u_\epsilon\} \cup \{u_\epsilon < -k < u\} \cup \{u < -k < u_\epsilon\}$. Thus,

$$
\left\{
\begin{aligned}
meas\left(\left\{ \left|\chi_{\{|u_\epsilon|<k\}} - \chi_{\{|u|<k\}}\right| \left|\frac{\partial}{\partial x_i} u_\epsilon\right| > \delta \right\}\right) \leq\ & meas\left(\{|u| = k\}\right) \\
& + meas\left(\{u_\epsilon < k < u\}\right) \\
& + meas\left(\{u < k < u_\epsilon\}\right) \\
& + meas\left(\{u_\epsilon < -k < u\}\right) \\
& + meas\left(\{u < -k < u_\epsilon\}\right).
\end{aligned}
\right. \tag{3.38}
$$

Note that
$$
\left\{
\begin{aligned}
& meas\left(\{|u| = k\}\right) \leq meas\left(\{k - h < u < k + h\}\right) \\
& + meas\left(\{-k - h < u < -k + h\}\right) \to 0 \text{ as } h \to 0 \text{ for a.e. } k > 0
\end{aligned}
\right. \text{, since } u \text{ is fixed function.}
$$
Next, $meas\left(\{u_\epsilon < k < u\}\right) \leq meas\left(\{k < u < k + h\}\right) + meas\left(\{|u_\epsilon - u| > h\}\right)$, for all $h > 0$.
Due to Proposition 3.2, we have for all fixed $h > 0$, $meas\left(\{|u_\epsilon - u| > h\}\right) \to 0$ as $\epsilon \to 0$. Since $meas\left(\{k < u < k + h\}\right) \to 0$ as $h \to 0$, for all $\vartheta > 0$, one can find N such that for all $n > N$, $meas\left(\{|u| = k\}\right) < \frac{\vartheta}{2} + \frac{\vartheta}{2} = \vartheta$ by choosing h and then N. Each of the other terms on the right-hand side of (3.38) can be treated in the same way as for $meas\left(\{u_\epsilon < k < u\}\right)$. Thus, $meas\left(\left\{ \left|\chi_{\{|u_\epsilon|<k\}} - \chi_{\{|u|<k\}}\right| \left|\frac{\partial}{\partial x_i} u_\epsilon\right| > \delta \right\}\right) \to 0$ as $\epsilon \to 0$. Finally, since $\frac{\partial}{\partial x_i} T_k(u_\epsilon) = \frac{\partial}{\partial x_i} u_\epsilon \chi_{\{|u_\epsilon|<k\}}$, (a) follows.

(b) Using the Proof of Proposition 3.2-(ii) we have $\frac{\partial}{\partial x_i} T_k(u_\epsilon) \rightharpoonup \frac{\partial}{\partial x_i} T_k(u)$ weakly in $L^{p_i^-}(\tilde{\Omega})$. The previous convergence also ensures that $\frac{\partial}{\partial x_i} T_k(u_\epsilon)$ converges to $\frac{\partial}{\partial x_i} T_k(u)$ weakly in $L^1(\Omega)$. On the other hand, by (a), $\frac{\partial}{\partial x_i} T_k(u_\epsilon)$ converges to $\nu_i \chi_{|u|<k\}}$ in measure. By Lemma 3.10, since $\frac{\partial}{\partial x_i} T_k(u_\epsilon)$ is uniformly bounded in $L^{p_i^-}(\tilde{\Omega})$ (see Lemma 3.5) hence in $L^{p_i^-}(\Omega)$, the convergence is actually strong in $L^1(\Omega)$; thus it is also weak in $L^1(\Omega)$. By the uniqueness of a weak L^1-limit, $\nu_i \chi_{\{|u|<k\}}$ coincides with $\frac{\partial}{\partial x_i} T_k(u)$.

(c) Let $0 < k < s$, and s be such that $\nu_i \chi_{\{|u|<s\}}$ coincides with $\frac{\partial}{\partial x_i} T_s(u)$. Then,

$$\begin{aligned} \frac{\partial}{\partial x_i} T_k(u) &= \frac{\partial}{\partial x_i} T_k(T_s(u)) \\ &= \frac{\partial}{\partial x_i} T_s(u) \chi_{\{|T_s(u)|<k\}} \\ &= \nu_i \chi_{\{|u|<s\}} \chi_{\{|u|<k\}} \\ &= \nu_i \chi_{\{|u|<k\}}. \end{aligned}$$

□

Now, we can end the proof of Lemma 3.8. Indeed, combining lemmas 3.12 (c) and 3.9; *(i)* follows. Next, by lemmas 3.10 and 3.12, we have for all $k > 0$, $i = 1, ..., N$, $a_i(x, \frac{\partial}{\partial x_i} T_k(u_\epsilon))$ converges to $a_i(x, \frac{\partial}{\partial x_i} T_k(u))$ in $L^1(\Omega)$ strongly. Indeed, Let $s, k > 0$, consider
$E_4 = \left\{ \left| \frac{\partial u_n}{\partial x_i} - \frac{\partial u_m}{\partial x_i} \right| > s, |u_n| \le k, |u_m| \le k \right\}$,
$E_5 = \left\{ \left| \frac{\partial u_m}{\partial x_i} \right| > s, |u_n| > k, |u_m| \le k \right\}$ and
$E_6 = \left\{ \left| \frac{\partial u_n}{\partial x_i} \right| > s, |u_n| \le k, |u_m| > k \right\}$.
We have

$$\left\{ \left| \frac{\partial T_k(u_n)}{\partial x_i} - \frac{\partial T_k(u_m)}{\partial x_i} \right| > s \right\} \subset E_4 \cup E_5 \cup E_6. \tag{3.39}$$

$\forall \vartheta > 0$, by Lemma 3.4, there exists $k(\vartheta)$ such that

$$meas(E_5) \le \frac{\vartheta}{3} \text{ and } meas(E_6) \le \frac{\vartheta}{3}. \tag{3.40}$$

Using (3.37)-(3.40), we get

$$meas \left(\left\{ \left| \frac{\partial}{\partial x_i} T_k(u_n) - \frac{\partial}{\partial x_i} T_k(u_m) \right| > s \right\} \right) \le \vartheta, \tag{3.41}$$

for all $n, m \ge n_1(s, \vartheta)$. Therefore, $\frac{\partial T_k(u_\epsilon)}{\partial x_i}$ converges in measure to $\frac{\partial T_k(u)}{\partial x_i}$. Using lemmas 3.5 and 3.10, we deduce that $\frac{\partial T_k(u_\epsilon)}{\partial x_i}$ converges to $\frac{\partial T_k(u)}{\partial x_i}$ *in* $L^1(\Omega)$. So, after passing to a suitable subsequence of $\left(\frac{\partial T_k(u_\epsilon)}{\partial x_i} \right)_{\epsilon > 0}$, we can assume that $\frac{\partial T_k(u_\epsilon)}{\partial x_i}$ converges to $\frac{\partial T_k(u)}{\partial x_i}$ $a.e.$ in Ω. By the continuity of $a_i(x, .)$, we deduce that $a_i \left(x, \frac{\partial T_k(u_\epsilon)}{\partial x_i} \right)$ converges to $a_i \left(x, \frac{\partial T_k(u)}{\partial x_i} \right)$ $a.e.$ *in* Ω. As Ω is bounded, this convergence is in measure. Using lemmas 3.10 and 3.12, we deduce that for all $k > 0$, $i = 1, ..., N$, $a_i(x, \frac{\partial}{\partial x_i} T_k(u_\epsilon))$ converges to $a_i(x, \frac{\partial}{\partial x_i} T_k(u))$ in $L^1(\Omega)$ strongly and $a_i(x, \frac{\partial}{\partial x_i} T_k(u_\epsilon))$ converges to $\chi_k \in L^{p_i'(\cdot)}(\Omega)$ weakly in $L^{p_i'(\cdot)}(\Omega)$. Since each of the convergences implies the weak L^1-convergence, χ_k can be identified with $a_i(x, \frac{\partial}{\partial x_i} T_k(u))$; thus, $a_i(x, \frac{\partial}{\partial x_i} T_k(u)) \in L^{p_i'(\cdot)}(\Omega)$

□

Lemma 3.13. *(Lebesgue generalized convergence theorem) Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of measurable functions and f a measurable function such that $f_n \to f$ a.e. in Ω.*

Let $(g_n)_{n\in\mathbb{N}} \subset L^1(\Omega)$ be such that $|f_n| \le g_n$ a.e. in Ω and $g_n \to g$ in $L^1(\Omega)$.Then,

$$\int_\Omega f_n dx \to \int_\Omega f dx.$$

Proposition 3.3. *For any $k > 0$ and any $i = 1, ..., N$, as ϵ tends to 0, we have*

(i) $\dfrac{\partial T_k(u_\epsilon)}{\partial x_i} \to \dfrac{\partial T_k(u)}{\partial x_i}$ *a.e. in Ω,*

(ii) $a_i(x, \dfrac{\partial T_k(u_\epsilon)}{\partial x_i})\dfrac{\partial T_k(u_\epsilon)}{\partial x_i} \to a_i(x, \dfrac{\partial T_k(u)}{\partial x_i})\dfrac{\partial T_k(u)}{\partial x_i}$ *a.e. in Ω and strongly in $L^1(\Omega)$,*

(iii) $\dfrac{\partial T_k(u_\epsilon)}{\partial x_i} \to \dfrac{\partial T_k(u)}{\partial x_i}$ *strongly in $L^{p_i(x)}(\Omega)$.*

Proof. We must show that

$$\lim_{\epsilon\to 0}\sum_{i=1}^N \int_\Omega \left(a_i(x, \frac{\partial T_k(u_\epsilon)}{\partial x_i}) - a_i(x, \frac{\partial T_k(u)}{\partial x_i})\right)\left(\frac{\partial T_k(u_\epsilon)}{\partial x_i} - \frac{\partial T_k(u)}{\partial x_i}\right) dx = 0. \tag{3.42}$$

Indeed, if the above equality yields, we can assert that for all $i = 1, ..., N$,

$$\lim_{\epsilon\to 0}\int_\Omega \left(a_i(x, \frac{\partial T_k(u_\epsilon)}{\partial x_i}) - a_i(x, \frac{\partial T_k(u)}{\partial x_i})\right)\left(\frac{\partial T_k(u_\epsilon)}{\partial x_i} - \frac{\partial T_k(u)}{\partial x_i}\right) dx = 0.$$

Let i be fixed; since

$$g^i_\epsilon(.) := \left(a_i(., \frac{\partial T_k(u_\epsilon)}{\partial x_i}) - a_i(., \frac{\partial T_k(u)}{\partial x_i})\right)\left(\frac{\partial T_k(u_\epsilon)}{\partial x_i} - \frac{\partial T_k(u)}{\partial x_i}\right) \ge 0,$$

up to a subsequence, we have $g^i_\epsilon(.) \to 0$ a.e. in Ω.
This implies that, there exists $Z \subset \Omega$ such that $meas(Z) = 0$ and $g^i_\epsilon(.) \to 0$ a.e. in $\Omega \setminus Z$.
Let $x \in \Omega \setminus Z$. Using the assumptions (H_2) and (H_4), it follows that the sequence $\left(\dfrac{\partial T_k(u_\epsilon(x))}{\partial x_i}\right)_{\epsilon>0}$ is bounded in $\mathbb{R}^N$. So, we can extract a subsequence which converges to some $\tilde{\xi}$ in $\mathbb{R}^N$. Passing to the limit in the expression of $g^i_\epsilon(x)$, we get

$$0 = \left(a_i(x, \tilde{\xi}) - a_i(x, \frac{\partial T_k(u(x))}{\partial x_i})\right)\left(\tilde{\xi} - \frac{\partial T_k(u(x))}{\partial x_i}\right).$$

This yields $\tilde{\xi} = \dfrac{\partial T_k(u(x))}{\partial x_i}$, $\forall x \in \Omega \setminus Z$. As the limit does not depend on the subsequence, the whole sequence $\left(\dfrac{\partial T_k(u_\epsilon(x))}{\partial x_i}\right)_{\epsilon>0}$ converges to $\tilde{\xi}$ in $\mathbb{R}^N$. This means that $\dfrac{\partial T_k(u_\epsilon)}{\partial x_i} \to \dfrac{\partial T_k(u)}{\partial x_i}$ a.e. in Ω.
We have to show (3.42). The proof consists in three steps.

Step 1. We prove that for every function $h \in W^{1,\infty}(\mathbb{R})$, $h > 0$ with compact support,

$$\begin{cases} \limsup\limits_{\epsilon\to 0}\sum\limits_{i=1}^N \displaystyle\int_\Omega \left(a_i(x, \frac{\partial u_\epsilon}{\partial x_i})\frac{\partial[h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))]}{\partial x_i}\right) dx \\ + \limsup\limits_{\epsilon\to 0}\sum\limits_{i=1}^N \displaystyle\int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial u_\epsilon}{\partial x_i}|^{p_i(x)-2}\frac{\partial u_\epsilon}{\partial x_i}\frac{\partial[h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))]}{\partial x_i}\right) dx \le 0. \end{cases} \tag{3.43}$$

Taking $\tilde{\xi} = h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))$ as test function in (3.19), we have

$$
\begin{cases}
\sum_{i=1}^{N} \int_{\Omega} \left(a_i(x, \frac{\partial}{\partial x_i} u_\epsilon) \frac{\partial}{\partial x_i} [h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))] \right) dx \\
+ \sum_{i=1}^{N} \int_{\tilde{\Omega} \setminus \Omega} \left(\frac{1}{\epsilon^{p_i(x)}} |\frac{\partial}{\partial x_i} u_\epsilon|^{p_i(x)-2} \frac{\partial}{\partial x_i} u_\epsilon \frac{\partial}{\partial x_i} [h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))] \right) dx \\
+ \int_{\Omega} b(u_\epsilon)[h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))] dx = \int_{\Omega} f_\epsilon [h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))] dx \\
+ \int_{\tilde{\Gamma}_{Ne}} (\tilde{d} - \tilde{\rho}(u_\epsilon)[h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))] d\sigma.
\end{cases} \tag{3.44}
$$

Note that

$$
\begin{aligned}
\int_{\Omega} b(u_\epsilon)[h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))] dx &= \int_{\Omega} [b(u_\epsilon) - b(u)] h(u_\epsilon)(T_k(u_\epsilon) - T_k(u)) dx \\
&+ \int_{\Omega} b(u) h(u_\epsilon)(T_k(u_\epsilon) - T_k(u)) dx.
\end{aligned}
$$

Since

$$\int_{\Omega} [b(u_\epsilon) - b(u)] h(u_\epsilon)(T_k(u_\epsilon) - T_k(u)) dx \geq 0$$

and

$$\lim_{\epsilon \to 0} \int_{\Omega} b(u) h(u_\epsilon)(T_k(u_\epsilon) - T_k(u)) dx = 0,$$

we get

$$\limsup_{\epsilon \to 0} \int_{\Omega} b(u_\epsilon)[h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))] dx \geq 0. \tag{3.45}$$

We also have

$$
\begin{aligned}
\int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}(u_\epsilon)[h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))] dx &= \int_{\tilde{\Gamma}_{Ne}} [\tilde{\rho}(u_\epsilon) - \tilde{\rho}(u)] h(u_\epsilon)(T_k(u_\epsilon) - T_k(u)) dx \\
&+ \int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}(u) h(u_\epsilon)(T_k(u_\epsilon) - T_k(u)) dx.
\end{aligned}
$$

Since

$$\int_{\tilde{\Gamma}_{Ne}} [\tilde{\rho}(u_\epsilon) - \tilde{\rho}(u)] h(u_\epsilon)(T_k(u_\epsilon) - T_k(u)) dx \geq 0$$

and

$$\lim_{\epsilon \to 0} \int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}(u) h(u_\epsilon)(T_k(u_\epsilon) - T_k(u)) dx = 0,$$

we get

$$\limsup_{\epsilon \to 0} \int_{\tilde{\Gamma}_{Ne}} \tilde{\rho}(u_\epsilon) h(u_\epsilon)(T_k(u_\epsilon) - T_k(u)) dx \geq 0. \tag{3.46}$$

By Lemma 3.13 , we have

$$\lim_{\epsilon \to 0} \int_{\Omega} f_\epsilon h(u_\epsilon)(T_k(u_\epsilon) - T_k(u)) dx = 0 \tag{3.47}$$

and

$$\lim_{\epsilon \to 0} \int_{\tilde{\Gamma}_{Ne}} \tilde{d} h(u_\epsilon)(T_k(u_\epsilon) - T_k(u)) dx = 0. \tag{3.48}$$

Letting ϵ goes to zero in (3.44) by using (3.45)-(3.48) we get (3.43).

Step 2. We prove that

$$\limsup_{l\to\infty}\limsup_{\epsilon\to 0}\sum_{i=1}^{N}\int_{[l<|u|<l+1]} a_i(x,\frac{\partial u_\epsilon}{\partial x_i})\frac{\partial u_\epsilon}{\partial x_i}\le 0. \tag{3.49}$$

Taking $w_l(u_\epsilon)$ as test function in (3.19), where $w_l(r)=T_1(r-T_l(r))$, we get

$$\begin{cases}\displaystyle\sum_{i=1}^{N}\int_{\Omega\cap[l<|u_\epsilon|<l+1]}\left(a_i(x,\frac{\partial}{\partial x_i}u_\epsilon)\frac{\partial}{\partial x_i}u_\epsilon\right)dx\\ \displaystyle+\sum_{i=1}^{N}\int_{\tilde\Omega\setminus\Omega\cap[l<|u_\epsilon|<l+1]}\left(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial}{\partial x_i}u_\epsilon|^{p_i(x)}\right)dx\\ \displaystyle+\int_\Omega b(u_\epsilon)(T_1(u_\epsilon-T_l(u))dx=\int_\Omega f_\epsilon(T_1(u_\epsilon-T_l(u))dx\\ \displaystyle+\int_{\tilde\Gamma_{Ne}}(\tilde d-\tilde\rho(u_\epsilon)(T_1(u_\epsilon-T_l(u))d\sigma.\end{cases} \tag{3.50}$$

In (3.50), the integrals $\displaystyle\sum_{i=1}^{N}\int_{\tilde\Omega\setminus\Omega\cap[l<|u_\epsilon|<l+1]}\left(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial}{\partial x_i}u_\epsilon|^{p_i(x)}\right)dx$, $\displaystyle\int_\Omega b(u_\epsilon)(T_1(u_\epsilon-T_l(u))dx$ and $\displaystyle\int_{\tilde\Gamma_{Ne}}\tilde\rho(u_\epsilon)T_1(u_\epsilon-T_l(u))d\sigma$ are non-negative and it is easy to see that

$$\limsup_{l\to\infty}\limsup_{\epsilon\to 0}\int_\Omega f_\epsilon(T_1(u_\epsilon-T_l(u))dx=0=\limsup_{l\to\infty}\limsup_{\epsilon\to 0}\int_{\tilde\Gamma_{Ne}}\tilde d(T_1(u_\epsilon-T_l(u))d\sigma.$$

Hence, (3.49) follows by passing to the limit in (3.50).

Step 3. We prove that for every $k>0$,

$$\limsup_{\epsilon\to 0}\sum_{i=1}^{N}\int_\Omega\left(a_i(x,\frac{\partial}{\partial x_i}u_\epsilon).\frac{\partial}{\partial x_i}(T_k(u_\epsilon)-T_k(u))\right)dx\le 0. \tag{3.51}$$

For all $l>0$, we define the function h_l by $h_l(r)=\inf\{1,(l+1-|r|)^+\}$.

For all $l>k$, we have

$$\begin{cases}\displaystyle\sum_{i=1}^{N}\int_\Omega\left(a_i(x,\frac{\partial}{\partial x_i}u_\epsilon)\frac{\partial}{\partial x_i}[h_l(u_\epsilon)(T_k(u_\epsilon)-T_k(u))]\right)dx\\ \displaystyle=\sum_{i=1}^{N}\int_{[|u_\epsilon|\le k]}\left(h_l(u_\epsilon)a_i(x,\frac{\partial}{\partial x_i}u_\epsilon)\frac{\partial}{\partial x_i}(T_k(u_\epsilon)-T_k(u))\right)dx\\ \displaystyle+\sum_{i=1}^{N}\int_{[|u_\epsilon|>k]}\left(h_l(u_\epsilon)a_i(x,\frac{\partial}{\partial x_i}u_\epsilon)\frac{\partial}{\partial x_i}(-T_k(u))\right)dx\\ \displaystyle+\sum_{i=1}^{N}\int_\Omega\left(h_l'(u_\epsilon)(T_k(u_\epsilon)-T_k(u))a_i(x,\frac{\partial}{\partial x_i}u_\epsilon)\frac{\partial u_\epsilon}{\partial x_i}\right)dx=\\ E_1+E_2+E_3.\end{cases}$$

Since $l>k$, on the set $[|u_\epsilon|\le k]$ we have $h_l(u_\epsilon)=1$ so that we can write (E_1) as

$$(E_1)=\sum_{i=1}^{N}\int_\Omega\left(a_i(x,\frac{\partial}{\partial x_i}T_k(u_\epsilon))\frac{\partial}{\partial x_i}(T_k(u_\epsilon)-T_k(u))\right)dx.$$

Hence, we obtain

$$\limsup_{\epsilon\to 0}(E_1)=\limsup_{\epsilon\to 0}\sum_{i=1}^{N}\int_\Omega\left(a_i(x,\frac{\partial}{\partial x_i}T_k(u_\epsilon))\frac{\partial}{\partial x_i}(T_k(u_\epsilon)-T_k(u))\right)dx.$$

Let us write the term (E_2) as

$$(E_2) = -\sum_{i=1}^{N}\int_{[|u_\epsilon|>k]}\Big(h_l(u_\epsilon)a_i(x,\frac{\partial}{\partial x_i}T_{l+1}(u_\epsilon))\frac{\partial}{\partial x_i}T_k(u)\Big)\,dx.$$

Using Lebesgue dominated convergence theorem, we get

$$\limsup_{\epsilon\to 0}(E_2) = -\sum_{i=1}^{N}\int_{[|u|>k]}\Big(h_l(u)a_i(x,\frac{\partial}{\partial x_i}T_{l+1}(u))\frac{\partial}{\partial x_i}T_k(u)\Big)\,dx = 0.$$

For the term (E_3), we have

$$\begin{cases}(E_3) \le \left|\displaystyle\sum_{i=1}^{N}\int_{\Omega}\Big(h_l'(u_\epsilon)(T_k(u_\epsilon)-T_k(u))a_i(x,\frac{\partial}{\partial x_i}u_\epsilon)\frac{\partial u_\epsilon}{\partial x_i}\Big)\,dx\right| \\ \le 2k\displaystyle\sum_{i=1}^{N}\int_{[l<|u|<l+1]}\Big(a_i(x,\frac{\partial}{\partial x_i}u_\epsilon)\frac{\partial u_\epsilon}{\partial x_i}\Big)\,dx.\end{cases}$$

Using the result of the Step 2 we deduce that

$$\limsup_{l\to\infty}\limsup_{\epsilon\to 0}(E_3) \le 0.$$

Applying (3.43) with h replaced by h_l, $l>k$ we get

$$\begin{cases}\displaystyle\limsup_{\epsilon\to 0}\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,\frac{\partial T_k(u_\epsilon)}{\partial x_i})\frac{\partial(T_k(u_\epsilon)-T_k(u))}{\partial x_i}\Big)\,dx \\ \displaystyle+\limsup_{\epsilon\to 0}\sum_{i=1}^{N}\int_{\tilde{\Omega}\setminus\Omega}\Big(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial u_\epsilon}{\partial x_i}|^{p_i(x)-2}\frac{\partial u_\epsilon}{\partial x_i}\frac{\partial[h_l(u_\epsilon)(T_k(u_\epsilon)-T_k(u))]}{\partial x_i}\Big)\,dx \le -\limsup_{\epsilon\to 0}E_3.\end{cases}$$

This last inequality imply that

$$\begin{cases}\displaystyle\limsup_{\epsilon\to 0}\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,\frac{\partial T_k(u_\epsilon)}{\partial x_i})\frac{\partial(T_k(u_\epsilon)-T_k(u))}{\partial x_i}\Big)\,dx \\ \displaystyle+\limsup_{\epsilon\to 0}\sum_{i=1}^{N}\int_{\tilde{\Omega}\setminus\Omega}\Big(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial T_k(u_\epsilon)}{\partial x_i}|^{p_i(x)-2}\frac{\partial T_k(u_\epsilon)}{\partial x_i}\frac{\partial[h_l(u_\epsilon)(T_k(u_\epsilon)-T_k(u))]}{\partial x_i}\Big)\,dx \\ \displaystyle\le -\limsup_{\epsilon\to 0}E_3.\end{cases} \tag{3.52}$$

Note that

$$\begin{cases}\displaystyle\limsup_{\epsilon\to 0}\sum_{i=1}^{N}\int_{\tilde{\Omega}\setminus\Omega}\Big(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial T_k(u_\epsilon)}{\partial x_i}|^{p_i(x)-2}\frac{\partial T_k(u_\epsilon)}{\partial x_i}\frac{\partial[h_l(u_\epsilon)(T_k(u_\epsilon)-T_k(u))]}{\partial x_i}\Big)\,dx \\ \displaystyle=\limsup_{\epsilon\to 0}\sum_{i=1}^{N}\int_{\tilde{\Omega}\setminus\Omega}\Big(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial T_k(u_\epsilon)}{\partial x_i}|^{p_i(x)-2}\frac{\partial T_k(u_\epsilon)}{\partial x_i}\frac{\partial(T_k(u_\epsilon)-T_k(u))}{\partial x_i}\Big)\,dx \\ \displaystyle=\limsup_{\epsilon\to 0}\sum_{i=1}^{N}\int_{\tilde{\Omega}\setminus\Omega}\Big(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial T_k(u_\epsilon)}{\partial x_i}|^{p_i(x)}\Big)\,dx \ge 0;\end{cases}$$

since $\forall i = 1,...,N,\ \dfrac{\partial T_k(u)}{\partial x_i} = 0$ on $\tilde{\Omega}\setminus\Omega$, where in the first Step we have used the fact that $h_l(u_\epsilon) = 1$ on $[|u_\epsilon|\le k]$. Hence from (3.52), we get

$$\limsup_{\epsilon\to 0}\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,\frac{\partial T_k(u_\epsilon)}{\partial x_i})\frac{\partial(T_k(u_\epsilon)-T_k(u))}{\partial x_i}\Big)\,dx \le -\limsup_{\epsilon\to 0}E_3. \tag{3.53}$$

By letting l goes to ∞ in (3.53), the inequality (3.51) follows.

Thanks to (3.51), we deduce that for all $k > 0$,

$$\lim_{\epsilon\to 0}\sum_{i=1}^{N}\int_{\Omega}\left(a_i(x,\frac{\partial T_k(u_\epsilon)}{\partial x_i})-a_i(x,\frac{\partial T_k(u)}{\partial x_i})\right)\left(\frac{\partial T_k(u_\epsilon)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\right)dx=0.$$

(ii) The continuity of $a_i(x,\xi)$ for any $i = 1, ..., N$ with respect to $\tilde{\xi} \in \mathbb{R}^N$ gives us

$$a_i\left(x,\frac{\partial T_k(u_\epsilon)}{\partial x_i}\right)\to a_i\left(x,\frac{\partial T_k(u)}{\partial x_i}\right)\ a.e.\ in\ \Omega,$$

for all $i = 1, ..., N$. Then, we obtain for all $i = 1, ..., N$,

$$a_i(x,\frac{\partial T_k(u_\epsilon)}{\partial x_i})\frac{\partial T_k(u_\epsilon)}{\partial x_i}\to a_i(x,\frac{\partial T_k(u)}{\partial x_i})\frac{\partial T_k(u)}{\partial x_i}\ a.e.\ in\ \Omega.$$

For i fixed, setting $y^i_\epsilon = a_i(x,\frac{\partial T_k(u_\epsilon)}{\partial x_i})\frac{\partial T_k(u_\epsilon)}{\partial x_i}$ and $y^i = a_i(x,\frac{\partial T_k(u)}{\partial x_i})\frac{\partial T_k(u)}{\partial x_i}$, we have $y^i_\epsilon \geq 0$, $y^i_\epsilon \to y^i$ $a.e.$ in Ω, $y^i \in L^1(\Omega)$, $\int_\Omega y^i_\epsilon dx \to \int_\Omega y^i dx$. Since $\int_\Omega |y^i_\epsilon - y^i|dx = 2\int_\Omega (y^i - y^i_\epsilon)^+ dx + \int_\Omega (y^i_\epsilon - y^i)dx$, and $(y^i - y^i_\epsilon)^+ \leq y^i$, it follow by the Lebesgue dominated convergence theorem that $\lim_{\epsilon\to 0}\int_\Omega |y^i_\epsilon - y^i|dx = 0$, which means that $a_i(x,\frac{\partial T_k(u_\epsilon)}{\partial x_i})\frac{\partial T_k(u_\epsilon)}{\partial x_i} \to a_i(x,\frac{\partial T_k(u)}{\partial x_i})\frac{\partial T_k(u)}{\partial x_i}$ strongly in $L^1(\Omega)$□

(iii) By (H_4), for all $i = 1, ..., N$, one has

$$|T_k(u_\epsilon)|^{p_i(x)} \leq a_i(x,\frac{\partial T_k(u_\epsilon)}{\partial x_i})\frac{\partial T_k(u_\epsilon)}{\partial x_i}.$$

Using the L^1-convergence of (ii) and the generalized dominated convergence Theorem, the result of (iii) follows

□

4. Existence and uniqueness of entropy solution

We are now able to prove Theorem 2.4.

Proof of Theorem 2.4. Thanks to the Proposition 3.2 and as $\forall k > 0$, $\forall i = 1, ..., N$, $\frac{\partial T_k(u)}{\partial x_i} = 0$ in $L^{p_i^-}(\tilde{\Omega}\setminus\Omega)$, then $\forall k > 0$, $T_k(u) = constant\ a.e.$ on $\tilde{\Omega}\setminus\Omega$. Hence, we conclude that $u \in \mathcal{T}^{1,\vec{p}(.)}_{Ne}(\Omega)$.

We already state that $b(u) \in L^1(\Omega)$.

To show that u is an entropy solution of $P(\rho, f, d)$, we only have to prove the inequality in (2.9).

Let $\varphi \in W^{1,\vec{p}(.)}_D(\Omega) \cap L^\infty(\Omega)$. We consider the function $\varphi_1 \in W^{1,\vec{p}(.)}_D(\tilde{\Omega}) \cap L^\infty(\Omega)$ such that

$$\varphi_1 = \varphi\chi_\Omega + \varphi_N\chi_{\tilde{\Omega}\setminus\Omega};$$

we set $\tilde{\xi} = T_k(u_\epsilon - \varphi_1)$ in (3.19) to get

$$(4.1)\qquad \begin{cases}\displaystyle\sum_{i=1}^{N}\int_{\Omega}\left(a_i(x,\frac{\partial}{\partial x_i}u_\epsilon).\frac{\partial}{\partial x_i}T_k(u_\epsilon-\varphi)\right)dx\\ \displaystyle+\sum_{i=1}^{N}\int_{\tilde{\Omega}\setminus\Omega}\left(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial}{\partial x_i}u_\epsilon|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon.\frac{\partial}{\partial x_i}T_k(u_\epsilon-\varphi_N)\right)dx\\ \displaystyle\int_\Omega b(u_\epsilon)T_k(u_\epsilon-\varphi)dx=\int_\Omega f_\epsilon\tilde{\xi}dx+\int_{\tilde{\Gamma}_{Ne}}(\tilde{d}-\tilde{\rho}(u_\epsilon))T_k(u_\epsilon-\varphi_N)d\sigma.\end{cases}$$

The following convergence result holds true.

Lemma 4.1. *For any $k > 0$, for all $i = 1, ..., N$, as $\epsilon \to 0$,*

$$\frac{\partial}{\partial x_i}T_k(u_\epsilon - \varphi) \to \frac{\partial}{\partial x_i}T_k(u - \varphi) \text{ strongly in } L^{p_i(\cdot)}(\Omega).$$

Proof. Let $k > 0$, $i = 1, ..., N$. We have

$$\begin{aligned}
\int_\Omega |\frac{\partial}{\partial x_i}T_k(u_\epsilon - \varphi) - \frac{\partial}{\partial x_i}T_k(u - \varphi)|^{p_i(x)}dx &= \int_{\Omega\cap[|u_\epsilon-\varphi|\le k, |u-\varphi|\le k]} |\frac{\partial}{\partial x_i}u_\epsilon - \frac{\partial}{\partial x_i}u|^{p_i(x)}dx \\
&\le \int_{\Omega\cap[|u_\epsilon|\le l, |u|\le l]} |\frac{\partial u_\epsilon}{\partial x_i} - \frac{\partial u}{\partial x_i}|^{p_i(x)}dx, \text{ with } l = k + \|\varphi\|_\infty \\
&= \int_\Omega |\frac{\partial}{\partial x_i}T_l(u_\epsilon) - \frac{\partial}{\partial x_i}T_l(u)|^{p_i(x)}dx \\
&\to 0 \text{ as } \epsilon \to 0 \text{ by Proposition } 3.3 - (iii).
\end{aligned}$$

□

This completes the proof of the Lemma 4.1.

We need to pass to the limit in (4.1) as $\epsilon \to 0$. We have

$$\sum_{i=1}^N \int_\Omega \left(a_i(x, \frac{\partial}{\partial x_i}u_\epsilon)\frac{\partial}{\partial x_i}T_k(u_\epsilon - \varphi)\right)dx = \sum_{i=1}^N \int_\Omega \left(a_i(x, \frac{\partial T_l(u_\epsilon)}{\partial x_i})\frac{\partial}{\partial x_i}T_k(u_\epsilon - \varphi)\right)dx,$$

with $l = k + \|\varphi\|_\infty$, then, by Lemma 3.8-(ii) and Lemma 4.1, we have

$$\lim_{\epsilon\to 0}\sum_{i=1}^N \int_\Omega \left(a_i(x, \frac{\partial T_l(u_\epsilon)}{\partial x_i})\frac{\partial}{\partial x_i}T_k(u_\epsilon - \varphi)\right)dx = \sum_{i=1}^N \int_\Omega \left(a_i(x, \frac{\partial T_l(u)}{\partial x_i})\frac{\partial}{\partial x_i}T_k(u - \varphi)\right)dx;$$

that is

$$\lim_{\epsilon\to 0}\sum_{i=1}^N \int_\Omega \left(a_i(x, \frac{\partial}{\partial x_i}u_\epsilon)\frac{\partial}{\partial x_i}T_k(u_\epsilon - \varphi)\right)dx = \sum_{i=1}^N \int_\Omega \left(a_i(x, \frac{\partial T_l(u)}{\partial x_i})\frac{\partial}{\partial x_i}T_k(u - \varphi)\right)dx. \tag{4.2}$$

For the second term in the left hand side of (4.1), we have

$$\limsup_{\epsilon\to 0}\sum_{i=1}^N \int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial}{\partial x_i}u_\epsilon|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon\frac{\partial}{\partial x_i}T_k(u_\epsilon - \varphi_N)\right)dx \ge 0. \tag{4.3}$$

Indeed

$$\begin{cases}
\sum_{i=1}^N \int_{\tilde{\Omega}\setminus\Omega} \left(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial}{\partial x_i}u_\epsilon|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon.\frac{\partial}{\partial x_i}T_k(u_\epsilon - \varphi_N)\right)dx \\
= \sum_{i=1}^N \int_{\tilde{\Omega}\setminus\Omega\cap[|u_\epsilon-\varphi|\le k]} \left(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial}{\partial x_i}u_\epsilon|^{p_i(x)-2}\frac{\partial}{\partial x_i}u_\epsilon.\frac{\partial}{\partial x_i}(u_\epsilon - \varphi_N)\right)dx \\
\sum_{i=1}^N \int_{\tilde{\Omega}\setminus\Omega\cap[|u_\epsilon-\varphi|\le k]} \left(\frac{1}{\epsilon^{p_i(x)}}|\frac{\partial}{\partial x_i}u_\epsilon|^{p_i(x)}\right)dx \ge 0.
\end{cases}$$

Hence, we get (4.3).

Let us examine the last term in the left hand side of (4.1).

we have

$$\int_\Omega b(u_\epsilon)T_k(u_\epsilon - \varphi)dx = \int_\Omega b(u_\epsilon) - b(\varphi))T_k(u_\epsilon - \varphi)dx + \int_\Omega b(\varphi)T_k(u_\epsilon - \varphi)dx.$$

As b is non-decreasing,

$$(b(u_\epsilon) - b(\varphi))T_k(u_\epsilon - \varphi) \ge 0 \text{ a.e. in } \Omega$$

and we get by Fatou's Lemma that

$$\liminf_{\epsilon\to 0}\int_\Omega (b(u_\epsilon)-b(\varphi))T_k(u_\epsilon-\varphi)dx \geq \int_\Omega (b(u)-b(\varphi))T_k(u-\varphi)dx.$$

As $\varphi\in L^\infty(\Omega)$, we obtain $b(\varphi)\in L^\infty(\Omega)$ and so $b(\varphi)\in L^1(\Omega)$ (as Ω is bounded) and by Lebesgue dominated convergence theorem, we deduce that

$$\lim_{\epsilon\to 0}\int_\Omega b(\varphi)T_k(u_\epsilon-\varphi)dx = \int_\Omega b(\varphi)T_k(u-\varphi)dx.$$

Consequently,

$$\limsup_{\epsilon\to 0}\int_\Omega b(u_\epsilon)T_k(u_\epsilon-\varphi)dx \geq \int_\Omega b(u)T_k(u-\varphi)dx. \tag{4.4}$$

It is easy to see by the Lebesgue generalized convergence theorem that

$$\begin{cases}\lim\limits_{\epsilon\to 0}\displaystyle\int_\Omega f_\epsilon T_k(u_\epsilon-\varphi)dx = \int_\Omega f T_k(u-\varphi)dx,\\ \lim\limits_{\epsilon\to 0}\displaystyle\int_{\tilde\Gamma_{Ne}} \tilde d_\epsilon T_k(u_\epsilon-\varphi_N)d\sigma = \int_\Omega \tilde d T_k(u-\varphi_N)d\sigma.\end{cases} \tag{4.5}$$

We know that $\forall k>0$, $T_k(u)=constant$ on $\tilde\Omega\setminus\Omega$, then, it yields that $u=constant$ on $\tilde\Omega\setminus\Omega$. So, one has

$$\lim_{\epsilon\to 0}\int_{\tilde\Gamma_{Ne}} \tilde d_\epsilon T_k(u_\epsilon-\varphi)dx = dT_k(u_N-\varphi_N). \tag{4.6}$$

At last, we have

$$\int_{\tilde\Gamma_{Ne}}\tilde\rho(u_\epsilon)T_k(u_\epsilon-\varphi_N)d\sigma = \int_{\tilde\Gamma_{Ne}}(\tilde\rho(u_\epsilon)-\tilde\rho(\varphi_N))T_k(u_\epsilon-\varphi_N)d\sigma + \int_{\tilde\Gamma_{Ne}}\tilde\rho(\varphi_N)T_k(u_\epsilon-\varphi_N)d\sigma.$$

As $\tilde\rho$ is non-decreasing,

$$(\tilde\rho(u_\epsilon)-\tilde\rho(\varphi_N))T_k(u_\epsilon-\varphi_N)\geq 0 \text{ a.e. on } \tilde\Gamma_{Ne}$$

and we get by Fatou's Lemma that

$$\begin{aligned}\liminf_{\epsilon\to 0}\int_{\tilde\Gamma_{Ne}}(\tilde\rho(u_\epsilon)-\tilde\rho(\varphi_N))T_k(u_\epsilon-\varphi_N)d\sigma &\geq \int_{\tilde\Gamma_{Ne}}(\tilde\rho(u_N)-\tilde\rho(\varphi_N))T_k(u_N-\varphi_N)d\sigma\\ &= (\rho(u_N)-\rho(\varphi_N))T_k(u_N-\varphi_N).\end{aligned}$$

As $\varphi_N\in L^\infty(\tilde\Gamma_{Ne})$, we obtain $\tilde\rho(\varphi_N)\in L^\infty(\tilde\Gamma_{Ne})$ and so $\tilde\rho(\varphi_N)\in L^1(\tilde\Gamma_{Ne})$ (as $\tilde\Gamma_{Ne}$ is bounded) and by the Lebesgue dominated convergence theorem, we deduce that

$$\lim_{\epsilon\to 0}\int_{\tilde\Gamma_{Ne}}\tilde\rho(\varphi_N)T_k(u_\epsilon-\varphi_N)d\sigma = \int_{\tilde\Gamma_{Ne}}\tilde\rho(\varphi_N)T_k(u_N-\varphi_N)d\sigma = \rho(\varphi_N)T_k(u_N-\varphi_N).$$

Hence,

$$\limsup_{\epsilon\to 0}\int_{\tilde\Gamma_{Ne}}\tilde\rho(u_\epsilon)T_k(u_\epsilon-\varphi_N)d\sigma \geq \rho(\varphi_N)T_k(u_N-\varphi_N). \tag{4.7}$$

Passing to the limit as $\epsilon\to 0$ in (4.1) and using (4.2)-(4.7), we see that u is an entropy solution of $P(\rho,f,d)$.

We now prove the uniqueness part of Theorem 2.4.

Let u and v be two entropy solutions of $P(\rho,f,d)$.

Let $h > 0$. For u, we take $\xi = T_h(v)$ as test function and for v, we take $\xi = T_h(u)$ as test function in (2.9), to get for any $k > 0$ with $k < h$,

$$\text{(4.8)} \qquad \begin{cases} \int_\Omega \left(\sum_{i=1}^N a_i(x, \frac{\partial}{\partial x_i}u) \frac{\partial}{\partial x_i} T_k(u - T_h(v)) \right) dx + \int_\Omega b(u) T_k(u - T_h(v)) dx \leq \\ \int_\Omega f T_k(u - T_h(v)) dx + (d - \rho(u_{Ne}) T_k(u_{Ne} - T_h(v)) \end{cases}$$

and

$$\text{(4.9)} \qquad \begin{cases} \int_\Omega \left(\sum_{i=1}^N a_i(x, \frac{\partial}{\partial x_i}v) \frac{\partial}{\partial x_i} T_k(v - T_h(u)) \right) dx + \int_\Omega b(v) T_k(v - T_h(u)) dx \leq \\ \int_\Omega f T_k(v - T_h(u)) dx + (d - \rho(v_{Ne}) T_k(v_{Ne} - T_h(u)). \end{cases}$$

By adding (4.8) and (4.9), we obtain

$$\text{(4.10)} \qquad \begin{cases} \int_\Omega \left(\sum_{i=1}^N a_i(x, \frac{\partial}{\partial x_i}u) \frac{\partial}{\partial x_i} T_k(u - T_h(v)) \right) dx \\ + \int_\Omega \left(\sum_{i=1}^N a_i(x, \frac{\partial}{\partial x_i}v) \frac{\partial}{\partial x_i} T_k(v - T_h(u)) \right) dx & := A(h,k) \\ + \int_\Omega b(u) T_k(u - T_h(v)) dx + \int_\Omega b(v) T_k(v - T_h(u)) dx & := B(h,k) \\ + \rho(u_{Ne}) T_k(u_{Ne} - T_h(v)) + \rho(v_{Ne}) T_k(v_{Ne} - T_h(u)) & := C(h,k) \\ \leq \int_\Omega f T_k(u - T_h(v)) dx + \int_\Omega f T_k(v - T_h(u)) dx & := D(h,k) \\ + d T_k(u_{Ne} - T_h(v)) + d T_k(v_{Ne} - T_h(u)) & := E(h,k). \end{cases}$$

Let us introduce the following subsets of Ω.

$$\begin{aligned} A_0 &:= [|u - v| < k, |u| < h, |v| < h] \\ A_1 &:= [|u - T_h(v)| < k, |v| \geq h] \\ A_1' &:= [|v - T_h(u)| < k, |u| \geq h] \\ A_2 &:= [|u - T_h(v)| < k, |u| \geq h, |v| < h] \\ A_2' &:= [|v - T_h(u)| < k, |v| \geq h, |u| < h]. \end{aligned}$$

We have

$$\begin{cases} A(h,k) = \\ \int_{A_0} \left(\sum_{i=1}^N \left(a_i(x, \frac{\partial}{\partial x_i}u) - a_i(x, \frac{\partial}{\partial x_i}v) \right) \frac{\partial}{\partial x_i}(u - v) \right) dx & := I_0(h,k) \\ + \int_{A_1} \left(\sum_{i=1}^N a_i(x, \frac{\partial}{\partial x_i}u) \frac{\partial}{\partial x_i} u \right) dx + \int_{A_1'} \left(\sum_{i=1}^N a_i(x, \frac{\partial}{\partial x_i}v) \frac{\partial}{\partial x_i} v \right) dx & := I_1(h,k) \\ + \int_{A_2} \left(\sum_{i=1}^N a_i(x, \frac{\partial}{\partial x_i}u) \frac{\partial}{\partial x_i}(u - v) \right) dx \\ + \int_{A_2'} \left(\sum_{i=1}^N a_i(x, \frac{\partial}{\partial x_i}v) \frac{\partial}{\partial x_i}(v - u) \right) dx & := I_2(h,k). \end{cases}$$

The term $I_1(h,k)$ is non-negative since each term in $I_1(h,k)$ is non-negative.

For the term $I_2(h,k)$, as

$$I_2(h,k) + \int_{A_2} \left(\sum_{i=1}^{N} a_i(x, \frac{\partial}{\partial x_i} u) \frac{\partial}{\partial x_i} v \right) dx + \int_{A_2'} \left(\sum_{i=1}^{N} a_i(x, \frac{\partial}{\partial x_i} v) \frac{\partial}{\partial x_i} u \right) dx = I_1(h,k),$$

so,

$$I_2(h,k) \geq - \left(\int_{A_2} \left(\sum_{i=1}^{N} a_i(x, \frac{\partial}{\partial x_i} u) \frac{\partial}{\partial x_i} v \right) dx + \int_{A_2'} \left(\sum_{i=1}^{N} a_i(x, \frac{\partial}{\partial x_i} v) \frac{\partial}{\partial x_i} u \right) dx \right).$$

Let us show that $-\left(\int_{A_2} \left(\sum_{i=1}^{N} a_i(x, \frac{\partial}{\partial x_i} u) \frac{\partial}{\partial x_i} v \right) dx \right)$ goes to 0 as $h \to \infty$.

We have

$$\begin{cases} \left| \int_{A_2} \left(\sum_{i=1}^{N} a_i(x, \frac{\partial}{\partial x_i} u) \frac{\partial}{\partial x_i} (v) \right) dx \right| \leq \\ C \sum_{i=1}^{N} \left(|j_i|_{p_i'(.)} + \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)-1}_{L^{p_i(.)}(\{h<|u|\leq h+k\})} \right) \left| \frac{\partial v}{\partial x_i} \right|_{L^{p_i(.)}(\{h-k<|v|\leq h\})} \end{cases}.$$

For all $i = 1, \dots N$, the quantity $\left(|j_i|_{p_i'(.)} + \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)-1}_{L^{p_i(.)}(\{h<|u|\leq h+k\})} \right)$ is finite since $u = T_{h+k}(u) \in \mathcal{T}^{1,\vec{p}(.)}_{Ne}(\Omega)$ and $j_i \in L^{p_i'(.)}(\Omega)$; then by Lemma 3.4, the last expression converges to zero as h tends to infinity.

Similarly we can show that $-\left(\int_{A_2} \left(\sum_{i=1}^{N} a_i(x, \frac{\partial}{\partial x_i} v) \frac{\partial}{\partial x_i} (u) \right) dx \right)$ goes to 0 as $h \to \infty$, hence, we obtain

$$\limsup_{h \to \infty} A(h,k) \geq \int_{[|u-v|<k]} \left[\sum_{i=1}^{N} \left(a_i(x, \frac{\partial}{\partial x_i} u) - a_i(x, \frac{\partial}{\partial x_i} v) \right) \frac{\partial}{\partial x_i} (u-v) \right] dx. \tag{4.11}$$

By using the Lebesgue dominated convergence theorem, it yields that

$$\lim_{h \to \infty} B(h,k) = \int_{\Omega} (b(u) - b(v)) T_k(u-v) dx \text{ and } \lim_{h \to \infty} D(h,k) = 0. \tag{4.12}$$

For h large enough, we get

$$\lim_{h \to \infty} C(h,k) = (\rho(u_N) - \rho(v_N)) T_k(u_N - v_N) \text{ and } \lim_{h \to \infty} E(h,k) = 0. \tag{4.13}$$

Letting h goes to ∞ in (4.10) and combining (4.12)-(4.13), we obtain

$$\begin{cases} \int_{[|u-v|<k]} \left[\sum_{i=1}^{N} \left(a_i(x, \frac{\partial}{\partial x_i} u) - a_i(x, \frac{\partial}{\partial x_i} v) \right) \frac{\partial}{\partial x_i} (u-v) \right] dx \\ + \int_{\Omega} (b(u) - b(v)) T_k(u-v) dx + (\rho(u_N) - \rho(v_N)) T_k(u_N - v_N) \leq 0. \end{cases} \tag{4.14}$$

All the terms in the left hand side of (4.14) are non-negative so that we get $\forall k > 0$,

$$\int_{[|u-v|<k]} \left[\sum_{i=1}^{N} \left(a_i(x, \frac{\partial}{\partial x_i} u) - a_i(x, \frac{\partial}{\partial x_i} v) \right) \frac{\partial}{\partial x_i} (u-v) \right] dx = 0 \tag{4.15}$$

and

$$\begin{cases} \int_{\Omega} (b(u) - b(v)) T_k(u-v) dx = 0 \\ (\rho(u_N) - \rho(v_N)) T_k(u_N - v_N) = 0. \end{cases} \tag{4.16}$$

Relation (4.15) gives $\frac{\partial}{\partial x_i}(u-v)=0$ a.e. in Ω; we deduce that there exists a constant c such that $u-v=c$ a.e. in Ω. We deduce from (4.16) $b(u)=b(v)=0$ a.e. in Ω. As b is invertible,

$$\begin{cases} u=v \text{ a.e. in } \Omega \\ \rho(u_N)=\rho(v_N); \end{cases}$$

which prove the uniqueness part.

References

[1] M. Bendahmane and K. H. Karlsen; Anisotropic nonlinear elliptic systems with measure data and anisotropic harmonic maps into spheres. Electron. J. Differ. Equ. 2006 (2006), 46.

[2] Ph. Bénilan, H. Brézis and M. G. Crandall; A semilinear equation in $L^1(\mathbb{R}^N)$. Ann. Scuola. Norm. Sup. Pisa, 2 (1975), 523-555.

[3] B. K. Bonzi, S. Ouaro and F. D. Y. Zongo; Entropy solution for nonlinear elliptic anisotropic homogeneous Neumann Problem, Int. J. Differ. Equ. 2003 (2013), Article ID 476781.

[4] M. M. Boureanu and V. D. Radulescu; Anisotropic Neumann problems in Sobolev spaces with variable exponent. Nonlinear Anal. TMA 75 (12) (2012), 4471-4482.

[5] Y. Ding, T. Ha-Duong, J. Giroire, V. Moumas, Modeling of single-phase flow for horizontal wells in a stratified medium. Comput. Fluids, 33 (2004), 715-727.

[6] J. Giroire, T. Ha-Duong, V. Moumas, A non-linear and non-local boundary condition for a diffusion equation in petroleum engineering. Math. Methods Appl. Sci. 28 (13) (2005), 1527-1552.

[7] P. Halmos, Measure Theory, D. Van Nostrand Company, New York, 1950.

[8] B. Koné, S. Ouaro and F. D. Y. Zongo; Nonlinear elliptic anisotropic problem with fourier boundary condition , Int. J. Evol. Equ. 8 (4) (2013), 305-328.

[9] X. Fan and D. Zhao; On the spaces $L^{p(\cdot)}(\Omega)$ and $W^{m,p(\cdot)}(\Omega)$. J. Math. Anal. Appl. 263 (2001), 424-446.

[10] X. Fan; Anisotropic variable exponent Sobolev spaces and $\vec{p}(.)$ -Laplacian equations. Complex Var. Elliptic Equ. 56 (7-9) (2011), 623-642.

[11] M. Sanchon and J. M. Urbano; Entropy solutions for the $p(x)$-Laplace Equation. Trans. Amer. Math. Soc. 361 (12) (2009), 6387-6405.

[12] M. Troisi; teoremi di inclusione per spazi di Sobolev non isotropi. Ric. Mat. 18 (1969), 3-24.

[13] R. E. Showalter; Monotone operators in Banach space and nonlinear partial differential equations, Mathematical Surveys and Monographs, vol. 49, American Mathematical Society, Providence, RI, 1997.