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A PRIORI ERROR ESTIMATE OF A DISCONTINUOUS FINITE VOLUME METHOD FOR THE OBSTACLE PROBLEM

YUPING ZENG*, TING LI

School of Mathematics, Jiaying University, Meizhou 514015, China

*Corresponding author: zeng_yuping@163.com

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Abstract. The present paper designs a discontinuous finite volume method for the numerical approximation of the obstacle problem. In addition, an optimal a priori error estimate in the energy norm is derived assuming some regularity conditions.

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1. Introduction

The obstacle problem describes the membrane deformation phenomenon that is an important example of a variational inequality of the first kind. The theory of variational inequalities has been made applications in diverse fields such as mechanics, physics, and operations research (cf. [27]). Due to the variational property, finite element (FE) method is a natural way to approximate the obstacle problem. In the last four decades, there has been made impressive progress in this field. More precisely, see [7, 10, 14, 22, 23, 40, 45, 56] for the conforming FE methods, and the nonconforming FE methods can be found in [11, 39, 46]. Recently, discontinuous Galerkin (DG) [1] methods have also been developed to solve the obstacle problem (cf. [2, 19, 24, 41, 42, 53]).

Finite volume (FV) method is an efficient discrete scheme for solving partial differential equations (PDEs). By integrating the PDEs on a control volume, the FV method satisfies some conservation property such as mass, momentum, or energy. Thus, FV method become

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a popular approach in computational fluid mechanics. For more details of FV methods, we refer to the monographs [21,32], the papers [4,5,8,13,15–17,20,25,26,36,37,48–50,54,55] and references therein. Discontinuous finite volume (DFV) methods were originally introduced by Ye [51] for solving the second order elliptic equations, therein a priori error estimate in a mesh-dependent norm was derived. Inheriting attractive features of both DG and FV methods, such methods can easily handle complicate geometries and inhomogeneous boundary conditions, and they satisfy some conservative properties. Another interesting feature of DFV methods is their localizability of discontinuous elements and the corresponding dual partitions, this make them suitable for parallel computing. In addition, compared to the classical conforming and nonconforming FV methods, DFV methods have small support in the control volume of the dual mesh. For these reasons, DFV methods have further been investigated to solve various PDEs, such as second order elliptic equations [3,12,28,35], Stokes equations [9,18,30,44,52], Darcy-Stokes problems [34,43], Biot equations [29], phase field model [33], optimal control problems [31,38] and so on.

In contrast to huge literature on FE methods for variational inequalities, the work on FV methods is considerably less developed. More recently, a conforming FV method is designed to solve two kinds of variational inequalities including the obstacle and simplified frictional problems [58]. Later, this method is extend to the Signorini problem and a super-close interpolation estimate was obtained (see [57]). The objective of this work is to apply the discontinuous finite volume (DFV) method to solve the obstacle problem and give an optimal error estimate in the energy norm. It is worth mentioning that we shall address two difficulties, one arises from the inherent nonlinearity of variational inequality, and the other comes from the complexity of bilinear form of DFV methods.

The rest of the paper is organized as follows. In Section 2, we introduce the model problem and state the corresponding DFV numerical scheme. Next, in Section 3 we give an optimal a priori error analysis in the mesh-dependent norm. Finally, some conclusions are made in Section 4.

2. The DFV method for the obstacle problem

2.1. **Obstacle problem.** We begin by introducing some notation. For $\mathcal{O} \in \mathbb{R}^2$, let $H^k(\mathcal{O})$, $k \geq 0$ be the usual Sobolev space with the norm $\|\cdot\|_{k,\mathcal{O}}$ and the seminorm $|\cdot|_{k,\mathcal{O}}$. When $\mathcal{O} = \Omega$, we omit the index Ω. For $k = 0$, $H^0(\mathcal{O})$ is reduced to the Lebesgue space $L^2(\mathcal{O})$.

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal domain with Lipschitz boundary Γ. For $f \in L^2(\Omega)$, we consider the elliptic obstacle problem: Find $u \in K$ such that

$$\int_\Omega \nabla u \cdot \nabla(v-u)dx \geq \int_\Omega f(v-u)dx \quad \forall v \in K, \tag{1}$$

with

$$K = \{v \in H_g^1(\Omega) : v \geq \psi \quad a.e. \quad \text{in} \quad \Omega\}.$$

Here $H_g^1(\Omega) = \{v \in H^1(\Omega) : v = g \text{ on } \Gamma\}$, with g being the restriction of an $H^2(\Omega)$ function to Γ. $\psi \in H^2(\Omega)$ is referred to the obstacle function that satisfies $\psi \leq g$ on Γ.

2.2. **DFV method.** Let $\mathcal{T}_h$ ba a shape-regular mesh which decomposes Ω into triangular elements $\{T\}$. Denote by $h_T = \text{diam}(T)$ and let $h = \max_{T\in\mathcal{T}_h} h_T$. $\mathcal{E}_h^I$ stands for the set of interior edges of elements in $\mathcal{T}_h$, and $\mathcal{E}_h^\partial$ is the set of edges on the boundary Γ. Thus, the set of all edges $\mathcal{E}_h = \mathcal{E}_h^I \cup \mathcal{E}_h^\partial$. We assign each edge $e \in \mathcal{E}_h$ an unit normal $\mathbf{n}$, such that on $e \in \mathcal{E}_h^\partial$, $\mathbf{n}$ refers to the outward unit normal. In addition, we divide each element T by connecting its vertices and barycenter to obtain a dual mesh $\mathcal{T}_h^*$, see Fig.1. In what follows, we shall use C, with or without subscripts to denote generic constants independent of h, but depend on the minimum angle of T.

For a discontinuous function v, on each interior edge $e \in \mathcal{E}_h^I$ shared by two elements T^+ and T^-, let $v^\pm = v|_{e\cap\partial T^\pm}$, we define the average and jump by

$$\{v\} = \frac{1}{2}(v^+ + v^-) \text{ and } [\![v]\!] = v^+ - v^-.$$

On a boundary edge $e \in \mathcal{E}_h^\partial$, we define

$$\{v\} = v \text{ and } [\![v]\!] = v.$$

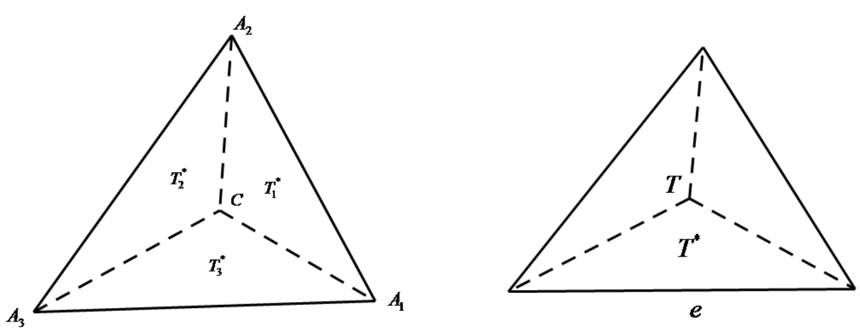

FIGURE 1. A triangular partition and its dual volume for DFV method.

We consider the discontinuous $\mathbb{P}_1$ finite dimensional space for trial functions associated with $\mathcal{T}_h$:

$$V_h = \{v \in L^2(\Omega) : v|_T \in \mathbb{P}_1(T) \quad \forall T \in \mathcal{T}_h\}.$$

where $\mathbb{P}_k(T)$ is the space of polynomials of total degree at most k on T. Then, an approximation of the set K is defined by

$$K_h = \{v_h \in V_h : v_h|_T(p) \geq \psi(p) \quad \forall p \in \mathcal{V}_T \quad \forall T \in \mathcal{T}_h\},$$

with $\mathcal{V}_T$ being the set of three vertices of T.

On the other hand, for test functions associated with dual mesh $\mathcal{T}_h^*$, we use the finite dimensional space:

$$V_h^* = \{v \in L^2(\Omega) : v|_{T^*} \in \mathbb{P}_0(T^*) \quad \forall T^* \in \mathcal{T}_h^*\}.$$

Let $V(h) = V_h + [H_g^1(\Omega) \cap H^2(\Omega)]$, in order to connect $V(h)$ to V_h^*, we define the mapping $\gamma_h : V(h) \to V_h^*$ by

$$\gamma_h v\big|_{T^*} = \frac{1}{h_e}\int_e v|_{T^*} ds \quad \forall T^* \in \mathcal{T}_h^*,$$

where h_e is the length of the edge e.

Following [51], we define the bilinear form with regard to the DFV method:

$$\begin{aligned} a_h(w,v) =& A(w,v) - \sum_{e\in\mathcal{E}_h}\int_e \{\nabla w \cdot \mathbf{n}\}[\![\gamma_h v]\!] ds - \sum_{e\in\mathcal{E}_h}\int_e \{\nabla v \cdot \mathbf{n}\}[\![\gamma_h w]\!] ds \\ &+ \sum_{e\in\mathcal{E}_h} \rho_e h_e^{-1}\int_e [\![\gamma_h w]\!][\![\gamma_h v]\!] ds, \end{aligned} \tag{2}$$

with $A(w,v) = -\sum_{T^*\in\mathcal{T}_h^*}\int_{\partial T^*}(\nabla w \cdot \mathbf{n})\gamma_h v ds + \sum_{T\in\mathcal{T}_h}\int_{\partial T}(\nabla w\cdot\mathbf{n})\gamma_h v ds.$

The DFVM for solving the obstacle problem (1) is to find $u_h \in K_h$ such that

$$\begin{aligned} a_h(u_h, v_h - u_h) \geq& \int_\Omega f(\gamma_h v_h - \gamma_h u_h) dx - \sum_{e\in\mathcal{E}_h^\partial}\int_e \gamma_h g(\nabla(v_h - u_h)\cdot\mathbf{n}) ds \\ &+ \sum_{e\in\mathcal{E}_h^\partial}\rho_e h_e^{-1}\int_e \gamma_h g(\gamma_h v_h - \gamma_h u_h) ds \quad \forall v_h \in K_h. \end{aligned} \tag{3}$$

3. A priori error estimates

This subsection is devoted to a priori error estimate of the numerical scheme (3). To proceed, we first define the mesh-dependent norm $\|\cdot\|_h$ on $V(h)$ (cf. [51]):

$$\|v\|_h = \left(\sum_{T\in\mathcal{T}_h}\|\nabla v\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h}[\![\gamma_h v]\!]^2 + \sum_{T\in\mathcal{T}_h} h_T^2|v|_{2,T}^2\right)^{1/2}. \tag{4}$$

It can be proved that $a_h(\cdot,\cdot)$ satisfies the following properties (see Lemmas 2.2 and 2.3 in [51] for more details).

Lemma 3.1. *There holds*

$$|a_h(w,v)| \leq C_\alpha \|w\|_h \|v\|_h \quad \forall w,v \in V(h). \tag{5}$$

Furthermore, we can choose large enough ρ_e to satisfy

$$a_h(v,v) \geq C_\beta \|v\|_h^2 \quad \forall v \in V_h. \tag{6}$$

We also need some useful results with respect to the mapping γ_h (cf. [17]).

Lemma 3.2.

$$\text{if } [\![v]\!] = 0, \quad \text{then} \quad [\![\gamma_h v]\!] = 0, \tag{7}$$

$$\|v - \gamma_h v\|_{0,T} \leq C h_T \|\nabla v\|_{0,T} \quad \forall v \in V(h), \tag{8}$$

Let u_I be the the continuous linear interpolation of u, it satisfies [6,47]:

$$\|u - u_I\|_{m,T} \leq C h^{2-m} |u|_{2,T} \quad m = 0,1,2. \tag{9}$$

In addition, we recall the following trace inequality (see e.g. [1]):

$$\|w\|_{0,e}^2 \leq C\big(h_e^{-1}\|w\|_{0,T}^2 + h_e \|\nabla w\|_{0,T}^2\big) \quad \forall w \in H^1(T), \tag{10}$$

with e any edge of the element T.

Observing that $[\![u]\!] = 0$ and $[\![u_I]\!] = 0$ on any $e \in \mathcal{E}_h^I$, from (7) we see that $[\![\gamma_h u]\!] = 0$ and $[\![\gamma_h u_I]\!] = 0$ on any $e \in \mathcal{E}_h^I$. Therefore,

$$\|u - u_I\|_h = \Bigg(\sum_{T \in \mathcal{T}_h} (\|\nabla(u-u_I)\|_{0,T}^2 + h_T^2 |u - u_I|_{2,T}^2) + \sum_{e \in \mathcal{E}_h^\partial} [\![\gamma_h (u - u_I)]\!]^2 \Bigg)^{1/2}. \tag{11}$$

In view of the definition of γ_h, Cauchy-Schwarz inequality and trace inequality (10), we deduce that

$$\begin{aligned}\sum_{e\in\mathcal{E}_h^\partial}[\![\gamma_h(u-u_I)]\!]^2 &= \sum_{e\in\mathcal{E}_h^\partial} h_e^{-2}\Big(\int_e (u-u_I)ds\Big)^2\\ &\le \sum_{e\in\mathcal{E}_h^\partial} h_e^{-2}\int_e 1^2 ds\int_e (u-u_I)^2 ds\\ &= \sum_{e\in\mathcal{E}_h^\partial} h_e^{-1}\|u-u_I\|_{0,e}^2\\ &\le C\sum_{T\in\mathcal{T}_h}\Big(h_e^{-2}\|u-u_I\|_{0,T}^2+\|\nabla(u-u_I)\|_{0,T}^2\Big). \end{aligned} \tag{12}$$

Inserting (12) into (11), and using (9) and the shape-regularity of the mesh, we infer that

$$\|u-u_I\|_h \le Ch|u|_2. \tag{13}$$

We are now in a position to state our main result. Applying some techniques developed in [7, 53], we prove that the DFV method has an optimal convergence rate in the mesh-dependent norm.

Theorem 3.3. *Let u and u_h be the solutions of (1) and (3), respectively. Assume that $u \in H^2(\Omega)$, it holds that*

$$\|u-u_h\|_h \le Ch\big(|u|_2+|\psi|_2+\|\Delta u+f\|_0\big). \tag{14}$$

Proof. The triangle inequality yields

$$\|u-u_h\|_h \le \|u-u_I\|_h+\|u_I-u_h\|_h. \tag{15}$$

The bound of $\|u-u_I\|_h$ have been obtained in (13), we only need to estimate $\|u_I-u_h\|_h$. It follows from (6) that

$$C_\beta\|u_I-u_h\|_h^2 \le a_h(u_I-u_h, u_I-u_h) \equiv \mathbb{B}_1+\mathbb{B}_2, \tag{16}$$

where

$$\mathbb{B}_1 = a_h(u_I-u, u_I-u_h),$$

$$\mathbb{B}_2 = a_h(u-u_h, u_I-u_h).$$

For the first term $\mathbb{B}_1$, application of (5) and Young's inequality to find that

$$\mathbb{B}_1 \le C_\alpha\|u_I-u\|_h\|u_I-u_h\|_h \le \frac{C_\alpha}{4\epsilon_1}\|u_I-u\|_h^2+C_\alpha\epsilon_1\|u_I-u_h\|_h^2. \tag{17}$$

It remains to estimate the second term $\mathbb{B}_2$. Noting that $[\![u]\!] = 0$ $(\forall e \in \mathcal{E}_h^I)$, this together with (7) gives $[\![\gamma_h u]\!] = 0$ $(\forall e \in \mathcal{E}_h^I)$. In addition, since V_h^* is piecewise constant space and $[\![\nabla u \cdot \mathbf{n}]\!] = 0$ $(\forall e \in \mathcal{E}_h^I)$, we use integration by parts to obtain

$$
\begin{aligned}
& a_h(u, u_I - u_h) \\
=& -\sum_{T^* \in \mathcal{T}_h^*} \int_{\partial T^*} (\nabla u \cdot \mathbf{n}) \gamma_h (u_I - u_h) ds + \sum_{T \in \mathcal{T}_h} \int_{\partial T} (\nabla u \cdot \mathbf{n}) \gamma_h (u_I - u_h) ds \\
& - \sum_{e \in \mathcal{E}_h} \int_e \{\nabla u \cdot \mathbf{n}\} [\![\gamma_h (u_I - u_h)]\!] ds - \sum_{e \in \mathcal{E}_h^\partial} \int_e \gamma_h g \nabla (u_I - u_h) ds \\
& + \sum_{e \in \mathcal{E}_h^\partial} \rho_e h_e^{-1} \int_e \gamma_h g \gamma_h (u_I - u_h) ds \\
=& \sum_{T^* \in \mathcal{T}_h^*} \int_{T^*} -\Delta u \gamma_h (u_I - u_h) dx + \sum_{e \in \mathcal{E}_h} \int_e \{\nabla u \cdot \mathbf{n}\} [\![\gamma_h (u_I - u_h)]\!] ds \\
& + \sum_{e \in \mathcal{E}_h^I} \int_e [\![\nabla u \cdot \mathbf{n}]\!] \{\gamma_h (u_I - u_h)\} ds - \sum_{e \in \mathcal{E}_h} \int_e \{\nabla u \cdot \mathbf{n}\} [\![\gamma_h (u_I - u_h)]\!] ds \\
& - \sum_{e \in \mathcal{E}_h^\partial} \int_e \gamma_h g \nabla (u_I - u_h) ds + \sum_{e \in \mathcal{E}_h^\partial} \rho_e h_e^{-1} \int_e \gamma_h g \gamma_h (u_I - u_h) ds \\
=& \sum_{T \in \mathcal{T}_h} \int_T -\Delta u \gamma_h (u_I - u_h) dx - \sum_{e \in \mathcal{E}_h^\partial} \int_e \gamma_h g \nabla (u_I - u_h) ds \\
& + \sum_{e \in \mathcal{E}_h^\partial} \rho_e h_e^{-1} \int_e \gamma_h g \gamma_h (u_I - u_h) ds.
\end{aligned} \tag{18}
$$

On the other hand, setting $v_h = u_I$ in (3) shows that

$$
\begin{aligned}
a_h(u_h, u_I - u_h) \geq & \int_\Omega f \gamma_h (u_I - u_h) dx - \sum_{e \in \mathcal{E}_h^\partial} \int_e \gamma_h g \nabla (u_I - u_h) ds \\
& + \sum_{e \in \mathcal{E}_h^\partial} \rho_e h_e^{-1} \int_e \gamma_h g \gamma_h (u_I - u_h) ds.
\end{aligned} \tag{19}
$$

Then, we infer from (18) and (19) that

$$
\begin{aligned}
\mathbb{B}_2 =& a_h(u - u_h, u_I - u_h) \\
\leq & \sum_{T \in \mathcal{T}_h} \int_T -(\Delta u + f) \gamma_h (u_I - u_h) dx
\end{aligned}
$$

$$
\begin{aligned}
&=\Big[\sum_{T\in\mathcal{T}_h}\int_T -(\Delta u+f)\gamma_h(u_I-u_h)dx-\sum_{T\in\mathcal{T}_h}\int_T -(\Delta u+f)(u_I-u_h)dx\Big]\\
&\quad+\sum_{T\in\mathcal{T}_h}\int_T -(\Delta u+f)(u_I-u_h)dx\\
&\equiv \mathbb{B}_{21}+\mathbb{B}_{22}. \qquad (20)
\end{aligned}
$$

For simplicity, let $\vartheta = u_I - u_h$, $\mathbb{B}_{21}$ can be rewritten as

$$\mathbb{B}_{21}=\sum_{T\in\mathcal{T}_h}\int_T -(\Delta u+f)(\gamma_h\vartheta-\vartheta)ds. \qquad (21)$$

In term of (8), Cauchy-Schwarz inequality and Young's inequality, we can estimate $\mathbb{B}_{21}$ by

$$
\begin{aligned}
\mathbb{B}_{21}&=\sum_{T\in\mathcal{T}_h}\int_T -(\Delta u+f)(\gamma_h\vartheta-\vartheta)ds\\
&\le\Big(\sum_{T\in\mathcal{T}_h}\|\Delta u+f\|_{0,T}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h}\|\gamma_h\vartheta-\vartheta\|_{0,T}^2\Big)^{1/2}\\
&\le\Big(\sum_{T\in\mathcal{T}_h}\|\Delta u+f\|_{0,T}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h}C_1h_T^2|\vartheta|_{1,T}^2\Big)^{1/2}\\
&\le C_2h\|\Delta u+f\|_0\|u_I-u_h\|_h\\
&\le\frac{C_2}{4\epsilon_2}h^2\|\Delta u+f\|_0+C_2\epsilon_2\|u_I-u_h\|_h^2. \qquad (22)
\end{aligned}
$$

Next, we shall give the bound for $\mathbb{B}_{22}$. First, we recall the following well-known result [7]

$$-\Delta u\ge f,\ u\ge\psi,\ (-\Delta u-f)(u-\psi)=0\ a.e.\ \text{in}\ \Omega. \qquad (23)$$

By adding and subtracting some terms, we have

$$
\begin{aligned}
\mathbb{B}_{22}&=\sum_{T\in\mathcal{T}_h}\int_T -(\Delta u+f)(u_I-u_h)dx\\
&=\sum_{T\in\mathcal{T}_h}\int_T -(\Delta u+f)(u_I-u)dx+\sum_{T\in\mathcal{T}_h}\int_T -(\Delta u+f)(u-\psi)dx\\
&\quad+\sum_{T\in\mathcal{T}_h}\int_T -(\Delta u+f)(\psi_I-u_h)dx+\sum_{T\in\mathcal{T}_h}\int_T -(\Delta u+f)(\psi-\psi_I)dx. \qquad (24)
\end{aligned}
$$

Due to (23), the second term in the right hand of (24) satisfies

$$\sum_{T\in\mathcal{T}_h}\int_T -(\Delta u+f)(u-\psi)dx=0. \qquad (25)$$

On the other hand, by the definition of K_h, it holds that $u_h|_T(p) \geq \psi(p) = \psi_I(p)$, $(\forall p \in \mathcal{V}_T,\ \forall T \in \mathcal{T}_h)$. Since we use linear finite element, it follows that $\psi_I - u_h \leq 0$ on any $T \in \mathcal{T}_h$. Consequently, we have

$$\sum_{T\in\mathcal{T}_h} \int_T -(\Delta u + f)(\psi_I - u_h)dx \leq 0. \tag{26}$$

Inserting (25) and (26) into (24) yields

$$\mathbb{B}_{22} \leq \sum_{T\in\mathcal{T}_h} \int_T -(\Delta u + f)(u_I - u)dx + \sum_{T\in\mathcal{T}_h} \int_T -(\Delta u + f)(\psi - \psi_I)dx. \tag{27}$$

Utilizing the interpolation error estimates (9) and Cauchy-Schwarz inequality implies that

$$\mathbb{B}_{22} \leq C_3 h^2 \|\Delta u + f\|_0 \Big(|u|_2 + |\psi|_2\Big). \tag{28}$$

Combining (16), (17), (20), (22) and (28), we infer that

$$\begin{aligned} &\Big(C_\beta - C_\alpha \epsilon_1 - C_2\epsilon_2)\|u_I - u_h\|_h^2 \\ &\leq \frac{C_\alpha}{4\epsilon_1}\|u_I - u\|_h^2 + \frac{C_2}{4\epsilon_2} h^2 \|\Delta u + f\|_0 + C_3 h^2 \|\Delta u + f\|_0 \big(|u|_2 + |\psi|_2\big). \end{aligned} \tag{29}$$

Choosing appropriate parameters ϵ_i $(i = 1, 2)$ such that $C_\beta - C_\alpha\epsilon_1 - C_2\epsilon_2 > 0$, and combining (13), (15) and (29), we obtain the desired estimate (14). □

4. Conclusion

We introduced and analyzed a discontinuous finite volume method to solve the obstacle problem. A detailed a priori error estimate in the energy norm was established. Nature extension of this work includes a posteriori error estimate, and we shall also extend such method to variational inequality of the second kind in the future work.

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