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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR NONLINEAR IMPLICIT HADAMARD FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS IN A WEIGHTED BANACH SPACE

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Abstract. The aim of this paper is to prove the existence and uniqueness of solutions for a nonlinear implicit Hadamard fractional differential equation with nonlocal conditions in a weighted Banach space. Our results are based on the Banach and Krasnoselskii fixed point theorems. An example is given to illustrate our obtained results.

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1. Introduction

Fractional differential equations arise from a variety of applications including in various fields of science and engineering. In particular, problems concerning qualitative analysis of fractional differential equations have received the attention of many authors, see [1]– [25], [27]– [31] and the references therein.

Fractional differential equations involving Riemann-Liouville and Caputo fractional derivatives have been studied extensively by several researchers. However, the literature on Hadamard differential equations is not yet as enriched. The fractional derivative due to Hadamard differs from the aforementioned derivatives in the sense that the kernel of the integral in the definition of Hadamard fractional derivative contains a logarithmic function of arbitrary exponent, see [4–6, 10, 16].

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In [20], Lachouri et al. discussed the existence and uniqueness of solutions of the following implicit fractional differential equation with nonlocal conditions

$$\begin{cases} D_{0^+}^{\alpha} x(t) = f\left(t, x(t), D_{0^+}^{\alpha} x(t)\right), \ t \in (0,T], \\ t^{1-\alpha} x(t)|_{t=0} = x_0 - g(x), \ x_0 \in \mathbb{R}, \end{cases}$$

where $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative of order $0 < \alpha < 1$, $f : (0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : C((0,T], \mathbb{R}) \to \mathbb{R}$ are continuous functions. By employing the Banach and Krasnoselskii fixed point theorems, the authors obtained existence and uniqueness results.

In this paper, we extend the results in [20] by proving the existence and uniqueness of solutions for the following implicit Hadamard fractional differential equation with nonlocal conditions in a weighted Banach space

$$\begin{cases} D_{1^+}^{\alpha} x(t) = f\left(t, x(t), D_{1^+}^{\alpha} x(t)\right), \ t \in (1,T], \\ (\log t)^{1-\alpha} x(t)\big|_{t=1} = x_0 - g(x), \ x_0 \in \mathbb{R}, \end{cases} \tag{1}$$

where $D_{1^+}^{\alpha}$ is the Hadamard fractional derivative of order $0 < \alpha < 1$, $f : (1,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : C((1,T], \mathbb{R}) \to \mathbb{R}$ are continuous nonlinear functions. To prove the existence and uniqueness of solutions in a weighted Banach space, we transform (1) into an equivalent integral equation and then use the Banach and Krasnoselskii fixed point theorems. Finally, we provide an example to illustrate our obtained results.

2. Preliminaries

Let $T > 1$, $J = [1,T]$. By $C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|x\|_{\infty} = \sup\{|x(t)| : t \in J\}.$$

In what follows $\gamma > 0$, we consider the weighted space of continuous functions

$$C_{\gamma}(J, \mathbb{R}) = \{x : (1,T] \to \mathbb{R} : (\log t)^{\gamma} x \in C(J, \mathbb{R})\},$$

with the norm

$$\|x\|_{C_{\gamma}} = \sup_{t \in J} |(\log t)^{\gamma} x(t)|.$$

Clearly $C_{\gamma}(J, \mathbb{R})$ is a Banach space.

Definition 1 ([16]). *The Hadamard fractional integral of order $\alpha > 0$ of a function $x : J \to \mathbb{R}$ is given by*

$$I_{1^+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} x(s) \frac{ds}{s},$$

where Γ is the gamma function.

Definition 2 ([16]). *The Hadamard fractional derivative of order $\alpha > 0$ of $x : J \to \mathbb{R}$ is defined by*

$$D_{1^+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} x(s) \frac{ds}{s},$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of real number α.

Lemma 1 ([16]). *The general solution of linear fractional differential equation*

$$D_{1^+}^{\alpha} x(t) = 0,$$

is given by

$$x(t) = c_1 (\log t)^{\alpha-1} + c_2 (\log t)^{\alpha-2} + c_3 (\log t)^{\alpha-3} + ... + c_n (\log t)^{\alpha-n}, \ c_i \in \mathbb{R}, \ i = 1, 2, ..., n,$$

where $n = [\alpha] + 1$.

Lemma 2 ([16]). *We have*

$$I_{1^+}^{\alpha} (\log t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (\log t)^{\alpha+\beta-1}, \ \alpha \geq 0, \ \beta > 0.$$

Theorem 1 (Banach's fixed point theorem [26]). *Let Ω be a non-empty closed convex subset of a Banach space $(S, \|.\|)$, then any contraction mapping Φ of Ω into itself has a unique fixed point.*

Theorem 2 (Krasnoselskii's fixed point theorem [26]). *Let Ω be a non-empty closed bounded convex subset of a Banach space $(S, \|.\|)$. Suppose that F_1 and F_2 map Ω into S such that*

(i) $F_1 x + F_2 y \in \Omega$ for all $x, y \in \Omega$,

(ii) F_1 is continuous and compact,

(iii) F_2 is a contraction with constant $l < 1$.

Then there is a $z \in \Omega$ with $F_1 z + F_2 z = z$.

3. Existence and uniqueness

Definition 3. *A function* $x \in C^1((1,T],\mathbb{R})$ *is said to be a solution of* (1) *if* x *satisfies* $D_{1+}^{\alpha}x(t) = f\left(t, x(t), D_{1+}^{\alpha}x(t)\right)$ *for any* $t \in (1,T]$ *and* $\left.(\log t)^{1-\alpha}x(t)\right|_{t=1} = x_0 - g(x)$.

For the existence of solutions for the problem (1), we need the following auxiliary lemma.

Lemma 3. *The function* x *solves* (1) *if and only if it is a solution of the integral equation*

$$x(t) = (\log t)^{\alpha-1}(x_0 - g(x)) + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f\left(s, x(s), D_{1+}^{\alpha}x(s)\right)\frac{ds}{s},\ t \in (1,T]. \tag{2}$$

Proof. Suppose the function x satisfies (1), then applying I_{1+}^{α} to both sides of (1), we get

$$I_{1+}^{\alpha}D_{1+}^{\alpha}x(t) = I_{1+}^{\alpha}f\left(t, x(t), D_{1+}^{\alpha}x(t)\right).$$

By using Lemma 1, we obtain

$$x(t) = c_1(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f\left(s, x(s), D_{1+}^{\alpha}x(s)\right)\frac{ds}{s}. \tag{3}$$

The condition $\left.(\log t)^{1-\alpha}x(t)\right|_{t=1} = x_0 - g(x)$ implies that

$$c_1 = x_0 - g(x). \tag{4}$$

Substituting (4) in (3) we get the integral equation (2). The converse can be proven by direct computations. The proof is completed. □

In the following subsections we prove existence, as well as existence and uniqueness results, for the problem (1) by using a variety of fixed point theorems.

The following assumptions will be used in our main results.

(H1) There exist constants $k_1 > 0$ and $k_2 \in (0,1)$ such that

$$\left|f(t,u,v) - f(t,u^*,v^*)\right| \le k_1|u-u^*| + k_2|v-v^*|,$$

for $t \in (1,T]$, $u, v, u^*, v^* \in \mathbb{R}$ and $f(.,0,0) \in C_{1-\alpha}(J,\mathbb{R})$.

(H2) There exist a constant $b \in (0,1)$ such that

$$\left|g(u) - g(u^*)\right| \le b\left\|u-u^*\right\|_{C_{1-\alpha}},$$

for $u, u^* \in C_{1-\alpha}(J,\mathbb{R})$.

3.1. **Existence and uniqueness results via Banach's fixed point theorem.**

Theorem 3. *Suppose that* (*H1*) *and* (*H2*) *hold. If*

$$b+\frac{\Gamma\left(\alpha\right)k_{1}\left(\log T\right)^{\alpha}}{\Gamma\left(2\alpha\right)\left(1-k_{2}\right)}<1, \tag{5}$$

then there exists a unique solution for (1) *in the space* $C_{1-\alpha}\left(J,\mathbb{R}\right)$.

Proof. We define the operator $\Phi:C_{1-\alpha}\left(J,\mathbb{R}\right)\rightarrow C_{1-\alpha}\left(J,\mathbb{R}\right)$ by

$$\left(\Phi x\right)\left(t\right)=\left(\log t\right)^{\alpha-1}\left(x_{0}-g\left(x\right)\right)+\frac{1}{\Gamma\left(\alpha\right)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}h\left(s\right)\frac{ds}{s},\ t\in\left(1,T\right],$$

where $h:\left(1,T\right]\rightarrow\mathbb{R}$ be a function satisfying the functional equation

$$h\left(t\right)=f\left(t,x\left(t\right),h\left(t\right)\right).$$

By Lemma 3, the fixed points of operator Φ are solutions of (1). The operator Φ is well define, i.e. for every $x\in C_{1-\alpha}\left(J,\mathbb{R}\right)$ and $t>1$, the integral

$$\frac{1}{\Gamma\left(\alpha\right)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}h\left(s\right)\frac{ds}{s}, \tag{6}$$

belongs to $C_{1-\alpha}\left(J,\mathbb{R}\right)$. Under the condition (H1), we get

$$\begin{aligned}\left|h\left(t\right)\right|&=\left|f\left(t,x\left(t\right),h\left(t\right)\right)\right|\\&\leq\frac{k_{1}}{1-k_{2}}\left|x\left(t\right)\right|+c\left(\log t\right)^{\alpha-1}\ \text{ for each } t\in\left(1,T\right],\end{aligned} \tag{7}$$

where $c=\frac{\sup_{t\in J}\left|\left(\log t\right)^{1-\alpha}f(t,0,0)\right|}{1-k_{2}}$. For every $x\in C_{1-\alpha}\left(J,\mathbb{R}\right)$, we obtain

$$\begin{aligned}&\left|\frac{\left(\log t\right)^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}h\left(s\right)\frac{ds}{s}\right|\\&\leq\frac{\left(\log t\right)^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\left|h\left(s\right)\right|\frac{ds}{s}\\&\leq\frac{\left(\log t\right)^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\frac{k_{1}}{1-k_{2}}\left|x\left(s\right)\right|+c\left(\log s\right)^{\alpha-1}\right)\frac{ds}{s}\\&\leq\frac{\left(\log t\right)^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\log s\right)^{\alpha-1}\left(\frac{k_{1}}{1-k_{2}}\left|\left(\log s\right)^{1-\alpha}x\left(s\right)\right|+c\right)\frac{ds}{s}\\&\leq\left(\frac{k_{1}}{1-k_{2}}\left\|x\right\|_{C_{1-\alpha}}+c\right)\left(\log t\right)^{1-\alpha}I_{1^{+}}^{\alpha}\left(\left(\log t\right)^{\alpha-1}\right).\end{aligned}$$

By Lemma 2, we have

$$\begin{aligned}&\left|\frac{(\log t)^{1-\alpha}}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}h(s)\frac{ds}{s}\right|\\&\leq\left(\frac{k_1}{1-k_2}\|x\|_{C_{1-\alpha}}+c\right)\frac{\Gamma(\alpha)(\log t)^{\alpha}}{\Gamma(2\alpha)}\\&\leq\left(\frac{k_1}{1-k_2}\|x\|_{C_{1-\alpha}}+c\right)\frac{\Gamma(\alpha)(\log T)^{\alpha}}{\Gamma(2\alpha)}.\end{aligned}$$

That is to say that the integral exists and belongs to $C_{1-\alpha}(J,\mathbb{R})$.

Let $x,y\in C_{1-\alpha}(J,\mathbb{R})$. Then for $t\in(1,T]$, we obtain

$$\begin{aligned}&|(\Phi x)(t)-(\Phi y)(t)|\\&\leq(\log t)^{\alpha-1}|g(x)-g(y)|+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|h_x(s)-h_y(s)|\frac{ds}{s}\end{aligned}$$

where $h_x,h_y\in C_{1-\alpha}(J,\mathbb{R})$ be such that

$$h_x(t)=f(t,x(t),h_x(t)),$$

and

$$h_y(t)=f(t,y(t),h_y(t)).$$

By (H1) we get

$$\begin{aligned}|h_x(t)-h_y(t)|&=|f(t,x(t),h_x(t))-f(t,y(t),h_y(t))|\\&\leq k_1|x(t)-y(t)|+k_2|h_x(t)-h_y(t)|.\end{aligned}$$

Then

$$|h_x(t)-h_y(t)|\leq\frac{k_1}{1-k_2}|x(t)-y(t)|.$$

Therefore, for each $t\in(1,T]$, we have

$$\begin{aligned}&|(\Phi x)(t)-(\Phi y)(t)|\\&\leq b(\log t)^{\alpha-1}\|x-y\|_{C_{1-\alpha}}+\frac{k_1}{\Gamma(\alpha)(1-k_2)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|x(s)-y(s)|\frac{ds}{s}\\&=(\log t)^{\alpha-1}b\|x-y\|_{C_{1-\alpha}}\\&+\frac{k_1}{\Gamma(\alpha)(1-k_2)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}(\log s)^{\alpha-1}\left|(\log s)^{1-\alpha}(x(s)-y(s))\right|\frac{ds}{s}\\&\leq(\log t)^{\alpha-1}b\|x-y\|_{C_{1-\alpha}}+\frac{k_1}{1-k_2}I_{1^+}^{\alpha}\left((\log t)^{\alpha-1}\right)\|x-y\|_{C_{1-\alpha}}.\end{aligned}$$

By Lemma 2, we get

$$\left|(\Phi x)(t) - (\Phi y)(t)\right| \leq (\log t)^{\alpha-1} b \|x-y\|_{C_{1-\alpha}} + \frac{\Gamma(\alpha) k_1 (\log t)^{2\alpha-1}}{\Gamma(2\alpha)(1-k_2)} \|x-y\|_{C_{1-\alpha}},$$

which implies that

$$\begin{aligned}
&\left|(\log t)^{1-\alpha}\left((\Phi x)(t) - (\Phi y)(t)\right)\right| \\
&\leq b\|x-y\|_{C_{1-\alpha}} + \frac{\Gamma(\alpha) k_1 (\log t)^{\alpha}}{\Gamma(2\alpha)(1-k_2)} \|x-y\|_{C_{1-\alpha}} \\
&\leq b\|x-y\|_{C_{1-\alpha}} + \frac{\Gamma(\alpha) k_1 (\log T)^{\alpha}}{\Gamma(2\alpha)(1-k_2)} \|x-y\|_{C_{1-\alpha}}.
\end{aligned}$$

Thus

$$\|\Phi x - \Phi y\|_{C_{1-\alpha}} \leq \left(b + \frac{\Gamma(\alpha) k_1 (\log T)^{\alpha}}{\Gamma(2\alpha)(1-k_2)}\right) \|x-y\|_{C_{1-\alpha}}.$$

From (5), Φ is a contraction. As a consequence of Banach's fixed point theorem, we get that Φ has a unique fixed point which is a unique solution of the problem (1). □

3.2. **Existence results via Krasnoselskii's fixed point theorem.**

Theorem 4. *Suppose that* (*H*1), (*H*2) *hold and the following hypothesis*

(*H*3) *There exist* $p_1 \in C_{1-\alpha}(J, \mathbb{R}^+)$, $p_2, p_3 \in C(J, \mathbb{R}^+)$ *with* $p_3^* = \sup_{t\in J} p_3(t) < 1$ *such that*

$$|f(t,u,v)| \leq p_1(t) + p_2(t)|u| + p_3(t)|v|,$$

for $t \in (1, T]$ *and each* $u, v \in \mathbb{R}$.

If

$$\lambda = b + \frac{p_2^* \Gamma(\alpha)(\log T)^{\alpha}}{(1-p_3^*)\Gamma(2\alpha)} < 1,$$

where $p_2^* = \sup_{t\in J} p_2(t)$. *Then* (1) *has at least one solution in* Ω.

Proof. Set

$$R = \frac{1}{1-\lambda}, \ \Lambda = |x_0| + Q + \frac{(\log T)^{\alpha} p_1^* \Gamma(\alpha)}{(1-p_3^*)\Gamma(2\alpha)},$$

where $p_1^* = \sup_{t\in J}\left\{(\log t)^{1-\alpha} p_1(t)\right\}$ and $Q = |g(0)|$. Let us fix

$$M \geq R\Lambda.$$

Consider the non-empty closed bounded convex subset $\Omega = \left\{x \in C_{1-\alpha}(J, \mathbb{R}) : \|x\|_{C_{1-\alpha}} \leq M\right\}$ and define two operators F_1 and F_2 on Ω, as follows

$$(F_1 x)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} h(s) \frac{ds}{s},$$

and

$$(F_2 x)(t) = (\log t)^{\alpha-1}(x_0 - g(x)),$$

where $h:(1,T] \to \mathbb{R}$ be a function satisfying the functional equation

$$h(t) = f(t, x(t), h(t)).$$

We shall use the Krasnoselskii fixed point theorem to show there exists at least one fixed point of the operator $F_1 + F_2$ in Ω. The proof will be given in several steps.

Step 1. We prove that $F_1 x + F_2 y \in \Omega$ for all $x, y \in \Omega$.

For any $x, y \in \Omega$ and $t \in (1,T]$, we have

$$\begin{aligned}
&|(F_1 x)(t) + (F_2 y)(t)| \\
&\leq \left| (\log t)^{\alpha-1}(x_0 - g(x)) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} h(s) \frac{ds}{s} \right| \\
&\leq (\log t)^{\alpha-1}|x_0| + (\log t)^{\alpha-1}|g(x) - g(0)| + (\log t)^{\alpha-1}|g(0)| \\
&+ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} (\log s)^{\alpha-1} \left|(\log s)^{1-\alpha} h(s)\right| \frac{ds}{s} \\
&\leq (\log t)^{\alpha-1}|x_0| + (\log t)^{\alpha-1} b \|x\|_{C_{1-\alpha}} + (\log t)^{\alpha-1} Q \\
&+ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} (\log s)^{\alpha-1} \left|(\log s)^{1-\alpha} h(s)\right| \frac{ds}{s} \\
&\leq (\log t)^{\alpha-1}|x_0| + (\log t)^{\alpha-1} bM + (\log t)^{\alpha-1} Q \\
&+ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} (\log s)^{\alpha-1} \left|(\log s)^{1-\alpha} h(s)\right| \frac{ds}{s}.
\end{aligned} \tag{8}$$

By (H3), for each $t \in (1,T]$, we obtain

$$\begin{aligned}
|h(t)| &= |f(t, x(t), h(t))| \\
&\leq p_1(t) + p_2(t)|x(t)| + p_3(t)|h(t)|.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\left|(\log t)^{1-\alpha} h(t)\right| &\leq (\log t)^{1-\alpha} p_1(t) + p_2(t)\left|(\log t)^{1-\alpha} x(t)\right| + p_3(t)\left|(\log t)^{1-\alpha} h(t)\right| \\
&\leq p_1^* + p_2^* M + p_3^* \left|(\log t)^{1-\alpha} h(t)\right|,
\end{aligned}$$

then, we have

$$\left|(\log t)^{1-\alpha} h(t)\right| \leq \frac{p_1^* + p_2^* M}{1 - p_3^*}. \tag{9}$$

Replacing (9) in the inequality (8) and with Lemma 2, we obtain

$$\begin{aligned}
&\left|\left(F_1x\right)(t)+\left(F_2y\right)(t)\right| \\
&\leq (\log t)^{\alpha-1}\left|x_0\right|+(\log t)^{\alpha-1} bM+(\log t)^{\alpha-1} Q \\
&+\left(\frac{p_1^*+p_2^*M}{1-p_3^*}\right)\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}(\log s)^{\alpha-1}\frac{ds}{s} \\
&\leq (\log t)^{\alpha-1}\left|x_0\right|+(\log t)^{\alpha-1} bM+(\log t)^{\alpha-1} Q+\left(\frac{p_1^*+p_2^*M}{1-p_3^*}\right)\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}(\log t)^{2\alpha-1}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left|(\log t)^{1-\alpha}\left(\left(F_1x\right)(t)+\left(F_2x\right)(t)\right)\right| \\
&\leq \left|x_0\right|+Q+\frac{(\log T)^{\alpha} p_1^*\Gamma(\alpha)}{\left(1-p_3^*\right)\Gamma(2\alpha)}+\left(b+\frac{p_2^*\Gamma(\alpha)(\log T)^{\alpha}}{\left(1-p_3^*\right)\Gamma(2\alpha)}\right)M.
\end{aligned}$$

Thus

$$\begin{aligned}
&\left\|F_1x+F_2x\right\|_{C_{1-\alpha}} \\
&\leq \Lambda+\lambda M\leq\frac{M}{R}+\left(1-\frac{1}{R}\right)M=M.
\end{aligned}$$

Hence $F_1x+F_2y\in\Omega$ for all $x,y\in\Omega$.

Step 2. We show that F_1 is continuous.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence such that $x_n\to x$ in $C_{1-\alpha}(J,\mathbb{R})$, then for each $t\in(1,T]$, we have

$$\left|\left(F_1x_n\right)(t)-\left(F_1x\right)(t)\right|\leq\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left|h_n(s)-h(s)\right|\frac{ds}{s}, \tag{10}$$

where $h_n,h\in C_{1-\alpha}(J,\mathbb{R})$ be such that

$$h_n(t)=f\left(t,x_n(t),h_n(t)\right),$$

and

$$h(t)=f(t,x(t),h(t)).$$

By (H1) we have

$$\begin{aligned}
\left|h_n(t)-h(t)\right|&=\left|f\left(t,x_n(t),h_n(t)\right)-f(t,x(t),h(t))\right| \\
&\leq k_1\left|x_n(t)-x(t)\right|+k_2\left|h_n(t)-h(t)\right|.
\end{aligned}$$

Then

$$\left|h_n(t)-h(t)\right|\leq\frac{k_1}{1-k_2}\left|x_n(t)-x(t)\right|. \tag{11}$$

By replacing (11) in inequality (10), we get

$$\begin{aligned}
&\left|\left(F_1 x_n\right)(t)-\left(F_1 x\right)(t)\right| \\
&\leq \frac{k_1}{\left(1-k_2\right) \Gamma(\alpha)} \int_1^t\left(\log \frac{t}{s}\right)^{\alpha-1}\left|x_n(t)-x(t)\right| \frac{d s}{s} \\
&=\frac{k_1}{\left(1-k_2\right) \Gamma(\alpha)} \int_1^t\left(\log \frac{t}{s}\right)^{\alpha-1}(\log s)^{\alpha-1}\left|(\log s)^{1-\alpha}\left(x_n(t)-x(t)\right)\right| \frac{d s}{s} \\
&\leq \frac{k_1}{1-k_2} I_{1^{+}}^\alpha\left((\log t)^{\alpha-1}\right)\left\|x_n-x\right\|_{C_{1-\alpha}} .
\end{aligned}$$

By Lemma 2, we obtain

$$\left|\left(F_1 x_n\right)(t)-\left(F_1 x\right)(t)\right| \leq \frac{\Gamma(\alpha) k_1(\log t)^{2 \alpha-1}}{\left(1-k_2\right) \Gamma(2 \alpha)}\left\|x_n-x\right\|_{C_{1-\alpha}},$$

which implies that

$$\begin{aligned}
\left|(\log t)^{1-\alpha}\left(\left(F_1 x_n\right)(t)-\left(F_1 x\right)(t)\right)\right| &\leq \frac{\Gamma(\alpha) k_1(\log t)^\alpha}{\left(1-k_2\right) \Gamma(2 \alpha)}\left\|x_n-x\right\|_{C_{1-\alpha}} \\
&\leq \frac{\Gamma(\alpha) k_1(\log T)^\alpha}{\left(1-k_2\right) \Gamma(2 \alpha)}\left\|x_n-x\right\|_{C_{1-\alpha}} .
\end{aligned}$$

Thus

$$\left\|F_1 x_n-F_1 x\right\|_{C_{1-\alpha}} \leq \frac{\Gamma(\alpha) k_1(\log T)^\alpha}{\left(1-k_2\right) \Gamma(2 \alpha)}\left\|x_n-x\right\|_{C_{1-\alpha}},$$

and hence

$$\left\|F_1 x_n-F_1 x\right\|_{C_{1-\alpha}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Consequently, F_1 is continuous.

Step 3. We prove that F_1 is compact.

For all $x \in \Omega$ and $t \in (1, T]$, we have

$$\left|\left(F_1 x\right)(t)\right| \leq \frac{1}{\Gamma(\alpha)} \int_1^t\left(\log \frac{t}{s}\right)^{\alpha-1}(\log s)^{\alpha-1}\left|(\log s)^{1-\alpha} h(s)\right| \frac{d s}{s}. \tag{12}$$

Replacing (9) in the inequality (12) and with Lemma 2, we obtain

$$\left|\left(F_1 x\right)(t)\right| \leq\left(\frac{p_1^*+p_2^* M}{1-p_3^*}\right) \frac{\Gamma(\alpha)}{\Gamma(2 \alpha)}(\log t)^{2 \alpha-1} .$$

Therefore

$$\left|(\log t)^{1-\alpha}\left(F_1 x\right)(t)\right| \leq\left(\frac{p_1^*+p_2^* M}{1-p_3^*}\right) \frac{\Gamma(\alpha)}{\Gamma(2 \alpha)}(\log T)^\alpha .$$

Thus

$$\left\|F_1 x\right\|_{C_{1-\alpha}} \leq\left(\frac{p_1^*+p_2^* M}{1-p_3^*}\right) \frac{\Gamma(\alpha)}{\Gamma(2 \alpha)}(\log T)^\alpha .$$

Hence $F_1(\Omega)$ is uniformly bounded.

It remains to show that $F_1(\Omega)$ is equicontinuous, let $1 \le t_1 < t_2 \le T$ and $x \in \Omega$. Then

$$\begin{aligned}
&\left|(\log t_2)^{1-\alpha}(F_1 x)(t_2) - (\log t_1)^{1-\alpha}(F_1 x)(t_1)\right| \\
&= \frac{1}{\Gamma(\alpha)}\left|\int_1^{t_1}(\log t_2)^{1-\alpha}\left(\log\frac{t_2}{s}\right)^{\alpha-1} h(s)\frac{ds}{s} + \int_{t_1}^{t_2}(\log t_2)^{1-\alpha}\left(\log\frac{t_2}{s}\right)^{\alpha-1} h(s)\frac{ds}{s}\right. \\
&\left.-\int_1^{t_1}(\log t_1)^{1-\alpha}\left(\log\frac{t_1}{s}\right)^{\alpha-1} h(s)\frac{ds}{s}\right| \\
&\le \frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left|(\log t_2)^{1-\alpha}\left(\log\frac{t_2}{s}\right)^{\alpha-1}(\log s)^{\alpha-1}\right. \\
&\left.-(\log t_1)^{1-\alpha}\left(\log\frac{t_1}{s}\right)^{\alpha-1}(\log s)^{\alpha-1}\right|\left|(\log s)^{1-\alpha}h(s)\right|\frac{ds}{s} \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(\log t_2)^{1-\alpha}\left(\log\frac{t_2}{s}\right)^{\alpha-1}(\log s)^{\alpha-1}\left|(\log s)^{1-\alpha}h(s)\right|\frac{ds}{s} \\
&\le \frac{p_1^*+p_2^*M}{1-p_3^*}\left(\frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left|(\log t_2)^{1-\alpha}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\right.\right. \\
&\left.\left.-(\log t_1)^{1-\alpha}\left(\log\frac{t_1}{s}\right)^{\alpha-1}\right|(\log s)^{\alpha-1}\frac{ds}{s}\right) \\
&+\frac{p_1^*+p_2^*M}{1-p_3^*}\left(\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(\log t_2)^{1-\alpha}\left(\log\frac{t_2}{s}\right)^{\alpha-1}(\log s)^{\alpha-1}\frac{ds}{s}\right).
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. That is to say that $F_1(\Omega)$ is equicontinuous, then by Ascoli-Arzela theorem, we can conclude that the operator F_1 is compact.

Step 4. We prove that $F_2 : \Omega \to C_{1-\alpha}(J, \mathbb{R})$ is a contraction mapping.

For all $x \in \Omega$ and from (H2), we get

$$\begin{aligned}
|(F_2 x)(t) - (F_2 y)(t)| &= \left|(\log t)^{\alpha-1}(g(x) - g(y))\right| \\
&\le (\log t)^{\alpha-1} b \|x-y\|_{C_{1-\alpha}}.
\end{aligned}$$

Therefore

$$\left|(\log t)^{1-\alpha}((F_2 x)(t) - (F_2 y)(t))\right| \le b\|x-y\|_{C_{1-\alpha}}.$$

Then

$$\|F_2 x - F_2 y\|_{C_{1-\alpha}} \le b\|x-y\|_{C_{1-\alpha}}.$$

Hence, the operator F_2 is a contraction.

Clearly, all the hypotheses of the Krasnoselskii fixed point theorem (see [26]) are satisfied. Thus there a fixed point $x \in \Omega$ such that $x = F_1 x + F_2 x$, which is a solution of the problem (1). □

4. Example

We consider the following fractional initial value problem

$$(13) \qquad \begin{cases} D_{1^+}^{\frac{3}{4}} x(t) = \frac{1}{5\exp(-t+3)\left(2+|x(t)|+\left|D_{1^+}^{\frac{3}{4}} x(t)\right|\right)} + \frac{1}{(\log t)^{\frac{1}{4}}}, \ t \in (1,e], \\ (\log t)^{\frac{1}{4}} x(t)\Big|_{t=0} = \frac{1}{2} - \sum_{i=1}^{n} c_i (\log t_i)^{\frac{1}{4}} x(t_i), \end{cases}$$

where $1 < t_1 < ... < t_n < e$ and c_i, $i = 1, ..., n$ are positive constants with

$$\sum_{i=1}^{n} c_i \leq \frac{1}{3}.$$

Set

$$f(t,u,v) = \frac{1}{5\exp(-t+3)(2+|u|+|v|)} + \frac{1}{(\log t)^{\frac{1}{4}}}, \ t \in (1,e], \ u,v \in \mathbb{R},$$

We have

$$C_{1-\alpha}([1,e],\mathbb{R}) = C_{\frac{1}{4}}([1,e],\mathbb{R}) = \left\{h : (1,e] \to \mathbb{R} : (\log t)^{\frac{1}{4}} h \in C([1,e],\mathbb{R})\right\},$$

with $\alpha = \frac{3}{4}$. Clearly the functions f and g are continuous, $f(.,0,0) \in C_{\frac{1}{4}}([1,e],\mathbb{R})$. For each $u, u^*, v, v^* \in \mathbb{R}$ and $t \in (1,e]$, we have

$$\begin{aligned} &|f(t,u,v) - f(t,u^*,v^*)| \\ &= \left|\frac{1}{5\exp(-t+3)}\left(\frac{1}{2+|u|+|v|} - \frac{1}{2+|u^*|+|v^*|}\right)\right| \\ &\leq \frac{|u-u^*|+|v-v^*|}{5\exp(-t+3)(2+|u|+|v|)(2+|u^*|+|v^*|)} \\ &\leq \frac{1}{20e^{-e+3}}(|u-u^*|+|v-v^*|), \end{aligned}$$

and

$$\begin{aligned} |g(u) - g(u^*)| &\leq \sum_{i=1}^{n} c_i (\log t_i)^{\frac{1}{4}} |u(t_i) - u^*(t_i)| \\ &\leq \sum_{i=1}^{n} c_i \|u-u^*\|_{C_{\frac{1}{4}}} \leq \frac{1}{3}\|u-u^*\|_{C_{\frac{1}{4}}}. \end{aligned}$$

Hence, conditions (H1) and (H2) are satisfied with $k_1 = k_2 = \frac{1}{20e^{-e+3}}$ and $b = \frac{1}{3}$. The condition

$$b + \frac{\Gamma(\alpha) k_1 (\log T)^{\alpha}}{\Gamma(2\alpha)(1-k_2)} = \frac{1}{3} + \frac{\frac{\Gamma\left(\frac{3}{4}\right)}{20e^{-e+3}}}{\Gamma\left(\frac{3}{2}\right)\left(1 - \frac{1}{20e^{-e+3}}\right)} \simeq 0.388 < 1,$$

is satisfied with $T = e$. It follows from Theorem 3 that the problem (13) has a unique solution in the space $C_{\frac{1}{4}}([1,e], \mathbb{R})$.

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