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# ON THE GENERALIZED COMPLEMENT OF SOME GRAPHS

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Abstract. In this paper we study the generalized complement of the graph $G_{m,n} = (V, E)$ for some values of $m,\ n$. We study the generalized complement of $G_{m,n}$ graphs with respect to the equal degree partition. The $2-$complement of $G_{m,n}$ graphs are also determined for $m = 2$, $n$ is even or odd. In particular, for some values of $m, n \in \mathbb{N}$, we studied the complement of $G_{m,n}$ graphs with respect to the equal degree partition and the $2-$complement of $G_{m,n}$ graphs. We determine the partitions $P_k, k \in \mathbb{N}$ of the vertex set $V$ such that the generalized complement of $G_{m,n}$ graph is a path graph and a comb graph.

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## 1. Introduction

The $k-$complement of a graph $\Gamma = (V_\Gamma, E_\Gamma)$ with respect to a partition $\mathcal{P}$ was defined by Sampathkumar and Pushpalatha in [10]. Let us recall from [10], the $k-$complement of a graph $\Gamma$ with respect to partition $\mathcal{P}$. Let $\mathcal{P} = \{V_1, V_2, \dots, V_k\}$ be a partition of $V_\Gamma$ of order $k \geq 2$. For all $i \neq j$, $V_i, V_j \in \mathcal{P}$, by removing the edges between $V_i$ and $V_j$ in $\Gamma$ and adding the edges between $V_i$ and $V_j$ which are not in $\Gamma$, we obtain the $k-$complement of a graph $\Gamma$ with respect to the partition $\mathcal{P}$. $\Gamma$ is $k-$self complementary if $\Gamma \cong \Gamma_k^{\mathcal{P}}$ for partition $\mathcal{P}$ of order $k$.

In [4], an undirected simple graph called $G_{m,n} = (V, E)$ was defined and studied on a finite subset of natural numbers. The vertex set of $G_{m,n}$ is $V = I_n = \{1, 2, 3, \dots, n\}$ and two distinct vertices $a, b \in V$ are adjacent if and only if $a \neq b$ and $a + b$ is not divisible by $m$, where $m \in \mathbb{N}$ and $m > 1$ [4]. Kauser et al. studied various domination numbers, domatic number of $G_{m,n}$

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graphs in [5–9]. Energy, Spectrum, Harary energy, Randic energy, Zagreb indices of $G_{m,n}$ graphs are studied by Anusha et al. in [1–3]. For convenience, we denote the graph $G_{m,n}$ by $\mathcal{G}$.

In this paper, we study the $k-$complement of $\mathcal{G}$ with respect to various partition of the vertex set $V$. We denote the $k-$ complement of $\mathcal{G}$ by $\overline{\mathcal{G}_k^P}$, where $P$ denote the partition. Throughout the paper we consider $k \in \mathbb{N}$ and by vertex $u$, we mean the label of the vertex is $u$. Readers can refer [11], for any undefined terms in graph theory.

## 2. $k-$COMPLEMENT OF $\mathcal{G}$

In this section, we analyse the nature of the $k$-complement of $\mathcal{G}$ with respect to the equal degree partition and $2-$complement of $\mathcal{G}$ with respect to the partition $P_p$.

### 2.1. $k$-complement of $\mathcal{G}$ with respect to the equal degree partition.

Consider the partition $P_d$ of the vertex set $V$ of $\mathcal{G}$, where $P_d = \{V_1, V_2, \ldots, V_k\}$ and each set in $P_d$ contains the vertices of same degree of $\mathcal{G}$.

**Theorem 1.** *Let $m = n$ and $n$ be odd, then $\overline{\mathcal{G}_k^{P_d}}$ is disconnected. Moreover $\overline{\mathcal{G}_2^{P_d}}$ is disjoint union of $K_1$ and $(n-3)-$regular graph.*

*Proof.* Let $m = n$, then in $\mathcal{G}$, $\deg(n) = n-1$ and $\deg(i) = n-2$ for all $i(\neq n) \in V$ [4]. Thus $P_d = \{V_1, V_2\}$ and $k = 2$. Assume that $V_1 = \{n\}$ and $V_2 = \{1, 2, \ldots, n-1\}$. The vertex $v = n$ is isolated in $\overline{\mathcal{G}_2^{P_d}}$ as in $\mathcal{G}$ the vertex $v = n$ is adjacent to all other vertices. Hence $\overline{\mathcal{G}_2^{P_d}}$ is disconnected.

The number of vertices in $V_2$ is $n-1$. The vertex $i \in V_2$ is not adjacent to itself and the vertex labeled as $n-i$. Thus in $\overline{\mathcal{G}_2^{P_d}}$, the vertices in $V_2$ are of degree $n-3$ as $\forall i \in V_2$, $i$ is not adjacent to $i, n-i$ and $n$. Hence $\overline{\mathcal{G}_2^{P_d}}$ is disjoint union of $K_1$ and $(n-3)-$regular graph with respect to the partition $P_d$. □

**Theorem 2.** *For $m = n$, where $n$ is an even integer, $\overline{\mathcal{G}_k^{P_d}}$ is disconnected. Further $\overline{\mathcal{G}_2^{P_d}}$ is disjoint union of $K_2$ and $(n-4)-$regular graph.*

*Proof.* Let $m = n$ and $n$ be even. In $\mathcal{G}$, the vertices $\frac{n}{2}$ and $n$ are of degree $n-1$ and any other vertex $i(\neq n, \frac{n}{2}) \in V$ is of degree $n-2$ [4]. Thus consider the partition $P_d$ of the vertex set $V$ into $V_1$ and $V_2$ such that $V_1 = \{\frac{n}{2}, n\}$, $V_2 = V \setminus V_1$. Clearly $k = 2$. The vertices $\frac{n}{2}$, $n$ are adjacent in $\mathcal{G}$, implying, in $\overline{\mathcal{G}_k^{P_d}}$, the vertices $\frac{n}{2}$ and $n$ are adjacent, which produces a subgraph isomorphic to $K_2$. Again the vertices in $V_2 = V \setminus V_1$ are adjacent to the vertices $n$ and $\frac{n}{2}$ in $\mathcal{G}$, which implies in $\overline{\mathcal{G}_k^{P_d}}$, the vertices in $V_2$ are not adjacent to the vertices $n$ and $\frac{n}{2}$. Again each vertex $i \in V_2$ is not adjacent to the vertex $(n-i) \in V_2$. Thus the degree of the vertices in $V_2$ reduces by 4, which gives $\deg(i) = n-4$ for $i(\neq \frac{n}{2}, n) \in V$. Hence $\overline{\mathcal{G}_2^{P_d}}$ is a disconnected graph with two components $K_2$ and a regular graph of degree $n-4$. □

**Theorem 3.** *For $m = 2$, $\mathcal{G} \cong \overline{\mathcal{G}_k^{P_d}}$, if $n$ is an even integer and $\overline{\mathcal{G}_k^{P_d}}$ is a null graph, if $n$ is an odd integer.*

*Proof.* Let $m = 2$ and $n$ be even. Then in $\mathcal{G}$ all the vertices are of degree $\frac{n}{2}$. Therefore the partition $P_d = \{V\}$. Thus in $\overline{\mathcal{G}_k^{P_d}}$, the adjacency of the vertices remain same as in $\mathcal{G}$. Hence $\mathcal{G} \cong \overline{\mathcal{G}_k^{P_d}}$ with respect to the partition $P_d$.

Let $m = 2$ and $n$ be odd. Then in $\mathcal{G}$ that the number of vertices labeled as an even integer is $\lfloor \frac{n}{2} \rfloor$ and the number of vertices labeled as an odd integer is $\lfloor \frac{n}{2} \rfloor + 1$. In $\mathcal{G}$, all the vertices labeled as an even integer are of degree $\lfloor \frac{n}{2} \rfloor + 1$ as any vertex labeled as an even integer is adjacent to all the vertices labeled as odd integers. Similarly, the vertices labeled as odd integers are adjacent to all the vertices labeled as even integers, which implies the degree of all the vertices labeled as odd integers are $\lfloor \frac{n}{2} \rfloor$. So the partition $P_d = \{V_1, V_2\}$, where $V_1 = \{i \in V : \text{the label of } i \text{ is even}\}$ and $V_2 = \{i \in V : \text{ the label of } i \text{ is odd}\}$. In $\mathcal{G}$, all the vertices of $V_1$ are adjacent to all the vertices of $V_2$ but no two vertices in $V_j, j = 1, 2$ are adjacent. Thus, for $k = 2-$complement of $\mathcal{G}$ with respect to the partition $P_d$, no two vertices are adjacent implying $\overline{\mathcal{G}_2^{P_d}}$ is a null graph. □

**Theorem 4.** *Let $n$ be even and $m = n+1$, then $\mathcal{G} \cong \overline{\mathcal{G}_k^{P_d}}$.*

*Proof.* Let $n$ be even and $m = n+1$. Then $\mathcal{G}$ is an $n-2$ regular graph, as each vertex $i \in V$ is not adjacent to the vertex $m-i$ and itself. Hence, with respect to the partition $P_d$, $\mathcal{G} \cong \overline{\mathcal{G}_k^{P_d}}$. □

2.2. $2-$**complement of $\mathcal{G}$ with respect to the partition $P_p$.** In this section we consider the partition $P_p$ of the vertex set $V$ of $\mathcal{G}$ such that $P_p = \{V_1, V_2\}$, where $V_1 = \{x \in V : x \text{ is a prime}\}$ and $V_2 = V \setminus V_1$. Since $k = 2$ for the partition $P_p$, we denote the $k = 2$ complement of $\mathcal{G}$ by $\overline{\mathcal{G}_2^{P_p}}$. It is known that the function designated $\pi(x)$ is the number of primes not exceeding $x$ [12].

**Theorem 5.** *For $m = 2$, $\overline{\mathcal{G}_2^{P_p}} \cong K_{s,t}$ where $s = \frac{n+4}{2} - \pi(n)$, $t = \frac{n-4}{2} + \pi(n)$, if $n$ is even and $s = \frac{n+3}{2} - \pi(n)$, $t = \frac{n-5}{2} + \pi(n)$, if $n$ is odd.*

*Proof.* For $m = 2$, in $\mathcal{G}$, all the vertices labeled as even integers are adjacent to all the vertices labeled as odd integers, but no two vertices of the same parity are adjacent.

Let us consider the partition $P_p$ of the vertex set $V$, where $P_p = \{V_1, V_2\}$. Then $V_1 = \{v \in V : v \text{ is a prime }\} = \{2\} \cup \{u \in V : u \text{ is an odd prime}\}$ and $V_2 = V \setminus V_1$. Let $U_1 = \{u \in V_1 : u \text{ is an odd prime}\}$. The cardinality of $U_1$, $V_1$, $V_2$ are $\pi(n) - 1$, $\pi(n)$, $n - \pi(n)$, respectively. Clearly, all the vertices of $V_1$ are odd integers except $v = 2$. Thus in $\mathcal{G}$, the vertex $v = 2$ is adjacent to all the vertices in $U_1$. Since all the vertices in $U_1$ are odd integers, so they are not adjacent with each other in $\mathcal{G}$. Again, $V_2 \subseteq V$ can be expressed as $V_2 = \{1\} \cup U_2 \cup U_3$, where $U_2 = \{x \in V_2 : x \text{ is even}\}$ and $U_3 = \{y \in V_2 : y (\neq 1) \text{ is odd but not prime}\}$. The vertex $w = 1$ is adjacent to all the vertices in $U_2$. Similarly, the vertices in $U_3$ are adjacent to all the vertices in $U_2$. The vertices in $U_1$ are adjacent to all the vertices in $U_2$. The vertex $v = 2 \in V_1$ is adjacent to the vertex $w = 1 \in V_2$ and all the vertices in $U_3$.

Now we discuss the adjacency of the vertices in $\overline{\mathcal{G}_2^{P_p}} = (\overline{V}, \overline{E})$. The vertex $v = 2 \in V_1$ is not adjacent to $w = 1 \in V_2$ and all the vertices in $U_3 \subseteq V_2$. Thus it is clear that, the vertices $v = 2$, $w = 1$ and the vertices in $U_3$ are non adjacent. Similarly, the vertices in $U_1$ and the vertices in $U_2$ are non-adjacent. So we write the vertex set of $\overline{\mathcal{G}_2^{P_p}}$ as $\overline{V} = V_a \cup V_b$, where $V_a = \{1\} \cup \{2\} \cup U_3$, $V_b = U_1 \cup U_2$. Clearly, $\overline{\mathcal{G}_2^{P_p}}$ is isomorphic to a complete bipartite graph $K_{s,t}$ with the partite sets $V_a$, $V_b$, where $s = |V_a|$, $t = |V_b|$. The cardinality of $U_1$ is $\pi(n) - 1$. To find the cardinality of $U_2$ and $U_3$, the following two cases may arise.

Case I. For even $n$, the cardinalities of $U_2$, $U_3$ are $\frac{n-2}{2}$, $\frac{n}{2} - \pi(n)$ respectively. Thus the cardinality of $V_a$ is $s = \frac{n+4}{2} - \pi(n)$ and the cardinality of $V_b$ is $t = \frac{n-4}{2} + \pi(n)$.

Case II. For odd $n$, the cardinalities of $U_2$, $U_3$ are $\frac{n-3}{2}$, $\frac{n+1}{2} - \pi(n)$ respectively. This implies the cardinalities of $V_a$ and $V_b$ are $s = \frac{n+3}{2} - \pi(n)$ and $t = \frac{n-5}{2} + \pi(n)$, respectively.

Hence the theorem is proved. □

In the next two theorems, we observe the relation between twin primes and $\overline{\mathcal{G}_2^{P_p}}$.

**Theorem 6.** *Let $n$ be an even integer and $m = n + 1$ be a prime number. Then the vertices labeled as primes in $\overline{\mathcal{G}_2^{P_p}}$ form an induced subgraph $K_{\pi(n)}$ if and only if $m - 2, m$ are not twin primes.*

*Proof.* Let $n$ be even, $m = n + 1$ be a prime. Then easily it can be determined that, $\mathcal{G} = (V, E)$ is a $n - 2$ regular graph, where each vertex is not adjacent to itself and the vertex labeled as $m - i = n + 1 - i$.

Let $m - 2$ be a non-prime number. Now consider $\overline{\mathcal{G}_2^{P_p}}$ with respect to the partition $P_p$ of the vertex set $V$, where $P_p = \{V_1, V_2\}$, $V_1 = \{x \in V : x \text{ is a prime}\}$ and $V_2 = V \setminus V_1$. In $V_1$ except the vertex $u = 2$, all other vertices are odd primes. Since $m > n$, so for any two distinct vertices $u_1, u_2 \in V_1$ where $u_1, u_2$ are distinct from 2, $u_1 + u_2$ is even and $u_1 + u_2 < 2m$, which implies,

$u_1 + u_2$ is not divisible by $m$. Therefore $u_1, u_2$ are adjacent in $\mathcal{G}$ as well as in $\overline{\mathcal{G}_2^{P_p}}$. Again, the vertex $w = m - 2 \in V_2$, as $m - 2$ is non-prime. Thus the vertex $u = 2 \in V_1$ is adjacent to all other vertices in $V_1$ as $2 + v_i < m$, where $v_i \in V_1, v_i \neq 2$ in the graph $\mathcal{G}$ and $\overline{\mathcal{G}_2^{P_p}}$. Consequently in $\overline{\mathcal{G}_2^{P_p}}$, the vertices in $V_1$ form a complete subgraph and the cardinality of $V_1$ is the number of primes less than or equal to $n$, that is $\pi(n)$.

Conversely, assume that the vertices labeled as primes form a complete subgraph in $\overline{\mathcal{G}_2^{P_p}}$, where $m - 2, m$ are twin primes. Then the vertices $a = 2, b = m - 2 \in V_1$. But it is absurd because the vertices $a$ and $b$ cannot be adjacent in $\mathcal{G}$ as $m|(a + b)$, implying the vertices $a, b$ are not adjacent in $\overline{\mathcal{G}_2^{P_p}}$. Thus $m - 2$ and $m$ cannot be twin primes. □

**Corollary 7.** *Let $n$ be an even integer and $m = n + 1$ be a prime number. Then the vertices labeled as primes in $\overline{\mathcal{G}_2^{P_p}}$ form an induced subgraph $K_{\pi(n)} \setminus \{e\}$ if and only if $m - 2, m$ are twin primes.*

*Proof.* The proof follows immediately from Theorem 6. □

## 3. Path graph from $\mathcal{G}$

In this section, we find the partition $\mathbf{P} = \{V_1, V_2, \ldots, V_k\}$ of the vertex set $V$ and the values of $m, n$ of $\mathcal{G}$ such that the $k-$complement of $\mathcal{G}$ is a path graph $P_n$.

**Theorem 8.** *For $m = n$, where $n$ is even, $\overline{\mathcal{G}_k^{\mathbf{P}}} \cong P_n$, if the partition $\mathbf{P} = \{V_1, V_2, \ldots, V_{\frac{n}{2}}\}$, where $V_i = \{i, n - 1 - i\}$, $i = 1, 2, \ldots, \frac{n}{2} - 1$ and $V_{\frac{n}{2}} = \{n, n - 1\}$.*

*Proof.* Let $n$ be even, $m = n$ and $v_i = i$ for $v_i \in V$, $i = 1, 2, \ldots, n$. The graph $\mathcal{G}$ is bi-regular, as the vertices $v_n, v_{\frac{n}{2}}$ are of degree $n - 1$ and all other vertices $v_i \in V$ where $i \neq \frac{n}{2}, n$ are of degree $n - 2$. Consider the partition $\mathbf{P} = \{V_1, V_2, \ldots, V_{\frac{n}{2}}\}$ of $V$ such that $V_i = \{i, n - 1 - i\}$, $i = 1, 2, \ldots, \frac{n}{2} - 1$ and $V_{\frac{n}{2}} = \{n, n - 1\}$. For each $V_i$, $i = 1, 2, \ldots, \frac{n}{2} - 1$, the vertices $i$ and $n - 1 - i$ are adjacent in $\mathcal{G}$ implying the vertices $i$, $n - 1 - i$ are adjacent in $\overline{\mathcal{G}_k^{\mathbf{P}}}$. Similarly, the vertices $n, n - 1 \in V_{\frac{n}{2}}$ are adjacent in $\mathcal{G}$ implying $n, n - 1$ are adjacent in $\overline{\mathcal{G}_k^{\mathbf{P}}}$. The vertex $i$ is not adjacent to $n - i$ for $i = 1, 2, \ldots, \frac{n}{2} - 1$ in $\mathcal{G}$, this implies the vertices $i$ and $n - i$ are adjacent in $\overline{\mathcal{G}_k^{\mathbf{P}}}$ as $v_i \in V_i$, $v_{n-i} \in V_{n-i}$, where $V_i \neq V_{n-i}$. Again the vertices $v_n$ and $v_{\frac{n}{2}}$ are adjacent to all other vertices in $\mathcal{G}$. Thus in $\overline{\mathcal{G}_k^{\mathbf{P}}}$, the vertices $v_n$ and $v_{\frac{n}{2}}$ are not adjacent to any other vertices except the vertices $v_{n-1}$ and $v_{\frac{n}{2}-1}$ respectively. Therefore the vertices in $\overline{\mathcal{G}_k^{\mathbf{P}}}$ form a path isomorphic to $P_n$ as follows: $(v_n, v_{n-1}, v_1, v_{n-2}, v_2, v_{n-3}, v_3, \cdots, v_{n-j}, v_j, \cdots, v_{\frac{n}{2}-1}, v_{\frac{n}{2}})$. □

**Theorem 9.** *For $m = n$, where $n$ is odd, $\overline{\mathcal{G}_k^{\mathbf{P}}} \cong P_n$, if the partition $\mathbf{P} = \{V_1, V_2, \ldots, V_{\frac{n+1}{2}}\}$, where $V_i = \{i, n - 1 - i\}$, $i = 1, 2, \ldots, \frac{n-3}{2}$, $V_{\frac{n-1}{2}} = \{\frac{n-1}{2}\}$ and $V_{\frac{n+1}{2}} = \{n - 1, n\}$.*

*Proof.* Let $m = n$, $n$ be odd, $v_i = i$ for $v_i \in V$, $i = 1, 2, \ldots, n$. $\mathcal{G}$ is bi-regular as $\deg(v_n) = n-1$ and $\deg(v_i) = n-2$ for $i = 1, 2, \ldots, n-1$. Consider the partition $\mathbf{P} = \{\mathbf{V_1}, \mathbf{V_2}, \ldots, \mathbf{V_{\frac{n+1}{2}}}\}$ of $V$ such that $V_i = \{i, n-1-i\}$ for $i = 1, 2, \ldots, \frac{n-3}{2}$, $V_{\frac{n-1}{2}} = \frac{n-1}{2}$, $V_{\frac{n+1}{2}} = \{n, n-1\}$. In each $V_i$, for $i = 1, 2, \ldots, \frac{n-3}{2}$, the vertices $v_i$ and $v_{n-1-i}$ are adjacent in $\mathcal{G}$ implying $v_i$ and $v_{n-1-i}$ are adjacent in $\overline{\mathcal{G}_k^{\mathbf{P}}}$ as $v_i, v_{n-1-i} \in V_i$. Similarly, the vertices $v_{n-1}$ and $v_n$ are adjacent in $\mathcal{G}$, which gives $v_{n-1}$ and $v_n$ are also adjacent in $\overline{\mathcal{G}_k^{\mathbf{P}}}$. Clearly, the vertices $v_i$ and $v_{n-i}$ for $i = 1, 2, \ldots, \frac{n-1}{2}$, are adjacent in $\overline{\mathcal{G}_k^{\mathbf{P}}}$ as $v_i$, $v_{n-i}$ are in different partite set and $v_i$, $v_{n-i}$ are not adjacent in $\mathcal{G}$. Thus $\overline{\mathcal{G}_k^{\mathbf{P}}}$ form a path isomorphic to $P_n$ as follows: $(v_n, v_{n-1}, v_1, v_{n-1-1}, v_2, v_{n-2-1}, v_3, v_{n-3-1}, \ldots, v_{\frac{n-1}{2}-1}, v_{\frac{n-1}{2}-1-1}, v_{\frac{n-1}{2}})$. □

## 4. Comb graph from $\mathcal{G}$

The *comb* $P_l \odot K_1$ is the graph obtained from a path $P_l$ by attaching a pendant edge to each vertex of the path.

In this section, we find the partition $P_c = \{V_1, V_2, \ldots, V_k\}$ of the vertex set $V$ and the values of $m, n$ of $\mathcal{G}$ such that the $k-$complement of $\mathcal{G}$ is a comb graph $P_l \odot K_1$.

**Theorem 10.** *For $n \equiv 0\,(mod\,4)$ and $m = \frac{n}{2}$, $\overline{\mathcal{G}_k^{P_c}} \cong P_l \odot K_1$, if $V = \{V_1, V_2, \ldots, V_k\} = \{\cup_{i=1}^{\frac{n}{4}} V_i\} \cup \{\cup_{j=1}^{\frac{n}{4}-1} V_j\} \cup \{V_{\frac{n}{2}}\} \cup \{V_n\}$ where $V_i = \{i, \frac{n}{2} - i\}$ for $i = 1, 2, \ldots, \frac{n}{4}$, $V_j = \{\frac{n}{2} + j, n-1-j\}$ for $j = 1, 2, \ldots, \frac{n}{4} - 1$, $V_{\frac{n}{2}} = \frac{n}{2}$, $V_n = \{n, n-1\}$ and $k = \frac{n}{2} + 1$.*

*Proof.* Let $n \equiv 0\,(mod\,4)$, $m = \frac{n}{2}$, $v_i = i$ for $v_i \in V, i = 1, 2, \ldots, n$. In $\mathcal{G}$, $\deg(n) = \deg(\frac{n}{2}) = n-2$, $\deg(v_i) = n-3$ for $i = 1, 2, \ldots, \frac{n}{2} - 1, \frac{n}{2} + 1, \ldots, n-1$. The vertices $n$ and $\frac{n}{2}$ are not adjacent. For each $i \in \{1, 2, \ldots, \frac{n}{2} - 1\}$, the vertices $i$, $\frac{n}{2} - i$ as well as the vertices $i$, $n-i$ are not adjacent.

Let us define the partition $P_c$ of the vertex set $V$ such that $V = \{V_1, V_2, \ldots, V_k\} = \{\cup_{i=1}^{\frac{n}{4}} V_i\} \cup \{\cup_{j=1}^{\frac{n}{4}-1} V_j\} \cup \{V_{\frac{n}{2}}\} \cup \{V_n\}$ where $V_i = \{i, \frac{n}{2} - i\}$ for $i = 1, 2, \ldots, \frac{n}{4}$, $V_j = \{\frac{n}{2} + j, n-1-j\}$ for $j = 1, 2, \ldots, \frac{n}{4} - 1$, $V_{\frac{n}{2}} = \frac{n}{2}$, $V_n = \{n, n-1\}$ and $k = \frac{n}{2} + 1$. Thus the $k = \frac{n}{2} + 1$ complement of $\mathcal{G}$, $\overline{\mathcal{G}_k^{P_c}}$ can be obtained as shown in the FIGURE 1, which is isomorphic to $P_l \odot K_1$, $l = \frac{n}{2}$.

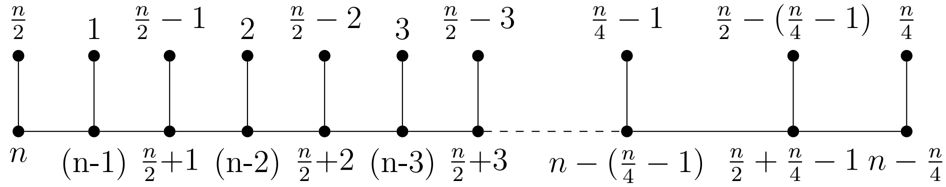


Figure 1. $\overline{\mathcal{G}_k^{P_c}} \cong P_n \odot K_1$

□

## 5. Conclusion

In this paper, we have studied the generalized complement of the graph $G_{m,n}$. We have determined the values of $m, n$ and the partitions of the vertex set $V$ such that the generalized complements of $G_{m,n}$ are self-complementary, bipartite, contain induced subgraph $K_{\pi(n)}$, path graph and comb graph.

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