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ON SOME SUBCLASSES OF BI-PSEUDO-STARLIKE FUNCTIONS DEFINED BY SĂLĂGEAN DIFFERENTIAL OPERATOR

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ABSTRACT. By applying Sălăgean operator, two new subclasses of bi-univalent functions associated with pseudo-starlike function in ∇ which are denoted by $\mathfrak{B}^{\phi,b}_{\mathfrak{E}}(\beta,\psi)$ and $\mathfrak{B}^{\phi,b}_{\mathfrak{E}}(\mu,\psi)$. Also, we investigated estimates on the coefficients $|m_2|$ and $|m_3|$ for functions in these new subclasses and significance of the results are indicated.

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1. Introduction

Let $\mathfrak{A}$ be the class of analytic function $f(z)$ in the open unit disk $\nabla = \{z : z \in \mathcal{C} : |z| < 1\}$, which is normalized by the conditions $f'(0) = 1$ and $f(0) = 0$ of the form

$$f(z) = z + \sum_{u=2}^{\infty} m_u z^u. \tag{1.1}$$

Further, let $\mathcal{S} \subset \mathfrak{A}$ which are univalent functions in ∇. let $S^*(\alpha)$ and $\mathcal{K}(\alpha)$ indicate the well known classes of starlike and convex functions of order α $(0 \le \alpha < 1)$ respectively (see [6]). Let $f^{-1}(z)$ be the inverse of the function $f(z)$ then we have

$$f^{-1}(f(z)) = z$$

and

$$f(f^{-1}(h)) = h, \quad |h| < r_0(f); r_0(f) \ge \frac{1}{4}$$

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where

$$g(h) = f^{-1}(h) = h - m_2h^2 + (2m_2^2 - m_3)h^3 - (5m_2^3 - 5m_2m_3 + m_4)h^4 + \cdots . \tag{1.2}$$

A function $f(z) \in \mathfrak{A}$ is said to be bi-univalent in ∇ if both $f(z)$ and $f^{-1}(z)$ are univalent in ∇. The class of analytic bi-univalent function in ∇ is denoted by $\mathfrak{E}$.

Examples of functions in the class $\mathfrak{E}$ are

$$-\log(1-z), \quad \frac{z}{1-z}, \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

and so on. However, the familiar Koebe function is not a member of $\mathfrak{E}$. Other common examples of functions in $\mathcal{S}$ such as

$$\frac{z}{1-z^2} \quad \textit{and} \quad z - \frac{z^2}{2}$$

are also not membars of $\mathfrak{E}$ (see [7,12]).

Lewin [10] (1967) investigated the bi-univalent function class $\mathfrak{E}$ and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [3] conjectured that $|a_2| \leq \sqrt{2}$. Netanyahu [12], on the other hand, showed that $|a_2| \leq \frac{4}{3}$. Brannan and Taha [5] (see also [21]) introduced certain subclasses of the bi-univalent function class $\mathfrak{E}$ similar to the familiar subclasses $S^*(\beta)$ and $\mathcal{K}(\beta)$ of starlike and convex functions of order β $(0 \leq \beta < 1)$, respectively (see [4]). Thus, following Brannan and Taha [5], a function $f \in \mathfrak{A}$ is in the class $S^*_{\mathfrak{E}}(\beta)$ of strongly bi-starlike of order β $(0 < \beta \leq 1)$, if

$$f \in \mathfrak{E}, \quad \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\beta\pi}{2}, \ z \in \nabla; \ \ 0 < \beta \leq 1$$

and

$$\left|\arg\left(\frac{hg'(h)}{g(h)}\right)\right| < \frac{\beta\pi}{2}, \ h \in \nabla; \ \ 0 < \beta \leq 1,$$

where the function g is given by (1.2).

Similarly, a function $f \in \mathfrak{A}$ is in the class $\mathcal{K}_{\mathfrak{E}}(\beta)$ of strongly bi-convex functions of order β $(0 < \beta \leq 1)$ if

$$f \in \mathfrak{E}, \quad \left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\beta\pi}{2}, \ z \in \nabla; \ \ 0 < \beta \leq 1$$

and

$$\left|\arg\left(1 + \frac{hg''(h)}{g'(h)}\right)\right| < \frac{\beta\pi}{2}, \ h \in \nabla; \ \ 0 < \beta \leq 1,$$

where the function g is given by (1.2).

The classes $S^*_{\mathfrak{E}}(\alpha)$ and $\mathcal{K}_{\mathfrak{E}}(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding (respectively) to the function classes $S^*(\alpha)$ and $\mathcal{K}(\alpha)$ were also introduced analogously. For each of the function classes $S^*_{\mathfrak{E}}(\beta)$ and $\mathcal{K}_{\mathfrak{E}}(\beta)$, Brannan and Taha [5] found non-sharp estimates on the first two Taylor-Maclaurin coefficient $|a_2|$ and $|a_3|$. But the coefficient estimates problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ for $n \in \mathcal{N}\backslash\{1,2\};\ \mathcal{N} := \{1,2,3,\cdots\}$ is presumably still an open problem. More details about certain subclasses of the bi-univalent function class $\mathfrak{E}$ see [10], [21], [18], [3], [1], [11], [13], [22], [12], [14], [17], [19].

These Functions and its various generalizations have large number of applications in problems of physical sciences, geometry and geometric function theory (for details see [20]). In [2] Babalola defined the class of ϕ-pseudo starlike functions of order ψ and prove that all Pseudo-starlike functions are Bazelivic of type $\left(1-\frac{1}{\phi}\right)$, order $\psi^{\frac{1}{\phi}}$ are univalent in ∇.
For $f(z) \in \mathfrak{A}$, Salagean [16] introduced the differential operator $\mathfrak{D}^b$ which is defined by

$$\mathfrak{D}^0 f(z) = f(z);$$

$$\mathfrak{D}^1 f(z) = \mathfrak{D} f(z) = z f'(z);$$

$$\mathfrak{D}^b f(z) = \mathfrak{D}(\mathfrak{D}^{b-1} f(z)), \quad b \in \mathcal{N} = 1,2,3,\cdots,$$

then,

$$\mathfrak{D}^b f(z) = z + \sum_{u=2}^{\infty} u^b m_u z^u$$

where $b \in \mathcal{N}_0 = \mathcal{N} \cup \{0\} = 0,1,2,3,\cdots$.
In this present paper, inspired by the earlier work of Babalola [2] and Joshi et. al. [8], we introduce the subclasses $\mathfrak{B}^{\phi,b}_{\mathfrak{E}}(\beta,\psi)$ and $\mathfrak{B}^{\phi,b}_{\mathfrak{E}}(\mu,\psi)$ of the function class $\mathfrak{E}$ associated with Salagean differential operator and determine the bounds on the initial coefficients $|m_2|$ and $|m_3|$. We need the following Lemma in other to establish our main results.

Lemma 1.1. [15] *If $r(z) \in \mathcal{P}$ and $z \in \nabla$, then $|w_n| \leq 2$ for each n. where $\mathcal{P}$ is the family of all function u analytic in ∇ for which $\Re(r(z)) > 0$,*

$$r(z) = 1 + w_1 z + w_2 z^2 + \cdots.$$

2. Coefficient Bounds for the Function Class $\mathfrak{B}_{\mathfrak{E}}^{\phi,b}(\beta,\psi)$

Definition 2.1. *A function $f(z)$ given by (1.1) is said to be in the class $\mathfrak{B}_{\mathfrak{E}}^{\phi,b}(\beta,\psi)$ if the following conditions are satisfied:*

$$\left|\arg\left[\frac{z[(\mathfrak{D}^b f(z))']^{\phi}}{(1-\psi)\mathfrak{D}^b f(z)+\psi\mathfrak{D}^{b+1}f(z)}\right]\right| < \frac{\beta\pi}{2} \quad z\in\nabla, \tag{2.1}$$

and

$$\left|\arg\left[\frac{h[(\mathfrak{D}^b g(h))']^{\phi}}{(1-\psi)\mathfrak{D}^b g(h)+\psi\mathfrak{D}^{b+1}g(h)}\right]\right| < \frac{\beta\pi}{2} \quad h\in\nabla, \tag{2.2}$$

where $f(z)\in\mathfrak{E}$, $\phi\geq 1$, $0<\beta\leq 1$, $0\leq\psi<1$ and

$$g(h)=h-m_2h^2+(2m_2^2-m_3)h^3-(5m_2^3-5m_2m_3+m_4)h^4+\cdots$$

Remark 2.1. *Taking $\psi=0$ in the class $\mathfrak{B}_{\mathfrak{E}}^{\phi,b}(\beta,\psi)$, we have $\mathfrak{B}_{\mathfrak{E}}^{\phi,b}(\beta,0)=\mathfrak{B}_{\mathfrak{E}}^{\phi,b}(\beta)$ and $f\in\mathfrak{B}_{\mathfrak{E}}^{\phi,b}(\beta)$ if the following conditions are satisfied:*

$$\left|\arg\left[\frac{z[(\mathfrak{D}^b f(z))']^{\phi}}{\mathfrak{D}^b f(z)}\right]\right| < \frac{\beta\pi}{2} \quad z\in\nabla, \tag{2.3}$$

and

$$\left|\arg\left[\frac{h[(\mathfrak{D}^b g(h))']^{\phi}}{\mathfrak{D}^b g(h)}\right]\right| < \frac{\beta\pi}{2} \quad h\in\nabla, \tag{2.4}$$

where $f(z)\in\mathfrak{E}$, $\phi\geq 1$, $0<\beta\leq 1$ and

$$g(h)=h-m_2h^2+(2m_2^2-m_3)h^3-(5m_2^3-5m_2m_3+m_4)h^4+\cdots$$

We note that for $b=0$, $\phi=1$ and $\psi=0$ the class $\mathfrak{B}_{\mathfrak{E}}^{1,0}(\beta,0)=S_{\mathfrak{E}}^*(\beta)$ is class of strongly bi-starlike functions of order β $(0<\beta\leq 1)$. When $b=1$, $\phi=1$ and $\psi=0$ the class $\mathfrak{B}_{\mathfrak{E}}^{1,1}(\beta,0)=\mathcal{K}_{\mathfrak{E}}^*(\beta)$ is class of strongly bi-convex functions of order β $(0<\beta\leq 1)$.

Remark 2.2. *For $b=0$ we have class introduced and studied in [8].*

Now we have the following theorem and the proof.

Theorem 2.1. *Let $f(z)$ given by (1.1) be in the class $\mathfrak{B}_{\mathfrak{E}}^{\phi,b}(\beta,\psi)$. Then*

$$|m_2|\leq\frac{2\beta}{\sqrt{\begin{aligned}3^b\beta(6\phi-4\psi-2)+2^{2b}[2\beta(2\phi^2-4\phi+\psi^2+2\phi\psi-2\psi+1)\\-(\beta-1)(2\phi-\psi-1)^2]\end{aligned}}} \tag{2.5}$$

and

$$|m_3|\leq\frac{4\beta^2}{2^{2b}(2\phi-\psi-1)^2}+\frac{2\beta}{3^b(3\phi-2\psi-1)}. \tag{2.6}$$

Proof. It follows from (2.1) and (2.2) that

$$\frac{z[(\mathfrak{D}^b f(z))']^{\phi}}{(1-\psi)\mathfrak{D}^b f(z)+\psi\mathfrak{D}^{b+1}f(z)} = [y(z)]^{\beta} \tag{2.7}$$

and

$$\frac{h[(\mathfrak{D}^b g(h))']^{\phi}}{(1-\psi)\mathfrak{D}^b g(h)+\psi\mathfrak{D}^{b+1}g(h)} = [x(h)]^{\beta} \tag{2.8}$$

where $y(z)$ and $x(u)$ are in the class $\mathcal{P}$ which is of the form

$$y(z) = 1 + y_1 z + y_2 z^2 + y_3 z^3 + \cdots \tag{2.9}$$

$$x(h) = 1 + x_1 h + x_2 h^2 + x_3 h^3 + \cdots \tag{2.10}$$

Hence,

$$[y(z)]^{\beta} = 1 + \beta y_1 z + \left(\beta y_2 + \frac{\beta(\beta-1)y_1^2}{2!}\right) z^2 + \cdots$$

$$[x(h)]^{\beta} = 1 + \beta x_1 h + \left(\beta x_2 + \frac{\beta(\beta-1)x_1^2}{2!}\right) h^2 + \cdots .$$

Now, equating the coefficient in (2.7) and (2.8) we get

$$(2\phi-\psi-1)2^b m_2 = \beta y_1, \tag{2.11}$$

$$2^{2b}(2\phi^2-4\phi+\psi^2+2\phi\psi-2\psi+1)m_2^2 + 3^b(3\phi-2\psi-1)m_3 = \beta y_2 + \frac{\beta(\beta-1)y_1^2}{2!}, \tag{2.12}$$

$$-(2\phi-\psi-1)2^b m_2 = \beta x_1, \tag{2.13}$$

$$3^b(2m_2^2-m_3)(3\phi-2\psi-1) + (2\phi^2-4\phi+\psi^2+2\phi\psi-2\psi+1)m_2^2 2^{2b} = \beta x_2 + \frac{\beta(\beta-1)x_1^2}{2!}. \tag{2.14}$$

From (2.11) and (2.13) we obtain

$$y_1 = -x_1 \tag{2.15}$$

and

$$2^{2b+1}(2\phi-\psi-1)^2 m_2^2 = \beta^2(y_1^2+x_1^2) \tag{2.16}$$

Also from (2.12), (2.14) and (2.16), we have

$$m_2^2 = \frac{\beta^2(y_2+x_2)}{3^b\beta(6\phi-4\psi-2)+2^{2b+1}\beta(2\phi^2-4\phi+\psi^2+2\phi\psi-2\psi+1) - (\beta-1)2^{2b}(2\phi-\psi-1)^2}$$

$$m_2^2 = \frac{\beta^2(y_2+x_2)}{3^b\beta(6\phi-4\psi-2)+2^{2b}[2\beta(2\phi^2-4\phi+\psi^2+2\phi\psi-2\psi+1) - (\beta-1)(2\phi-\psi-1)^2]}$$

Applying Lemma (1.1) for the coefficients y_2 and x_2, we get

$$|m_2| \leq \frac{2\beta}{\sqrt{3^b\beta(6\phi-4\psi-2)+2^{2b}[2\beta(2\phi^2-4\phi+\psi^2+2\phi\psi-2\psi+1) - (\beta-1)(2\phi-\psi-1)^2]}}.$$

Which gives us the desired estimate on $|m_2|$ as asserted in (2.5).

Next, in order to find the bounds on $|m_3|$, subtracting (2.14) from (2.12) we get

$$3^b(6\phi-4\psi-2)m_3 - 3^b(6\phi-4\psi-2)m_2^2 = \beta(y_2-x_2) + \frac{\beta(\beta-1)}{2!}(y_1^2-x_1^2) \tag{2.17}$$

It follows from (2.15), (2.16) and (2.17) that

$$m_3 = \frac{\beta^2(y_1^2+x_1^2)}{2^{2b+1}(2\phi-\psi-1)^2} + \frac{\beta(y_2-x_2)}{3^b(6\phi-4\psi-2)}$$

Applying Lemma 1.1 for the coefficients y_1, y_2, x_1 and x_2, we have

$$|m_3| \leq \frac{4\beta^2}{2^{2b}(2\phi-\psi-1)^2} + \frac{2\beta}{3^b(3\phi-2\psi-1)}$$

We get the desired estimate $|m_3|$ as asserted in (2.6).

□

Putting $\phi = 1$ in Theorem 2.1, we have the following corollary.

Corollary 2.1. *Let $f(z)$ given by (1.1) be in the class $\mathfrak{B}_{\mathfrak{E}}^{1,b}(\beta,\psi)$. Then*

$$|m_2| \leq \frac{2\beta}{\sqrt{4\beta(1-\psi)3^b+2^{2b}[2\beta(\psi^2-1)-(\beta-1)(1-\psi)^2]}}$$

and

$$|m_3| \leq \frac{4\beta^2}{2^{2b}(1-\psi)^2} + \frac{\beta}{3^b(1-\psi)}.$$

which is the results obtain by Jothibasu [9].

Putting $\psi = 0$ in Corollary (2.1), we have the following corollary.

Corollary 2.2. *Let $f(z)$ given by (1.1) be in the class $\mathfrak{B}^b_{\mathfrak{E}}(\beta, 0)$. Then*

$$|m_2| \leq \frac{2\beta}{\sqrt{4\beta 3^b + 2^{2b}(1-3\beta)}}$$

and

$$|m_3| \leq \frac{4\beta^2}{2^{2b}} + \frac{\beta}{3^b}.$$

Now putting $b = 0$ in Corollary (2.2), we obtain the coefficient estimate for well-known class $\mathfrak{B}^0_{\mathfrak{E}}(\beta, 0)$= $S^*_{\mathfrak{E}}(\beta)$ of strongly bi-starlike functions of order β as in [5]. Also when $b = 1$ in Corollary (2.2), we obtain well-known class $\mathfrak{B}^1_{\mathfrak{E}}(\beta, 0)$= $\mathcal{K}_{\mathfrak{E}}(\beta)$ of strongly bi-convex function of order β and have the same results in [5].

3. Coefficient Bounds for the Function Class $\mathfrak{B}^{\phi,b}_{\mathfrak{E}}(\mu, \psi)$

Definition 3.1. *A function $f(z)$ given by (1.1) is said to be in the class $\mathfrak{B}^{\phi,b}_{\mathfrak{E}}(\mu, \psi)$ if the following conditions are satisfied:*

$$\Re\left[\frac{z[(\mathfrak{D}^b f(z))']^\phi}{(1-\psi)\mathfrak{D}^b f(z) + \psi \mathfrak{D}^{b+1} f(z)}\right] > \mu \quad z \in \nabla, \tag{3.1}$$

and

$$\Re\left[\frac{h[(\mathfrak{D}^b g(h))']^\phi}{(1-\psi)\mathfrak{D}^b g(h) + \psi \mathfrak{D}^{b+1} g(h)}\right] > \mu \quad h \in \nabla, \tag{3.2}$$

where $f(z) \in \mathfrak{E}$, $\phi \geq 1$, $0 \leq \mu < 1$, $0 \leq \psi < 1$ and

$$g(h) = h - m_2h^2 + (2m_2^2 - m_3)h^3 - (5m_2^3 - 5m_2m_3 + m_4)h^4 + \cdots .$$

Remark 3.1. *Taking $\psi = 0$ in the class $\mathfrak{B}^{\phi,b}_{\mathfrak{E}}(\mu, \psi)$, we have $\mathfrak{B}^{\phi,b}_{\mathfrak{E}}(\mu, 0) = \mathfrak{B}^{\phi,b}_{\mathfrak{E}}(\mu)$ and $f \in \mathfrak{B}^{\phi,b}_{\mathfrak{E}}(\mu)$ if the following conditions are satisfied:*

$$\Re\left[\frac{z[(\mathfrak{D}^b f(z))']^\phi}{\mathfrak{D}^b f(z)}\right] > \mu \quad z \in \nabla, \tag{3.3}$$

and

$$\Re\left[\frac{h[(\mathfrak{D}^b g(h))']^\phi}{\mathfrak{D}^b g(h)}\right] > \mu \quad h \in \nabla, \tag{3.4}$$

where $f(z) \in \mathfrak{E}$, $\phi \geq 1$, $0 \leq \mu < 1$ and

$$g(h) = h - m_2h^2 + (2m_2^2 - m_3)h^3 - (5m_2^3 - 5m_2m_3 + m_4)h^4 + \cdots .$$

We note that for $b = 0$, $\phi = 1$ and $\psi = 0$ the class $\mathfrak{B}^{1,0}_{\mathfrak{E}}(\mu, 0) = S^*_{\mathfrak{E}}(\mu)$ is class of strongly bi-starlike functions of order μ $(0 \leq \mu < 1)$. When $b = 1$, $\phi - 1 = \psi = 0$ and the class $\mathfrak{B}^{1,1}_{\mathfrak{E}}(\mu, 0) = \mathcal{K}^*_{\mathfrak{E}}(\mu)$ is class of strongly bi-convex functions of order μ $(0 \leq \mu < 1)$.

Remark 3.2. *For $b = 0$ we have class introduced and studied in [8].*

Now we have the following theorem and the proof.

Theorem 3.1. *Let $f(z)$ given by (1.1) be in the class $\mathfrak{B}_{\mathfrak{E}}^{\phi,b}(\mu,\psi)$. Then*

$$|m_2| \le \sqrt{\frac{2(1-\mu)}{2^{2b}(2\phi^2-4\phi+\psi^2+2\phi\psi-2\psi+1)+(3\phi-2\psi-1)3^b}} \tag{3.5}$$

and

$$|m_3| \le \frac{4(1-\mu)^2}{2^{2b}(2\phi-\psi-1)^2} + \frac{2(1-\mu)}{3^b(3\phi-2\psi-1)}. \tag{3.6}$$

Proof. It follows from (3.3) and (3.4) that there exist $y, x \in \mathcal{P}$ such that

$$\frac{z[(\mathfrak{D}^b f(z))']^\phi}{(1-\psi)\mathfrak{D}^b f(z) + \psi\mathfrak{D}^{b+1} f(z)} = \mu + (1-\mu)y(z) \tag{3.7}$$

and

$$\frac{h[(\mathfrak{D}^b g(h))']^\phi}{(1-\psi)\mathfrak{D}^b g(h) + \psi\mathfrak{D}^{b+1} g(h)} = \mu + (1-\mu)x(h) \tag{3.8}$$

where $y(z)$ and $x(h)$ in $\mathcal{P}$ given by (2.9) and (2.10), that is

$$\mu + (1-\mu)y(z) = 1 + (1-\mu)y_1 z + (1-\mu)y_2 z^2 + \cdots$$

and

$$\mu + (1-\mu)x(h) = 1 + (1-\mu)x_1 h + (1-\phi)x_2 h^2 + \cdots$$

Equating the coefficients of (3.7) and (3.8) we get

$$(2\phi-\psi-1)2^b m_2 = (1-\mu)y_1, \tag{3.9}$$

$$2^{2b}(2\phi^2 - 4\phi + \psi^2 + 2\phi\psi - 2\psi + 1)m_2^2 + 3^b(3\phi - 2\psi - 1)m_3 = (1-\mu)y_2, \tag{3.10}$$

$$-(2\phi-\psi-1)2^b m_2 = (1-\mu)x_1, \tag{3.11}$$

$$3^b(2m_2^2 - m_3)(3\phi-2\psi-1) + (2\phi^2-4\phi+\psi^2+2\phi\psi-2\psi+1)m_2^2 2^{2b} = (1-\mu)x_2. \tag{3.12}$$

From (3.9) and (3.11) we get

$$y_1 = -x_1 \tag{3.13}$$

and

$$2^{2b+1}(2\phi-\psi-1)^2m_2^2=(1-\mu)^2(y_1^2+x_1^2) \tag{3.14}$$

Now adding (3.10), (3.12) and (3.14), we deduce that

$$m_2^2=\frac{(1-\mu)(y_2+x_2)}{2^{2b+1}(2\phi^2-4\phi+\psi^2+2\phi\psi-2\psi+1)+2(3\phi-2\psi-1)3^b}$$

$$|m_2^2|\leq\frac{(1-\mu)(|y_2|+|x_2|)}{2^{2b+1}(2\phi^2-4\phi+\psi^2+2\phi\psi-2\psi+1)+2(3\phi-2\psi-1)3^b} \tag{3.15}$$

Applying Lemma 1.1 for the coefficients y_2 and x_2, we have

$$|m_2|\leq\sqrt{\frac{2(1-\mu)}{2^{2b}(2\phi^2-4\phi+\psi^2+2\phi\psi-2\psi+1)+(3\phi-2\psi-1)3^b}} \tag{3.16}$$

which gives us the desired estimate on $|m_2|$ as asserted in (3.5).

Hence in order to get the bound on $|m_3|$, by subtracting (3.12) from (3.10), we get

$$3^b(6\phi-4\psi-2)m_3-3^b(6\phi-4\psi-2)m_2^2=(1-\mu)(y_2-x_2) \tag{3.17}$$

$$m_3=m_2^2+\frac{(1-\mu)(y_2-x_2)}{3^b(6\phi-4\psi-2)} \tag{3.18}$$

then from (3.14), we have

$$m_3=\frac{(1-\mu)^2(y_1^2+x_1^2)}{2^{2b+1}(2\phi-\psi-1)^2}+\frac{(1-\mu)(y_2-x_2)}{3^b(6\phi-4\psi-2)} \tag{3.19}$$

Applying Lemma 1.1 for the coefficients y_1, y_2, x_1 and x_2, we have

$$|m_3|\leq\frac{4(1-\mu)^2}{2^{2b}(2\phi-\psi-1)^2}+\frac{2(1-\mu)}{3^b(3\phi-2\psi-1)}. \tag{3.20}$$

We get desired estimate on $|m_3|$ as asserted in (3.6). □

Putting $\phi=1$ in Theorem 3.1, we have the following corollary.

Corollary 3.1. *Let $f(z)$ given by (1.1) be in the class $\mathfrak{B}_{\mathfrak{E}}^{1,b}(\mu,\psi)$. Then*

$$|m_2|\leq\sqrt{\frac{2(1-\mu)}{2^{2b}(\psi^2-1)+2(1-\psi)3^b}}$$

and

$$|m_3|\leq\frac{4(1-\mu)^2}{2^{2b}(1-\psi)^2}+\frac{2(1-\mu)}{3^b(1-\psi)}$$

which is the results obtain by Jothibasu [9].

Putting $\psi = 0$ in Corollary (3.1), we have the following corollary.

Corollary 3.2. *Let $f(z)$ given by (1.1) be in the class $\mathfrak{B}^b_{\mathfrak{E}}(\mu, 0)$. Then*

$$|m_2| \leq \sqrt{\frac{1-\mu}{3^b - 2^{2b-1}}}$$

and

$$|m_3| \leq \frac{4(1-\mu)^2}{2^{2b}} + \frac{2(1-\mu)}{3^b}.$$

Now putting $b = 0$ in Corollary (3.2), we obtain the coefficient estimate for well-known class $\mathfrak{B}^0_{\mathfrak{E}}(\mu, 0) = S^*_{\mathfrak{E}}(\mu)$ of bi-starlike functions of order μ as in [5]. Also when $b = 1$ in Corollary (3.2), we obtain well-known class $\mathfrak{B}^1_{\mathfrak{E}}(\mu, 0)= \mathcal{K}_{\mathfrak{E}}(\mu)$ of bi-convex function of order μ and have the same results in [5].

Remark 3.3. *When $b = 0$, the results acquired in this paper corresponds with the results considered in [8]. Also, for the different pick of b the results considered in this paper would pilot to many known and new results.*

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