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RESULTS OF SEMIGROUP OF LINEAR OPERATOR IN SPECTRAL THEORY

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Abstract. This paper consists of results ω-order preserving partial contraction mapping (ω-OCP_n) as a Semigroup of a linear operator in spectral theory. We consider $A \in \omega - OCP_n$ as the infinitesimal generator of a C_0-semigroup using the Spectral Mapping Theorem (SMT) to establish the relationships between the spectrum of A and the spectrum of each of the operators $T(t),\ t \geq 0$.

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1. Introduction

The study of spectral theory in mathematics is an inclusive term for theories extending the eigenvector and eigenvalue theory of a single square matrix to a much broader theory of the structure of operators in a variety of mathematical spaces. Assume $X_n \subseteq X$ is a finite set, $(T(t))_{t\geq0}$ the C_0-semigroup, $\omega - OCP_n$ the ω-order preserving partial contraction mapping, M_m be a matrix, $L(X)$ be a bounded linear operator on X, P_n a partial transformation semigroup, $\rho(A)$ a resolvent set, $\sigma(A)$ a spectrum of A and $A \in \omega - OCP_n$ is a generator of C_0-semigroup. This paper consist of results of ω-preserving partial contraction mapping generating a spectral results. Balakrishnan [1], deduced some fractional powers of closed operators. Banach [2], established and introduced the concept of Banach spaces. Engel and Nagel [3], obtained one-parameter semigroup for linear evolution equations. Greiner *et al.* [4], proved some results on

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the spectral bond generator of semigroup of positive operators. Hasegawa [5], presesented some results on the convergence of resolvents of operators. Pazy [6], introduced semigroup of linear operators and applications to partial differential equations. Rauf and Akinyele [7], introduced ω-order-preserving partial contraction mapping and established its properties, also in [8], Rauf *et al.* deduced some results of stability and spectra properties on semigroup of linear operator. Slemrod [9], explained asymptotic behavior of C_0-semigroup as determined by the spectrum of the generator. Vrabie [10], proved some results of C_0-semigroup and its applications. Yosida [11], established and proved some results on differentiability and representation of one-parameter semigroup of linear operators.

2. Preliminaries

Definition 2.1 (C_0-Semigroup) [10]
A C_0-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.

Definition 2.2 (ω-OCP_n) [7]
A transformation $\alpha \in P_n$ is called ω-order-preserving partial contraction mapping if $\forall x, y \in$Domα : $x \leq y \implies \alpha x \leq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.

Definition 2.3 (Resolvent Set) [3]
We define the resolvent set of A denoted by $\rho(A)$ set of all $\lambda \in \mathbb{C}$ such that λI - A is one-to-one with range equal to X

Definition 2.4 (Spectrum) [10]
The spectrum of A denoted by $\sigma(A)$ is defined as the complement of the resolvent set.

Example 1
Let X be the Banach space of Continuous function on [0,1] which are equal to zero at $x = 1$ with the supremum norm. Define

$$(T(t)f)(x) = \begin{cases} f(x+t) \; if \; x+t \leq 1 \\ 0 \; if \; x+t > 1 \end{cases}$$

$T(t)$ is obviously a C_0-Semigroup of Contractions on X. Its infinitesimal generator $A \in \omega$-OCP_n is given by

$$D(A) = \{f : f \in C'([0,1]) \cap X_1 f' \in X\}$$

and

$$Af = f' \; for \; f \in D(A).$$

one checks easily that for every $\lambda \in \mathbb{C}$ and $g \in X$ the equation $\lambda f - f' = g$ has a unique solution $f \in X$ given by

$$f(t) = \int_t^1 e^{\lambda(t-s)} g(s) ds.$$

Therefore $\sigma(A) = \phi$. on the other hand, since for every $t \geq 0,\ T(t)$ is a bounded linear operator, $\sigma(T(t)) \neq \phi$ for all $t \geq 0$ and the relation $\sigma(T(t)) = exp\{t\sigma(A)\}$ does not hold for any $t \geq 0$.

Theorem 2.1(Hille-Yoshida) []

A linear operator $A : D(A) \subseteq X \to X$ is the infinitesimal generator for a C_0-semigroup of contraction if and only if

i. A is densely defined and closed; and

ii. $(0, +\infty) \subseteq \rho(A)$ and for each $\lambda > 0$, we have

$$\|R(\lambda, A)\|_{L(X)} \leq \frac{1}{\lambda}. \tag{2.1}$$

3. Main Results

In this section, differentiable and analytic results on ω-OCP_n in semigroup of linear operator (C_0-semigroup) were established:

Theorem 3.1

This section presents results of spectral theory generated by ω-OCP_n which were established on semigroup of linear operator:

Theorem 3.1

Assume $T(t)$ is a C_0-semigroup and let $A \in w - OCP_n$ be its infinitesimal generator. If

$$B_\lambda(t)x = \int_0^t e^{\lambda(t-s)} T(s) x ds \tag{3.1}$$

then

$$(\lambda I - A) B_\lambda(t) x = e^{\lambda t} x - T(t) x \ for\ every\ x \in X \tag{3.2}$$

and

$$B_\lambda(t)(\lambda I - A) x = e^{\lambda t} x - T(t) x \ for\ every\ x \in D(A) \tag{3.3}$$

and $A \in w - OCP_n$.

Proof:

For every fixed λ *and* t, $B_\lambda(t)$ defined by (3.1) is a bounded linear operator on X. Moreover, for every $x \in X$ and $A_1B \in w - OCP_n$ we have

$$\frac{T(h)-1}{h}B_\lambda(t)x = \frac{e^{\lambda h}-1}{h}\int_h^t e^{\lambda(t-s)}T(s)xds + \frac{e^{\lambda h}}{h}\int_t^{t+h} e^{\lambda(t-s)}T(s)xds - \frac{1}{h}\int_0^h e^{\lambda(t-s)}T(s)xds. \tag{3.4}$$

As $h \to 0$ in the right-hand side of (3.4) converges to $\lambda B_\lambda(t)x + T(t)x - e^{\lambda t}x$ and consequently $B_\lambda(t)x \in D(A)$ and

$$AB_\lambda(t)x = \lambda B_\lambda(t)x + T(t)x - e^{\lambda t}x \tag{3.5}$$

which implies (3.2). From the definition of $B_\lambda(t)$, it is clear that for $x \in D(A)$ and $A \in w-OCP_n$, we have

$$AB_\lambda(t)x = B_\lambda(t)Ax$$

which implies

$$B_\lambda(t)(\lambda I - A)x = e^{\lambda t}x - T(t)x.$$

Hence, the prove is completed.

Theorem 3.2

Let $T(t)$ be a C_0-semigroup and $A \in w - OCP_n$ be the infinitesimal generator of semigroup of linear operator. Then

$$\sigma(T(t)) \supset e^{t\sigma(A)} \; for \; t \geq 0. \tag{3.6}$$

Proof

Let $e^{\lambda t} \in \rho(T(t))$ and let $Q = (e^{\lambda t}I - T(t))^{-1}$. The operators $B_\lambda(t)$, defined by (3.1), and Q clearly commute. From (3.2) and (3.3) we deduce

$$(\lambda I - A)B_\lambda(t)Qx = x \tag{3.7}$$

for every $x \in X$ and $A \in w - OCP_n$, and

$$QB_\lambda(t)(\lambda I - A)x = x \tag{3.8}$$

for every $x \in D(A)$ *and* $A \in w - OCP_n$.

Since $B_\lambda(t)$ and Q commute, we also have

$$B_\lambda(t)Q(\lambda I - A)x = x \tag{3.9}$$

for every $x \in D(A)$ *and* $A \in w - OCP_n$.

Therefore, $\lambda \in \rho(A), B_\lambda(t)Q = (\lambda I - A)^{-1} = R(\lambda; A)$ and $\rho(T(t)) \subset exp(t\rho(A))$ which implies

$$\sigma(T(t)) \supset e^{t\sigma(A)}$$

we recall that the spectrum of A consists of three mutually exclusive part; the point spectrum $\sigma_p(A)$, the continuous spectrum $Q_c(A)$ and the residual spectrum $\sigma_t(A)$. These are defined as follows: $\lambda \in \sigma_p(A)$ if $\lambda I - A$ is not one-to-one, $\lambda \in \sigma_c(A)$ if $\lambda I - A$ is one-to-one and its range is not dense in X. From these definitions it is clear that $\sigma_p(A), \sigma_c(A)$ and $\sigma_t(A)$ are mutually exclusive and that their union is $\sigma(A)$, therefore the proof is achieved.

Theorem 3.3

Suppose $T(t)$ is a C_0-semigroup and let $A \in w - OCP_n$ be its infinitesimal generator. Then

$$e^{t\sigma_p(A)} \subset \sigma_p(T(t)) \subset e^{t\sigma_p(A)} \cup \{0\}. \tag{3.10}$$

More precisely if $\lambda \in \sigma_p(A)$ then $e^{\lambda t} \in \sigma_p(T(t))$ and if $e^{\lambda t} \in \sigma_p(T(t))$ then there exists a $k, k \in \mathbb{N}$ such that $\lambda_k = \lambda + 2\pi ik/t \in \sigma_p(A)$.

Proof:

Suppose $\lambda \in \sigma_p(A)$ then there is $x_0 \in D(A),\ x_0 \neq 0$, such that $(\lambda I - A)x_0 = 0$. From (3.3) it then follows that $(e^{\lambda t}I - T(t))x_0 = 0$ and therefore $e^{\lambda t} \in \sigma_p(T(t))$ which proves the first inclusion. To prove the second inclusion, let $e^{\lambda t} \in \sigma_p(T(t))$ and let $x_0 \neq 0$ satisfy $(e^{\lambda t}I - T(t))x_0 = 0$. This implies that the continuous function $s \to e^{-\lambda s}T(s)x_0$ is periodic with period t and since it does not varnish identically and one of its Fourier Coefficients must be different from zero. Therefore is a $k, k \in \mathbb{N}$ such that

$$x_k = \frac{1}{t}\int_0^t e^{-(2\pi ik/t)s}(e^{-\lambda s}T(s)x_0)ds \neq 0 \tag{3.11}$$

We will show that $\lambda_k = \lambda + 2\pi ik/t$ is an eigen value of A. Let $||T(t)|| \leq Me^{wt}$. For $Re\mu > w$ we have

$$\begin{aligned} R(\mu; A)x_0 &= \int_0^\infty e^{-\mu s}T(s)x_0 ds = \sum_{n=0}^{\infty}\int_{nt}^{(n+1)t} e^{-\mu s}T(s)x_0 ds \\ &= \sum_{n=0}^{\infty} e^{n(\lambda-\mu)t}\int_0^t e^{-\mu s}T(s)x_0 ds \\ &= (1 - e^{(\lambda-\mu)t})^{-1}\int_0^t e^{-\mu s}T(s)x_0 ds \end{aligned} \tag{3.12}$$

where we used the periodicity of $e^{-\lambda s}T(s)x_0$. The integral on the right-hand side of (3.12) is clearly an entire function and therefore $R(\mu; A)x_0$ can be extended by (3.12) to a meromorphic function with possible poles at

$$\lambda_n = \lambda + 2\pi in/t,\ n \in \mathbb{N}.$$

Using (3.12), it is easy to show that

$$\lim_{\mu \to \lambda_k} (\mu - \lambda_k) R(\mu : A) x_0 = x_k \tag{3.13}$$

and

$$\lim_{\mu \to \lambda_k} (\lambda_k - A)[(\mu - \lambda_k) R(\mu; A) x_0] = 0 \tag{3.14}$$

From the closedness of A and (3.13), (3.14) it follows that $x_k \in D(A)$ and that $(\lambda_k I - A)x_k = 0$, i.e; $\lambda_k \in \sigma_p(A)$. Hence, the proof is complete.

Theorem 3.4

Assume $T(t)$ is a C_0-semigroup and let $A \in w - OCP_n$ be the infinitesimal generator of semigroup f linear operator. Then

(i) if $\lambda \in \sigma_t(A)$ and none of the $\lambda_n = \lambda + 2\pi in/t, n \in \mathbb{N}$ is in $\sigma_p(A)$ then $e^{\lambda t} \in \sigma_t(T(t))$; and

(ii) if $e^{\lambda t} \in \sigma_t(T(t))$ then none of the $\lambda_n = \lambda + 2\pi in/t, n \in \mathbb{N}$ is in $\sigma_p(A)$ and there exists a $K, K \in \mathbb{N}$ such that $\lambda_k \in \sigma_t(A)$.

Proof:

If $\lambda \in \sigma_t(A)$ then there is a $x^* \in X^*,\ x^* \neq 0$, such that $< x^*, (\lambda I - A)x >= 0$ for all $x \in D(A)$. From (3.2) it then follows that

$$< x^*, (e^{\lambda t} I - T(t))x >= 0$$

for all $x \in X$ and therefore the range of $e^{\lambda t} I - T(t)$ is not dense in X. If $e^{\lambda t} I - T(t)$ is not one-to-one then by Theorem 3.3, there is a $k \in \mathbb{N}$ such that $\lambda_k \in \sigma_p(A)$ contradicting our assumption that $\lambda_n \notin \sigma_p(A)$. Therefore $e^{\lambda t} I - T(t)$ is one-to-one and $e^{\lambda t} \in \sigma_t(T(t))$ which concludes the proof of (i).

To prove (ii), we note first that if for some k, $\lambda_k = \lambda + 2\pi ik/t \in \sigma_p(A)$ then by Theorem 3.3 $e^{\lambda t} \in \sigma_p(T(t))$ contradicting the assumption that $e^{\lambda t} \in \sigma_t(T(t))$. It suffices therefore to show that for some $k \in \mathbb{N},\ \lambda_k \in \sigma_t(A)$. This follows at once if we show that $\{\lambda_n\} \subset \rho(A) \cup \sigma_p(A)$ is impossible. From (3.3) we have

$$(e^{\lambda_n t} I - T(t))x = B_{\lambda_n}(t)(\lambda_n I - A)x \tag{3.15}$$

for $x \in D(A),\ A \in w - OCP_n$ and $n \in \mathbb{N}$.

Since by our assumption $e^{\lambda t} = e^{\lambda_n t} \in \sigma_t(T(t))$, the left-hand side of (3.15) belongs to a fixed non dense linear subspace Y of X. On the other hand if $\lambda_n \in \rho(A) \cup \sigma_c(A)$ then the range of $\lambda_n I - A$ is dense in X which implies by (3.15) that the range of $B_{\lambda_n}(t)$ belongs to Y for every $n \in \mathbb{N}$. Writing the Fourier series of the continuous function $e^{-\lambda s}T(s)x$ we have

$$e^{-\lambda s}T(s)x \sim \frac{1}{t}\sum_{n=-\infty}^{\infty} e^{(2\pi i n/t)} B_{\lambda_n}(t)x \tag{3.16}$$

and each term on the right-hand side of (3.16) belongs to Y. As in the classical numerical case the series (3.16) is summable to $e^{-\lambda s}T(s)x$ for $0 < s < t$ and therefore for $0 < s < t, e^{-\lambda s}T(s)x \in Y$. Letting $s \to 0$ it follows that every $x \in D(A)$ satisfies $x \in \overline{Y}$ which is impossible since $\overline{Y}$ is a proper closed subspace of X and D(A) is dense in X and the prove is achieved.

Theorem 3.5

Suppose $T(t)$ is C_0-semigroup and let A be its infinitesimal generator. If $\lambda \in \sigma_c(A)$ and if none of the $\lambda_n = \lambda + 2\pi i n/t$ in $\sigma_p(A) \cup \sigma_r(A)$ then $e^{\lambda t} \in \sigma_c(T(t))$.

Proof:

From 3.2 it follows that if $\lambda \in \sigma_c(A)$ then $e^{\lambda t} \in \sigma(T(t))$.

If $e^{\lambda t} \in \sigma_p(T(t))$, then by Theorem 3.3 some $\lambda_k \in \sigma_p(A)$ and therefore $e^{\lambda t} \notin \sigma_p(T(t))$.

Similarly if $e^{\lambda t} \in \sigma_r(T(t))$ then some $\lambda_k \in \sigma_r(A)$ and again $e^{\lambda t} \notin \sigma_r(T(t))$.

Hence, the prof is complete.

4. Conclusion

In this paper, it has been established that ω-order preserving partial contraction mapping (ω-OCP_n) generates results on spectral theory as a semigroup of linear operator.

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