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ON THE CLASS OF NULL ALMOST L-WEAKLY AND NULL ALMOST M-WEAKLY COMPACT OPERATORS AND THEIR WEAK COMPACTNESS

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Abstract. In this paper, we introduce new concepts of Null almost L-weakly and Null almost M-weakly compact operators. We attempts to deal with some characterizations of these operators. We investigate conditions on a pair of Banach lattices E and F that gives when every null almost L-weakly compact (resp. null almost M- weakly compact) operator $T : E \to F$ is weakly compact. We also studied the inverse sense.

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1. Introduction

Throughout this paper, X and Y will denote real Banach spaces, E and F will denote real Banach lattices. B_X (resp. B_E) is the closed unit of the Banach space X (resp. Banach lattice E). We will use term operator $T : X \longrightarrow Y$ between two Banach spaces to mean a bounded linear mapping, we refer to [1,10,13] for unexplained terminology of the Banach lattice theory and positive operators. Following Bouras et al. [6], an operator T from a Banach space X into a Banach lattice F is called almost L-weakly compact if T carries relatively weakly compact subsets of X onto L-weakly compact subsets of F. The authors proved that an operator T is almost L-weakly compact, if and only if $T(X) \subseteq F^a$, and $f_n(T(x_n)) \to 0$ for every weakly null sequence (x_n) of X and every disjoint sequence (f_n) of $B_{F'}$ (see, Proposition 1 of [9]), and an

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operator T from a Banach lattice E into a Banach space Y is called almost M-weakly compact if for every disjoint sequence (x_n) in B_E and every weakly convergent sequence (f_n) of Y', we have $f_n(T(x_n)) \to 0$ if and only if $T'(Y') \subseteq (E')^a$, and $f_n(T(x_n)) \to 0$ for every disjoint sequence (x_n) of B_E and every weakly null sequence (f_n) of Y'(see, Proposition 2 of [9]). For more information about these classes of operators, (see, [6,9]). An operator between two Banach spaces is called Dunford-Pettis whenever it maps weakly null sequences into norm null sequences. Alternatively, an operator $T : X \to F$ is called Dunford-Pettis if, and only if $|T(x_n)| \to 0$ for $\sigma(F, F')$ and $f_n(T(x_n)) \to 0$ for every weakly null sequence (x_n) of X and each disjoint sequence (f_n) of $B_{F'}$ (see, corollary 2.9 of [2]). Recall that an operator T from a Banach space X into a Banach lattice E is called L-weakly compact if for each disjoint bounded sequence (y_n) in the solid hull of $T(B_X)$, we have $\|y_n\| \longrightarrow 0$. An operator $T : E \longrightarrow Y$ is almost Dunford-Pettis, if the sequence $(T(x_n))$ converges to 0 for every weakly null sequence (x_n) of E consisting of pairwise disjoint elements, equivalently, $(T(x_n))$ converges to 0 for every weakly null sequence (x_n) of E^+. A Banach lattice E has the positive Schur property if weakly null sequences with positive terms are norm null, and a Banach lattice E has the weak Dunford-Pettis property if and only if $f_n(x_n) \to 0$ for every weakly null sequence $(x_n) \subset E^+$, with pairwise disjoint terms and for all weakly null sequence $(f_n) \subset E'$. We will use the term operator $T : E \longrightarrow F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. Note that each positive linear mapping on a Banach lattice is continuous. If $T : E \longrightarrow F$ is a positive operator between two Banach lattices, then its adjoint $T' : F' \longrightarrow E'$ is likewise positive where T' is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$. A linear operator $T : E \to F$ is called disjointness preserving if $Tx \perp Ty$ for all $x, y \in E$ satisfying $x \perp y$. A positive linear operator $T : E \to F$ is called interval preserving if $[0, Tx] = T[0, x]$ for every $x \in E_+$. A positive linear operator $T : E \to F$ is called almost interval preserving if $T[0, x]$ is dense in $[0, Tx]$ for every $x \in E_+$. We refer readers to [1] for unexplained terminology on Banach lattice theory and positive operators.

In this paper, using the class of almost L-weakly (resp. M-weakly) compact operators, we introduce a new class of operators that we call Null almost L-weakly (resp. M-weakly) compact operators. An operator T from a Banach lattice X into a Banach lattice F is called Null almost L-weakly compact if for every weakly null sequence (x_n) of X and every disjoint sequence (f_n) of $B_{F'}$, we have $f_n(T(x_n)) \to 0$. Note that the class of Null almost L-weakly operators contains that of almost L-weakly compact (resp. Dunford-Pettis) operators. However, a Null almost

L-weakly compact operator is not necessarily an almost L-weakly compact (resp. Dunford-Pettis) operator. For instance (see, remark 2.1 and remark 2.3). On the other hand, an operator T from a Banach lattice E into a Banach space Y is called Null almost M-weakly compact if $f_n(T(x_n)) \to 0$, for every disjoint sequence (x_n) of B_E and every weakly null sequence (f_n) of Y'. We also note that the latest defined class contains that of almost M-weakly compact. However, a Null almost M-weakly compact is not necessarily almost M-weakly compact (see, remark 2.5). In addition, if $T : E \to Y$ be an operator such that its adjoint T' is Dunford-Pettis, then T is null almost M-weakly compact. But the converse is not true in general (see, remark 2.6).

In this work, we give sufficient conditions under which Null almost L-weakly compact (resp. M-weakly compact) operators between Banach lattice to be almost L-weakly compact (resp. M-weakly compact). As consequences, we establish some interesting results. Precisely, for an operator $T : E \longrightarrow Y$, its adjoint T' is Null almost L-weakly compact, if only if T is Null almost M-weakly compact, and for an operator $T : X \longrightarrow F$, if its adjoint T' is Null almost M-weakly compact, then T is Null almost L-weakly compact (see, Theorem 2.1). We also study the connection between the Null almost L-weakly compact and almost Dunford-Pettis operators. We prove that if an operator $T : E \longrightarrow F$ is Null almost L-weakly compact then, T is almost Dunford-Pettis (see, Propositon 2.3). However, the converse is not true (see, Remark 2.7). Furthermore, we establish some conditions on a pair of Banach lattices E and F that tells us when every null almost L-weakly compact operator $T : E \longrightarrow F$ is weakly compact. More precisely, we prove that every null almost L-weakly compact operator $T : E \longrightarrow F$ is weakly compact if and only if the norm of E' is order continuous or F is reflexive (see, Theorem 2.2). We also prove that for a Banach lattice E and Banach space F, if every weakly compact operator $T : E \to F$ is null almost M-weakly compact, then one of the conditions is valid: the norm of E' is order continuous or F has the weak Dunford-Pettis property (see, Theorem 2.3). Moreover, we prove that for a Banach lattice E and Banach space F, if every weakly compact operator $T : E \to F$ is null almost L-weakly compact, then one of the conditions is valid: F is a KB-space or E has the weak Dunford-Pettis property (see, Theorem 2.4). In addition, we prove that for a Banach lattices E, F and G, if S is Null almost L-weakly compact operator, then $S \circ T$ is null almost L-weakly compact. And if T is a positive Null almost L-weakly compact and S is almost interval preserving operator, then $S \circ T$ is null almost L-weakly compact (see, Theorem 2.6).

2. Main Results

We start this section by the following definitions and propositions.

Definition 2.1. *An operator T from a Banach space X into a Banach lattice F is called Null almost L-weakly compact if for every weakly null sequence (x_n) of X and every disjoint sequence (f_n) of $B_{F'}$, we have $f_n(T(x_n)) \to 0$.*

Remark 2.1. *Clearly, every almost L-weakly compact operator is null almost L-weakly compact, but the converse is not true in general. For instance, consider the operator $T : \ell^1 \longrightarrow \ell^\infty$, defined by*

$$T(\lambda_1, \lambda_2, \dots) = \left(\sum_{n=1}^{\infty} \lambda_n, \sum_{n=1}^{\infty} \lambda_n, \dots\right) = \left(\sum_{n=1}^{\infty} \lambda_n\right) e$$

where $e = (1, 1, 1, \dots)$. Clearly, T is a compact operator (it has rank one) then T is the Dunford-Pettis, and hence null almost L-weakly compact operator but is neither almost L-weakly compact. To see that T is not almost L-weakly compact, let (e_n) be the sequence of the standard unit vectors of ℓ^1 and note that the singleton $\{e_n\}$ is a (weakly) compact subset of ℓ^1 but $\{T(e_n)\} = \{e\}$ is not a L-weakly compact subset of ℓ^∞ (because $e \notin (\ell^\infty)^a = c_0$). This shows that T is not almost L-weakly compact.

Remark 2.2. *If F has order continuous norm, then the classes of almost Dunford-Pettis, almost L-weakly compact, and null almost L-weakly compact operators are coincides.*

Remark 2.3. *Note that every Dunford- Pettis operator from a Banach lattice X into another F is null almost L-weakly compact, but the converse is false in general. In fact, since the Banach lattice $L^1([0,1])$ has the positive Schur property, its identity operator $Id_{L^1([0,1])} : L^1([0,1]) \to L^1([0,1])$ is null almost L-weakly compact(see, proposition 2.2). But it is not Dunford-Pettis, because $L^1([0,1])$ does not have the Schur property.*

Remark 2.4. *If the lattice operations in E or F are weakly sequentially continuous, then the classes of positive Dunford-Pettis operators and null almost L-weakly compact operators are coincides.*

Definition 2.2. *An operator T from a Banach lattice E into a Banach space Y is called Null almost M-weakly compact if $f_n(T(x_n)) \to 0$ for every disjoint sequence (x_n) of B_E and every weakly null sequence (f_n) of Y' .*

Remark 2.5. *Clearly, every almost M-weakly compact operator is null almost M-weakly compact, but the converse is not true in general. For instance, consider the operator $T : \ell^1 \longrightarrow \ell^\infty$, defined by*

$$T(\lambda_1, \lambda_2, \dots) = \left(\sum_{n=1}^{\infty} \lambda_n, \sum_{n=1}^{\infty} \lambda_n, \dots\right) = \left(\sum_{n=1}^{\infty} \lambda_n\right) e$$

where $e = (1, 1, 1, \ldots)$. Clearly, T is a compact operator (it has rank one) by Schauder's theorem ([1], Theorem 5.2) T' is compact, then T' is the Dunford-Pettis, and hence T' is null almost L-weakly compact operator, according to the theorem 2.1, T is null almost M-weakly compact operator but is neither almost M-weakly compact. In fact, the sequence (e_n) is a norm bounded disjoint sequence of ℓ^1 satisfying $T(e_n) = e$ for each n. Let κ be the continuous linear functional on ℓ^∞ defined by $\kappa(\lambda_1, \lambda_2, \ldots) = \lambda_1$ and let (κ_n) be the constant sequence defined by $\kappa_n = \kappa$ for all n (which is clearly a weak convergent sequence on $(\ell^\infty)'$). From $\kappa_n(T(e_n)) = 1 \nrightarrow 0$, we see that T fails to be almost M-weakly compact.

Remark 2.6. *Let $T : E \to Y$ be an operator such that its adjoint T' is Dunford-Pettis, then T is null almost M-weakly compact. But the converse is not true in general. For instance, the identity operator $Id_{l_\infty} : l_\infty \to l_\infty$ is null almost M-weakly compact because the Banach lattice $(l_\infty)'$ has the positive schur property (see, corollary 2.1), but its adjoint Id'_{l_∞} is not Dunford-Pettis operator.*

Proposition 2.1. *The following statements hold:*

(1) *The set of all Null almost L-weakly compact operators from X to F is a closed vector subspace of $L(X, F)$.*
(2) *The set of all Null almost M-weakly compact operators from E to Y is a closed vector subspace of $L(E, Y)$.*

Proof. (1) Let $T_1, T_2 \in NaLW(X, F)$ and $\alpha \in \mathbb{R}$. Let (x_n) be a weakly null sequence of X and (f_n) a disjoint sequence of $B_{F'}$. Since $T_1, T_2 \in NaLW(X, F)$, from definition 2.1, we have

$$f_n((\alpha T_1 + T_2)(x_n)) = \alpha f_n(T_1(x_n)) + f_n(T_2(x_n)) \longrightarrow 0.$$

Then, $\alpha T_1 + T_2 \in NaLW(X, F)$. Thus, $NaLW(X, F)$ is a vector subspace of $L(X, F)$. It is also a closed vector subspace of $L(X, F)$. Indeed, let T be in the closure of $NaLW(X, F)$, (x_n) be a weakly null sequence of X and (f_n) a disjoint sequence of $B_{F'}$. We need to show that $f_n(T(x_n)) \longrightarrow 0$. To this end, let $\varepsilon > 0$. Pick a Null almost L-weakly compact operator $S : X \longrightarrow F$ with $\|T - S\| < \varepsilon$. We have

$$\begin{aligned} \mid f_n(T(x_n)) \mid &\leq \mid f_n((T-S)(x_n)) \mid + \mid f_n(S(x_n)) \mid, \\ &\leq \parallel f_n \parallel \parallel T - S \parallel \parallel (x_n) \parallel + \mid f_n(S(x_n)) \mid . \end{aligned} \tag{1}$$

It follows from the inequality (1) that

$$\limsup |f_n(T(x_n))| \leq \varepsilon \|x_n\|_\infty.$$

By the arbitrariness of ε, we conclude that $f_n(T(x_n)) \longrightarrow 0$.

(2) $NaMW(E,Y)$ is a vector subspace of $L(E,Y)$. To see that it is also a closed vector subspace of $L(E,Y)$, let T be in the closure of $NaMW(E,Y)$. Assume that (x_n) is a disjoint sequence of B_E, and (f_n) a weakly null sequence of Y'. We show that $f_n(T(x_n)) \longrightarrow 0$. To this end, let $\varepsilon > 0$ and choose a Null almost M-weakly compact operator $S : E \longrightarrow Y$ with $\|T - S\| < \varepsilon$, we have

$$\begin{aligned} \mid f_n(T(x_n)) \mid &\leq \mid f_n((T-S)(x_n)) \mid + \mid f_n(S(x_n)) \mid, \\ &\leq \parallel f_n \parallel \parallel T - S \parallel \parallel x_n \parallel + \mid f_n(S(x_n)) \mid . \end{aligned} \tag{2}$$

So, we infer from (2) that

$$\limsup |f_n(T(x_n))| \leq \varepsilon \|f_n\|_\infty.$$

Since ε is arbitrary, we deduce that $f_n(T(x_n)) \longrightarrow 0$ holds. As desired.

□

In the following proposition, We give a characterization of the positive Schur property. For this, we need the following lemma.

Lemma 2.1. [*8, Corollary 2.6*]
Let E be a Banach lattice. A weak null sequence (x_n) in E_+ is norm convergent to 0 if and only if $f_n(x_n) \to 0$ for each disjoint bounded sequence (f_n) in E'_+.

Proposition 2.2. *For a Banach lattice E the following statements are equivalent.*

(1) *$Id_E \in aLW(E)$.*
(2) *$Id_E \in NaLW(E)$.*
(3) *$f_n(x_n) \to 0$ for every weakly null sequence (x_n) of E and every disjoint sequence (f_n) of $B_{E'}$.*
(4) *E has the positive Schur property.*

Proof.

$1 \Rightarrow 2$ Obvious.

$2 \Leftrightarrow 3$ This follows from Definition 2.1

$3 \Rightarrow 4$ This follows from Lemma 2.1

$4 \Rightarrow 1$ This follows from proposition 2.2 of [6]

□

As an immediate consequence of proposition 2.2 and corollary 2.2 of [6], we obtain the following characterization of Banach space E' with the positive Schur property.

Corollary 2.1. *For a Banach lattice E the following statements are equivalent.*

(1) *E' has the positive Schur property.*
(2) $Id_{E'} \in aLW(E')$.
(3) $Id_{E'} \in NaLW(E')$.
(4) $Id_E \in aMW(E)$.
(5) $Id_E \in NaMW(E)$.

Proposition 2.3. *Let T be a positive operator from E into F. If $T \in Nalw(E,F)$, then $T \in aDP(E,F)$.*

Proof. Let (x_n) be a weakly null sequence of E^+ and (f_n) a disjoint sequence of $B_{F'}$. The positive sequence $(T(x_n))$ converges weakly to zero. Since $T \in Nalw(E,F)$, then $f_n(T(x_n)) \to 0$. Thus, Lemma 2.1, implies that $\|T(x_n)\| \to 0$. This shows that $T \in aDP(E,F)$. □

Remark 2.7. *Note that the inverse of Proposition 2.3 is not always true. that is, a positive almost Dunford-Pettis operator is not necessarily a Null almost L-weakly compact. As counter example, let $E = L^1[0,1]$ and $F = c$. Since E has not weakly sequentialty continuous lattice operations and the norm of F is not order continuous, it follows from ([11], Theorem 2) that there exist two operators S, T such that $0 \leq T \leq S$ with S is Dunford-Pettis (almost Dunford-Pettis) and T is not Dunford-Pettis. Since the class of almost Dunford-Pettis operators satisfies the domination problem, then T is almost Dunford-Pettis. The operator $T : E = L^1[0,1] \longrightarrow F = c$ is not Dunford-Pettis, and F has weakly sequentialty continuous lattice operations, then by ([2], Corollary 2.9) there exist a weakly null sequence (x_n) of E and a disjoint sequence (f_n) of $B^+_{F'}$ such that $|T(x_n)| \to 0$ and $f_n(T(x_n)) \nrightarrow 0$. Hence T is not Null almost L-weakly compact.*

In the following result, we will use the following lemmas:

Lemma 2.2. ([6], *Lemma 2.4) For every nonempty bounded subset $A \subset E$, the following assertions are equivalent.*

(1) *A is L-weakly compact.*

(2) $f_n(x_n) \to 0$ *for every sequence* (x_n) *of* A *and every disjoint sequence* (f_n) *of* $B_{E'}$

Lemma 2.3. ([6], *Lemma 2.5*) *For every nonempty bounded subset* $A \subset E'$ *the following assertions are equivalent.*

(1) A *is L-weakly compact.*

(2) $f_n(x_n) \to 0$ *for every sequence* (f_n) *of* A *and every disjoint sequence* (x_n) *of* B_E

For the duality of classes Null almost L-weakly and Null almost M-weakly compact operators, we have the following result.

Theorem 2.1. *The following statements hold:*

(1) *An operator* $T : E \longrightarrow Y$ *is Null almost* M*-weakly compact if and only if its adjoit* T' *is Null almost* L*-weakly compact.*

(2) *For an operator* $T : X \longrightarrow F$*, if its adjoit* T' *is Null almost* M*-weakly compact, then* T *is Null almost* L*-weakly compact.*

Proof. (1) Let $T : E \longrightarrow Y$ an operator. By Definition 2.1, T' is Null almost L-weakly compact if and only if $T'(f_n)(x_n) \to 0$ for every weakly null sequence (f_n) of Y' and every disjoint sequence (x_n) of $B_{E''}$. This is implies $f_n(T(x_n)) \to 0$ for every weakly null sequence (f_n) of Y' and every disjoint sequence (x_n) of B_E. Then T is Null almost M-weakly compact.

For the other sense, assume that $T : E \to Y$ is Null almost M-weakly compact operator, we have to prove that $T' : Y' \to E'$ is Null almost L-weakly compact. By definition 2.2 the operator T from a Banach lattice E into a Banach space Y is Null almost M-weakly compact if and only if $f_n(T(x_n)) \to 0$ for every disjoint sequence (x_n) of B_E and every weakly null sequence (f_n) of Y'. According to lemme 2.3, we deduce that $T'(A)$ is L-weakly compact in E' with $A = \{f_n, n \in \mathbb{N}\}$, and according to lemme 2.2, we conclude that $X_n(g_n) \to 0$ for every sequence (g_n) of $T'(A)$ and every disjoint sequence (X_n) of $B_{E''}$, this is equivalent to $X_n(T'(f_n)) \to 0$ for every disjoint sequence (X_n) of $B_{E''}$. so by definition 2.1, $T' : Y' \to E'$ is Null almost L-weakly compact.

(2) Let $T : X \longrightarrow F$ be an operator such that T' is Null almost M-weakly compact. Let (x_n) be a weakly null sequence of X and (f_n) a disjoint sequence of $B_{F'}$. Let $J : X \longrightarrow X''$ be the canonical embedding of X into X''. Since $T' : F' \longrightarrow X'$ is Null almost M-weakly compact, and the sequence $(J(x_n))$ of X'' is weakly null, then $J(x_n)(T'(f_n)) = f_n(T(x_n)) \to 0$. Hence T is Null almost L-weakly compact.

□

Remark 2.8. *However, in general:*

$$T \text{ is null almost } L\text{-weakly compact} \not\Rightarrow T' \text{ is null almost } M\text{-weakly compact.}$$

Indeed, consider the operator $Id_{\wedge(\omega,1)} : \wedge(\omega,1) \to \wedge(\omega,1)$. Since the Lorentz space $\wedge(\omega,1)$ has the positive Schur property, $Id_{\wedge(\omega,1)}$ is null almost L-weakly compact (see, proposition 2.2). On the other hand $Id'_{\wedge(\omega,1)}$ is not null almost M-weakly compact, because the bidual of Lorentz space $\wedge(\omega,1)$ does not have the positive Schur property (see, corollary 2.1).

The next result tells us when every Null almost L-weakly compact operator is weakly compact.

Theorem 2.2. *Let E and F be two Banach lattices. The following assertions are equivalent.*

(1) *Every Null almost L-weakly compact operator $T : E \longrightarrow F$ is weakly compact.*

(2) *One of the conditions is valid:*

 (a) *the norm of E' is order continuous.*

 (b) *F is reflexive.*

Proof. (1) $\Rightarrow$ (2) We proceed by contradiction. Assume that the norm of E' is not order continuous and F is not reflexive. To finish the proof, we have to construct a Null almost L-weakly compact operator from E into F that is not weakly compact. Since E' does not have an order continuous norm then by Meyer-Nieberg ([10], theorem 2.4.14), we may assume that l^1 is a closed sublattice of E and it follows from Meyer-Nieberg ([10], proposition 2.3 .11) that there is a positive projection P from E onto l^1. On the other hand, since F is not reflexive, then the closed unit ball B_F of F is not weakly compact. Thus, by Theorem 3.40 (Eberlein-Smulien) of [1], there exists a sequence $(y_n) \subset B_F$ without any weakly convergent subsequences. we consider the operator S defined by

$$\begin{array}{ccc} S : l^1 & \longrightarrow & F \\ (\lambda_n) & \longmapsto & S\left((\lambda_n)\right) = \sum_{n=1}^{\infty} \lambda_n y_n \end{array}$$

Note that $\sum_{n=1}^{\infty} \|\lambda_n y_n\| \leq \sum_{n=1}^{\infty} |\lambda_n| < \infty$, so S is well defined. Next, we consider the composed operator $T = S \circ P : E \to \ell^1 \to F$. Now we claim that T is a Null almost L-weakly compact operator. Indeed, let $x_n \to 0$ be a weakly null sequence of E and (f_n) a disjoint sequence of $B_{F'}$. It is obvious that $(P(x_n))$ is a weakly null sequence in ℓ^1 which has the Schur property,

$\|P(x_n)\| \to 0$. Hence $\|T(x_n)\| = \|S(P(x_n))\| \to 0$. Now, it follows from the inequality

$$|f_n(T(x_n))| \leq \|f_n\|\|T(x_n)\| \leq \|T(x_n)\|$$

that T is a Null almost L-weakly compact operator. But T is not weakly compact. To see this, note that $T(e_n) = S \circ P\,(e_n) = y_n$ for all n where e_n is the sequence with the n'th entry equals to 1 and all others are zero. Thus, since (y_n) has no weakly convergent subsequences, we conclude that T is not weakly compact.

$(2.a) \Rightarrow (1)$ From Proposition 2.3 the operator T is almost Dunford-Pettis, since the norm of E' is order continuous, it follows from Theorem 2.1 of [5] that T is weakly compact.

$(2.b) \Rightarrow (1)$ If F is reflexive, then every operator from E into F is weakly compact. □

Using Theorem 2.2 we have:

Corollary 2.2. *For a Banach lattice E the following assertions are equivalent:*

(1) *Every Null almost L-weakly compact operator $T : E \longrightarrow c_0$ is weakly compact.*

(2) *The norm of E' is order continuous.*

Corollary 2.3. *For a Banach lattice E the following assertions are equivalent:*

(1) *Every Null almost L-weakly compact operator $T : E \longrightarrow l^\infty$ is weakly compact.*

(2) *The norm of E' is order continuous.*

Corollary 2.4. *For a Banach lattice F the following assertions are equivalent:*

(1) *Every Null almost L-weakly compact operator $T : l^1 \longrightarrow F$ is weakly compact.*

(2) *F is reflexive.*

As a consequence, we obtain a characterization of the order continuity of the norm of the topological dual of a Banach lattice.

Corollary 2.5. *Let E be a Banach lattice. Then following assertions are equivalent.*

(1) *Every Null almost L-weakly compact operator $T : E \longrightarrow E$ is weakly compact.*

(2) *The norm of E' is order continuous.*

The following result tells us when every weakly compact operator $T : E \to F$ is always null almost M-weakly compact.

Theorem 2.3. *Let E and F two Banach lattices. If every weakly compact operator $T : E \to F$ is null almost M-weakly compact, then one of the conditions is valid:*

(1) *the norm of E' is order continuous;*

(2) *F has the weak Dunford-Pettis property.*

Proof. Assume by way of contradiction that the norm of E' is not order continuous and F does not has the weak Dunford-Pettis property. To finish the proof, we have to construct a weakly compact operator from E into F that is not null almost M-weakly compact.

Since the norm of E' is not order continuous, it follows from Theorem 116.1 of [13] that there is a norm bounded disjoint sequence (u_n) of positive elements of E which does not converge weakly to zero. Hence, we may assume that $\|u_n\| \leqslant 1$ for all n and also that for some $0 \leqslant \Phi \in E'$ and $\epsilon > 0$ we have $\Phi(u_n) > \epsilon$ for all n. Then it follows from Theorem 116.3 of [13] that the components Φ_n of Φ in the carriers C_{u_n} form an order bounded disjoint sequence in $(E')^+$ such that $\Phi_n(u_n) = \Phi(u_n)$ for all n and $\Phi_n(u_m) = 0$ if $n \neq m$. Define the operator $P : E \to \ell^1$ by

$$P(x) = \left(\frac{\Phi_n(x)}{\Phi(u_n)}\right)_{n=1}^{\infty} \text{ for all } x \in E$$

since $\sum_{n=1}^{\infty} \left|\frac{\Phi_n(x)}{\Phi(u_n)}\right| \leqslant \frac{1}{\epsilon}\sum_{n=1}^{\infty} \Phi_n(|x|) \leqslant \frac{1}{\epsilon}\Phi(|x|)$ holds for each $x \in E$ the operator P is well defined. As F does not have weak Dunford-Pettis property, there exist a weakly null sequence (y_n) in F and a disjoint weakly null sequence (f_n) in F' such that $(f_n(y_n))$ does not converge to zero. Then, by passing to a subsequence, we can assume that for some $\epsilon > 0$ we have

$$|f_n(y_n)| > \epsilon$$

for all n. By Theorem 5.26 of [1], the operator $S : \ell^1 \to F$ defined by

$$S\left((\lambda_n)_n\right) = \sum_{n=1}^{\infty} \lambda_n y_n$$

is weakly compact. Now, the composed operator $T = S \circ P$ is weakly compact but not null almost M-weakly compact. Indeed, note that (u_n) is a disjoint sequence of B_E and (f_n) is a weakly null sequence in F'. From $|f_n(T(u_n))| = |f_n(y_n)| > \epsilon$ for all n, we conclude that $f_n(T(u_n)) \nrightarrow 0$, and the proof is finished. □

The following result tells us when every weakly compact operator $T : E \to F$ is always null almost L-weakly compact.

Theorem 2.4. *Let F be a Banach lattice and E be a non zero Banach lattice. If every weakly compact operator $T : E \to F$ is null almost L-weakly compact, then one of the conditions is valid:*

(1) *F is a KB-space;*

(2) *E has the weak Dunford-Pettis property.*

Proof. By hypothesis, every weakly compact operator $T : E \to F$ is null almost L-weakly compact, to complete the proof, it is enough to establish that if F is not a KB space then E has the weak Dunford-Pettis property. Now we suppose that F is not a KB-space, so by Theorem 2.4.12 of [10] (see, also Theorem 7.1 of [12]) F contains a closed order copy G of c_0. In virtue of Proposition 0.5.1 of [12], we have $G = \overline{\text{span}}\{y_n : n \in \mathbb{N}\}$ for some sequence $(y_n) \subset F$ of positive disjoint elements with $d = \inf \|y_n\| > 0$ and $c = \sup_k \left\|\sum_{n=1}^{k} y_k\right\| < \infty$. Moreover, the operator $S : c_0 \to F$ defined by

$$S\left((\lambda_n)_n\right) = \sum_{n=1}^{\infty} \lambda_n y_n$$

is a lattice homomorphism from c_0 onto F (see, proof of Proposition 0.5.1 of [12]). On the other hand, according to Theorem 116.3 of [13] there exists a disjoint sequence (f_n) of positive elements in the unit ball of F' such that $f_n(y_n) > d$ for all n and $f_n(y_m) = 0$ for $m \neq n$. In order to complete the proof, we have to show that E has the weak Dunford-Pettis property. Assume $x_n \xrightarrow{w} 0$ in E and (x'_n) is a disjoint weakly null sequence in E'. Consider the operator $R : E \to c_0$ defined by

$$R(x) = (x'_n(x))_{n=1}^{\infty}$$

By Theorem 5.26 of [1] the operator R is weakly compact, and so is the operator $T = S \circ R : E \to F$. From our hypothesis T is also a null almost L-weakly compact operator. So $f_n(T(x_n)) \longrightarrow 0$. Now for each n we have

$$\begin{aligned} f_n(T(x_n)) &= f_n\left(\sum_{k=1}^{\infty} x'_k(x_n)\, y_k\right) \\ &= \sum_{k=1}^{\infty} x'_k(x_n)\, f_n(y_k) \\ &= x'_n(x_n)\, f_n(y_n) \end{aligned}$$

and so

$$|f_n(T(x_n))| \geq |x'_n(x_n)|\, d$$

from which we conclude that $x'_n(x_n) \longrightarrow 0$. Thus, E has the weak Dunford-Pettis property and the proof is finished. □

As a consequence of this theorem, we deduce the following characterization of weak Dunford-Pettis property.

Corollary 2.6. *For a Banach space E the following assertions are equivalent:*

(1) *Every weakly compact operator $T : E \to c_0$ is null almost L -weakly compact.*

(2) *E has the weak Dunford-Pettis property.*

Proof. (1) $\Rightarrow$ (2) Follows immediately from Theorem 2.4 because c_0 is not a KB-space.

(2) $\Rightarrow$ (1) since X has the weak Dunford-Pettis property, every weakly compact operator $T : E \to c_0$ is an almost Dunford-Pettis operator, so by Theorem 2.2 of [3] and the fact that c_0 discrete and its norm is order continuous, T is Dunford-Pettis, then T is null almost L-weaky compact. □

Before state our main result, we need the definition and theorem from [10].

Definition 2.3. ([10], *Definition 1.4.18*)

i. *A positive linear operator $T : E \to F$ is called interval preserving if $T[0, x] = [0, Tx]$ for every $x \in E_+$*

ii. *Assume that E and F are normed Riesz spaces. A positive linear operator $T : E \to F$ is called almost interval preserving if $T[0, x]$ is dense in $[0, Tx]$ for every $x \in E_+$*

The adjoint of an interval preserving operator is always a lattice homomorphism.

Theorem 2.5. ([10], *Theorem 1.4.19.*)
For every normed Riesz spaces E and F and all $T \in \mathcal{L}(E, F)$ the following assertion hold. T is almost interval preserving if and only if T' is a lattice homomorphism.

Other properties of Null Almost L-weakly compact operators are given by the following theorem.

Theorem 2.6. *Consider the schema of operators $E \xrightarrow{T} F \xrightarrow{S} G$ where E, F and G are Banach lattices.*

(i) *If S is Null almost L-weakly compact operator, then $S \circ T$ is null almost L-weakly compact.*

(ii) *If T is a positive Null almost L-weakly compact and S is almost interval preserving operator, then $S \circ T$ is null almost L-weakly compact.*

Proof. (i) Obvious.

(ii) Let (x_n) be a weakly null sequence of E and (f_n) a disjoint sequence of $B_{G'}$. we have to show that $f_n(S \circ T(x_n)) \longrightarrow 0$. Since $S : F \longrightarrow G$ is almost interval preserving operator, then according to theorem 2.5, the operator $S^{'} : G^{'} \longrightarrow F^{'}$ is a lattice homomorphism. This is equivalent that the operator $S^{'}$ is disjointness preserving. So the sequence

$(S^{'}(f_n))$ of $B_{F'}$ is disjoint. On the other hand, since T is positive Null almost L-weakly compact, we have $S^{'}(f_n)(T(x_n)) \longrightarrow 0$. That is $f_n(S \circ T(x_n)) \longrightarrow 0$. The proof is completed.

□

References

[1] C.D. Aliprantis, O. Burkinshaw, Positive operators. Reprint of the 1985 original. Springer, Dordrecht, 2006.

[2] B. Aqzzouz, K. Bouras, (L) sets and almost (L)sets in Banach lattices, Quaest. Math. 36 (2013), 107-118. https://doi.org/10.2989/16073606.2013.779962.

[3] B. Aqzzoz, K. Bouras, Weak and almost Dunford–Pettis operators on Banach lattices, Demonstr. Math. 46 (2013), 165-179. https://doi.org/10.1515/dema-2013-0431.

[4] K. Bouras, M. Moussa, On the class of positive almost weak Dunford–Pettis operators, Positivity. 17 (2013), 589-600. https://doi.org/10.1007/S11117-012-0190-8.

[5] B. Aqzzouz, A. Elbour, O. Aboutafail, The weak compactness of almost Dunford-Pettis operators, Comment. Math. Univ. Carolin. 52 (2011), 31-35. https://dml.cz/dmlcz/141426.

[6] K. Bouras, D. Lhaimer, M. Moussa, On the class of almost L-weakly and almost M-weakly compact operator, Positivity. 22 (2018), 1433-1443. https://doi.org/10.1007/s11117-018-0586-1.

[7] N. Cheng, Z.L. Chen, b-AM-compact operators on Banach lattice, Chinese J. Eng. Math. 27 (2010), 753-756.

[8] P.G. Dodds, D.H. Fremlin, Compact operators on Banach lattices, Israel J. Math. 34 (1979), 287-320. https://doi.org/10.1007/BF02760610.

[9] A. Elbour, F. Afkir, M. Sabiri, Some properties of almost L-weakly and almost M-weakly compact operators, Positivity. 24 (2020), 141-149. https://doi.org/10.1007/s11117-019-00671-7.

[10] P. Meyer-Nieberg, Banach lattices. Universitext. Springer-Verlag, Berlin, 1991.

[11] A.W. Wickstead, Converses for the Dodds-Fremlin and Kalton-Saab theorems, Math. Proc. Camb. Phil. Soc. 120 (1996), 175-179. https://doi.org/10.1017/S0305004100074752.

[12] W. Wnuk, Banach lattices with order continuous norms, Polish Scientific Publishers PWN, Warsaw, 1999.

[13] A.C. Zaanen, Riesz spaces II, North Holland Publishing Company, Amsterdam (1983).